Konuralp Journal of Mathematics, *11 (1) (2023) 20-23*

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

A Solution Form of a Rational Difference Equation

Ramazan Karatas[1]

[1]*Akdeniz University Education Faculty 07058 Konyaaltı Antalya Turkey*

Abstract

This paper shows the solution form of the rational difference equation

$$x_{n+1} = \frac{ax_{n-(2k+3)}}{-a \mp x_{n-(k+1)}x_{n-(2k+3)}}\ ,\ n = 0, 1, \ldots$$

where k is a positive integer a and initial conditions are non-zero real numbers with $x_{n-(k+1)}x_{n-(2k+3)} \neq \mp a$ for all $n \in N_0$.

Keywords: *Rational difference equation; Solutions; Boundedness; Periodicity.*
2010 Mathematics Subject Classification: *39A10*

1. Introduction

The theory of difference equations play important role in applicable analysis. Rational difference equations are an important class of difference equations which have many applications in real life, for example the difference equation $x_{n+1} = \frac{a+bx_n}{cx_n}$,which is known as Riccati Difference Equation, has applications in optics and mathematical biology (see 15). Many researchers have investigated the solution of rational difference equations. For example see Refs. [1-16].
El-Sayed et al. [6] obtained the formulas of the recursive sequences

$$x_n = \frac{x_n x_{n-5}}{x_{n-4}(\pm 1 \pm x_n x_{n-5})}.$$

Gelişken and Karatas[2] studied the solutions of the difference equation

$$x_{n+1} = \frac{a_n x_{n-2k}}{b_n + c_n \prod_{i=0}^{2k} x_{n-i}}.$$

Simsek and Abdullayev [5] studied a solution of the difference equation

$$x_{n+1} = \frac{x_{n-(k+1)}}{1 + x_n x_{n-1} \ldots x_{n-k}}.$$

Karatas [12] studied the global behavior of the nonnegative equilibrium points of the difference equation

$$x_{n+1} = \frac{Ax_{n-m}}{B + C \prod_{i=0}^{2k+1} x_{n-i}}.$$

Çinar et al. [3] investigated the solutions of the difference equation

$$x_n = \frac{x_{n-3k}x_{n-4k}x_{n-5k}}{x_{n-k}x_{n-2k}\left(\pm 1 \pm x_{n-3k}x_{n-4k}x_{n-5k}\right)}$$

***Email address*:** **rkaratas@akdeniz.edu.tr (Ramazan Karatas)**

Abo-Zeid [10] investigated the global behavior of all solutions of the difference equation

$$x_{n+1}=\frac{Ax_{n-k}}{B+C\prod_{i=0}^{k}x_{n-i}}.$$

Karataş and Gelişken [11] investigated the solutions of the difference equation

$$x_{n+1}=\frac{(-1)^{n}x_{n-2k}}{a+(-1)^{n}\prod_{i=0}^{2k}x_{n-i}}.$$

Our aim in this paper is to obtain the solutions of the difference equation

$$x_{n+1}=\frac{ax_{n-(2k+3)}}{-a\mp x_{n-(k+1)}x_{n-(2k+3)}}\ ,\ n=0,1,... \tag{1.1}$$

where k is a positive integer, a and initial conditions are non zero real numbers with $x_{n-(k+1)}x_{n-(2k+3)}\neq\mp a$.

Definition 1.1. *Let I be some interval of real numbers and let $f:I^{k+1}\to I$ be a continuously differentiable function. Then for every set of initial conditions $x_{-k},x_{-(k+1)},...,x_{0}\in I$, the difference equation*

$$x_{n+1}=f\left(x_{n},x_{n-1},...,x_{n-k}\right),\ n=0,1,...$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.2. *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if*

$x_{n+p}=x_n$ for all $n\geq -k$.

2. Main Results

Theorem 2.1. *Let $\{x_n\}_{n=-(2k+3)}^{\infty}$ be a solution of $Eq.(1.1)$. Assume that $x_{n-(k+1)}x_{n-(2k+3)}\neq\mp a$. Then for $n=0,1,...$, all solutions of $Eq.(1.1)$ are of the form*

$$x_{(2k+4)n+i}=\begin{cases}\dfrac{a^{n+1}x_{-(2k+4-i)}}{\left[-a\mp x_{-(k+2-i)}x_{-(2k+4-i)}\right]^{n+1}}, & \text{for } i=1,2,...,k+2\\ \dfrac{1}{a^{n+1}}x_{-(2k+4-i)}\left[-a\mp x_{-(2k+4-i)}x_{-(3k+6-i)}\right]^{n+1}, & \text{for } i=k+3,k+4,...,2k+4\end{cases}$$

Proof. For $n=0$,the result holds. Assume that $n>0$ and that our assumption holds $n-1$. That is

$$x_{(2k+4)(n-1)+i}=\begin{cases}\dfrac{a^{n}x_{-(2k+4-i)}}{\left[-a\mp x_{-(k+2-i)}x_{-(2k+4-i)}\right]^{n}}, & \text{for } i=1,2,...,k+2\\ \dfrac{1}{a^{n}}x_{-(2k+4-i)}\left[-a\mp x_{-(2k+4-i)}x_{-(3k+6-i)}\right]^{n}, & \text{for } i=k+3,k+4,...,2k+4\end{cases} \tag{2.1}$$

Firstly, for $i=1$ it follows from $Eq.(1.1)$ and $Eq.(2.1)$ that

$$x_{(2k+4)n+1}=\frac{ax_{(2k+4)n-(2k+3)}}{-a\mp x_{(2k+4)n-(k+1)}x_{(2k+4)n-(2k+3)}}$$

$$=\frac{a\dfrac{a^{n}x_{-(2k+3)}}{\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^{n}}}{-a\mp\frac{1}{a^{n}}x_{-(k+1)}\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^{n}\dfrac{a^{n}x_{-(2k+3)}}{\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^{n}}}$$

$$=\frac{\dfrac{a^{n+1}x_{-(2k+3)}}{\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^{n}}}{-a\mp x_{-(k+1)}x_{-(2k+3)}}$$

$$=\frac{a^{n+1}x_{-(2k+3)}}{\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^{n+1}}.$$

That is, for $i=1$

$$x_{(2k+4)n+1}=\frac{a^{n+1}x_{-(2k+3)}}{\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^{n+1}}. \tag{2.2}$$

Similarly one can obtain other cases for $i=2,3,...,k+2$.

Secondly, we will show for $i=k+3$. It follows from $Eq.(1.1)$, $Eq.(2.1)$ and $Eq.(2.2)$ that

$$x_{(2k+4)n+k+3}=\frac{ax_{(2k+4)n-(k+1)}}{-a\mp x_{(2k+4)n+1}x_{(2k+4)n-(k+1)}}$$

$$= \frac{a\frac{1}{a^n}x_{-(k+1)}\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^n}{-a\mp\frac{a^{n+1}x_{-(2k+3)}}{\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^{n+1}}\frac{1}{a^n}x_{-(k+1)}\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^n}$$
$$= \frac{\frac{ax_{-(k+1)}\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^n}{a^n}}{-a\mp\frac{ax_{-(k+1)}x_{-(2k+3)}}{-a\mp x_{-(k+1)}x_{-(2k+3)}}}$$
$$= \frac{ax_{-(k+1)}\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^{n+1}}{a^{n+2}}.$$
So, for $i=k+3$

$$x_{(2k+4)n+k+3} = \frac{1}{a^{n+1}}x_{-(k+1)}\left[-a\mp x_{-(k+1)}x_{-(2k+3)}\right]^{n+1}. \tag{2.3}$$

Similarly one can obtain the other cases for $i=k+4,k+5,...,2k+4$.
The proof is completed. □

Theorem 2.2. *Assume that* $x_0x_{-(k+2)} = x_{-1}x_{-(k+3)} = ... = x_{-(k+1)}x_{-(2k+3)} = \mp 2a$.
Then every solution of $Eq.(1.1)$ *is periodic with period* $(2k+4)$.

Proof. In view of Theorem 1 and from our assumption, we get
$x_{(2k+4)n+1} = x_{-(2k+3)}, x_{(2k+4)n+2} = x_{-(2k+2)}, ..., x_{(2k+4)n+2k+4} = x_0$.
It is obvious that every solution of $Eq.(1.1)$ is periodic with period $(2k+4)$. □

Corollary 1. *Let* $\{x_n\}_{n=-(2k+3)}^{\infty}$ *be a solution of equation* $x_{n+1} = \frac{ax_{n-(2k+3)}}{-a+x_{n-(k+1)}x_{n-(2k+3)}}$. *Assume that*
$a=1, x_{-(2k+3)}, x_{-(2k+2)}, ..., x_0 > 0$
and
$x_0x_{-(k+2)} > 2, x_{-1}x_{-(k+3)} > 2, ..., x_{-(k+1)}x_{-(2k+3)} > 2$.
Then
$\lim_{n\to\infty} x_{(2k+4)n+i} = 0 \quad (i=1,2,...,k+2)$,
$\lim_{n\to\infty} x_{(2k+4)n+i} = \infty \quad (i=k+3,k+4,...2k+4)$.

Proof. Let $a=1, x_{-(2k+3)}, x_{-(2k+2)}, ..., x_0 > 0$
and
$x_0x_{-(k+2)} > 2, x_{-1}x_{-(k+3)} > 2, ..., x_{-(k+1)}x_{-(2k+3)} > 2$.
So we can write
$-1+x_0x_{-(k+2)} > 1, -1+x_{-1}x_{-(k+3)} > 1, ..., -1+x_{-(k+1)}x_{-(2k+3)} > 1$.
From Theorem 1 we get
for $i=1,2,...,k+2$,
$\lim_{n\to\infty} x_{(2k+4)n+i} = \lim_{n\to\infty} \frac{x_{-(2k+4-i)}}{\left[-1+x_{-(k+2-i)}x_{-(2k+4-i)}\right]^{n+1}} = 0$
and
for $i=k+3,k+4,...,2k+4$,
$\lim_{n\to\infty} x_{(2k+4)n+i} = \lim_{n\to\infty} x_{-(2k+4-i)}\left[-1\mp x_{-(2k+4-i)}x_{-(3k+6-i)}\right]^{n+1} = \infty$.
The poof is completed. □

Corollary 2. *Let* $\{x_n\}_{n=-(2k+3)}^{\infty}$ *be a solution of equation* $x_{n+1} = \frac{ax_{n-(2k+3)}}{-a+x_{n-(k+1)}x_{n-(2k+3)}}$. *Assume that*
$a=1, x_{-(2k+3)}, x_{-(2k+2)}, ..., x_0 < 0$
and
$x_0x_{-(k+2)} > 2, x_{-1}x_{-(k+3)} > 2, ..., x_{-(k+1)}x_{-(2k+3)} > 2$.
Then
$\lim_{n\to\infty} x_{(2k+4)n+i} = 0 \quad (i=1,2,...,k+2)$,
$\lim_{n\to\infty} x_{(2k+4)n+i} = -\infty \quad (i=k+3,k+4,...2k+4)$.

Proof. The proof is similar to Corollay 1. □

The following corollaries can be written from view of Theorem 1.

Corollary 3. *Let* $\{x_n\}_{n=-(2k+3)}^{\infty}$ *be a solution of equation* $x_{n+1} = \frac{ax_{n-(2k+3)}}{-a-x_{n-(k+1)}x_{n-(2k+3)}}$. *Assume that*
$a>0$ *and* $x_{-(2k+3)}, x_{-(2k+2)}, ..., x_0 < 0$.
Then all solutions of $Eq.(1.1)$ *are positive.*

Corollary 4. *Let* $\{x_n\}_{n=-(2k+3)}^{\infty}$ *be a solution of equation* $x_{n+1} = \frac{ax_{n-(2k+3)}}{-a+x_{n-(k+1)}x_{n-(2k+3)}}$. *Assume that*
$a>0, x_{-(2k+3)}, x_{-(2k+2)}, ..., x_0 > 0$
and
$x_0x_{-(k+2)} > a, x_{-1}x_{-(k+3)} > a, ..., x_{-(k+1)}x_{-(2k+3)} > a$.
Then all solutions of $Eq.(1.1)$ *are positive.*

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] Gelişken, A., *On a system of rational difference equation*, Journal of Computational Analysis and Applications, 23(4), 593-606, 2017.
[2] Gelişken, A., Karataş, R., *On a solvable difference equation with sequence coefficients*, Advances and Applications in Discrete Mathematics, 30, 27-33, 2022.
[3] Çinar, G., Gelişken, A., "Ozkan, O., *Well-defined solutions of the difference equation* $x_n = \frac{x_{n-3k}x_{n-4k}x_{n-5k}}{x_{n-k}x_{n-2k}(\pm 1 \pm x_{n-3k}x_{n-4k}x_{n-5k})}$, Asian-European Journal of Mathematics, 12(6), 2019.
[4] Simsek, D., Abdullayev, F., *On the recursive sequence* $x_{n+1} = \frac{x_{n-(4k+3)}}{1+\prod_{t=0}^{2} x_{n-(k+1)t-k}}$, Journal of Mathematical Sciences, 6(222), 762-771, 2017.
[5] Simsek, D., Abdullayev, F., *On the recursive sequence* $x_{n+1} = \frac{x_{n-(k+1)}}{1+x_n x_{n-1} \cdots x_{n-k}}$, Journal of Mathematical Sciences, 234(1), 73-81, 2018.
[6] Elsayed, E. M., Alzahrani, F., Alayachi, H. S., *Formulas and properties of some class of nonlinear difference equation*, Journal of Computational Analysis and Applications, 24(8), 1517-1531, 2018.
[7] Almatrafi, M. B., Elsayed, E. M., Alzahrani, F., *Investigating some properties of a fourth order difference equation*, Journal of Computational Analysis and Applications, 28(2), 243-253, 2020.
[8] Ari, M., Gelişken, A., *Periodic and asymptotic behavior of a difference equation*, Asian-European Journal of Mathematics, 12(6), 2040004, 10pp, 2019.
[9] "Ozkan, O., Kurbanlı, A. S., *On a system of difference equations*, Discrete Dynamics in Nature and Society, 2013.
[10] Abo-Zeid, R., *Behavior of solutions of higher order difference equation*, Alabama Journal of Mathematics, 42, 1-10, 2018.
[11] Karataş, R., Gelişken, A., A solution form of a higher order difference equation, *Korunalp Journal of Mathematics*, 9(2), 316-323, 2021.
[12] Karatas, R., Global behavior of a higher order difference equation, *Computers and Mathematics with Applications*, 60, 830-839, 2010.
[13] Karatas, R., On the solutions of the recursive sequence $x_{n+1} = \frac{\alpha x_{n-(2k+1)}}{-a+x_{n-k}x_{n-(2k+1)}}$, *Fasciculi Mathematici*, 45, 37-45, 2010.
[14] Ergin, S., Karatas, R., On the solutions of the recursive sequence $x_{n+1} = \frac{\alpha x_{n-k}}{a-\prod_{i=0}^{k} x_{n-i}}$, *Thai Journal of Mathematics*, 14(2), 391-397, 2016.
[15] Saary, T.I., *Modern Nonlinear Equations*, McGraw Hill, Newyork, 1967.
[16] Kocic, V.L., Ladas, G., *Global Behavior of Nonlinear Difference Equations of High Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.