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Stability analysis for a recovered fracturing fluid model in the wellbore of shale gas reservoir

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Abstract

This paper is concerned with the study of stability analysis to a complicated recovered fracturing fluid model (RFFM, for short), which consists of a stationary incompressible Stokes equation involving multivalued and nonmonotone boundary conditions, and a reaction-diffusion equation with Neumann boundary conditions. Firstly, we introduce a family of perturbated problems corresponding to (RFFM) and deliver the variational formulation of perturbated problem which is a hemivariational inequality coupled with a variational equation. Then, we prove that the existence of weak solutions to perturbated problems and the solution sequence to perturbated problems are uniformly bounded. Finally, via employing Mosco convergent approach and the theory of nonsmooth, a stability result to (RFFM) is established.

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1. Introduction

Recently, in [5], the authors applied various constitutive laws, such as the law of conservation of mass, Fick's diffusion law, inflow law, friction law and etc, for introducing a complicated recovered fracturing fluid model, in order to study the flow behavior of recovered fluid and the reaction-diffusion phenomenon of contaminants in the wellbore of shale gas reservoir during the early stage of fracturing fluid flowback. Indeed, the recovered fracturing fluid model in [5] is exactly formulated by a stationary incompressible Stokes equation involving multivalued and nonmonotone boundary conditions, and a reaction-diffusion equation with Neumann boundary condition.

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Problem 1.1. Find a velocity field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$, a pressure field $p\colon \Omega \to \mathbb{R}$ and a concentration field $y\colon \Omega \to \mathbb{R}$ such that

$$-\mu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D, \tag{1.3}$$

$$\begin{cases} u_\nu = 0, \\ -\boldsymbol{\tau}_\tau(\boldsymbol{u}) \in \delta(y)\partial j(\boldsymbol{x}, \boldsymbol{u}_\tau), \end{cases} \quad \text{on} \quad \Gamma_{C_1}, \tag{1.4}$$

$$\begin{cases} u_\nu \leq 0, \\ \tau_\nu(\boldsymbol{u}, p) \leq 0, \\ u_\nu \tau_\nu(\boldsymbol{u}, p) = 0, \\ \boldsymbol{\tau}_\tau(\boldsymbol{u}) = \boldsymbol{0}, \end{cases} \quad \text{on} \quad \Gamma_{C_2}, \tag{1.5}$$

$$\begin{cases} u_\nu \geq 0, \\ \tau_\nu(\boldsymbol{u}, p) = -\phi, \\ \boldsymbol{\tau}_\tau(\boldsymbol{u}) = \boldsymbol{0}, \end{cases} \quad \text{on} \quad \Gamma_{C_3}, \tag{1.6}$$

and

$$\begin{cases} -\operatorname{div}(\beta(\boldsymbol{u})\nabla y) + g(\boldsymbol{x}, y) = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial y}{\partial \nu_\beta} := (\beta(\boldsymbol{u})\nabla y) \cdot \boldsymbol{\nu} = h\chi_{\Gamma_{C_2} \cup \Gamma_{C_3}} & \text{on} \quad \Gamma, \end{cases} \tag{1.7}$$

where the boundary Γ is assumed to be divided into fourth disjoint parts Γ_D, Γ_{C_1}, Γ_{C_2} and Γ_{C_3} with Γ_D having positive measure, $\boldsymbol{\nu}$ is the unit outward normal on the boundary Γ, $\chi_{\Gamma_{C_2} \cup \Gamma_{C_3}}$ stands for the characteristic function of Γ_{C_2} and Γ_{C_3}, $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$ are the normal and tangential components of velocity field $\boldsymbol{u}$ on the boundary Γ, $\tau_\nu(\boldsymbol{u}, p) = \boldsymbol{\tau}(\boldsymbol{u}, p) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\tau}_\tau(\boldsymbol{u}) = \boldsymbol{\tau}(\boldsymbol{u}, p) - \tau_\nu(\boldsymbol{u}, p)\boldsymbol{\nu}$ are the normal and tangential components to traction vector field $\boldsymbol{\tau}$.

In Problem 1.1, condition (1.3) models that the velocity $\boldsymbol{u}$ satisfies homogeneous Dirichlet condition on Γ_D; condition (1.4) reflects that there is no phenomenon of osmosis to the recovered fracturing fluid and the tangential component of the traction vector $\boldsymbol{\tau}_\tau(\boldsymbol{u})$ satisfies a multivalued and nonmonotone friction constitutive law on Γ_{C_1}; condition (1.5) indicate that recovered fracturing fluid satisfies the inflow boundary condition on Γ_{C_2}; condition (1.6) represent that recovered fracturing fluid satisfies the outflow boundary condition on Γ_{C_3}; condition $(1.7)_2$ show that contaminants satisfies the nonhomogeneous Neumann boundary condition on Γ_{C_2} and Γ_{C_3}.

Since the physical quantities, data and coefficients in recovered fracturing fluid measured by various devices are not precise in general. Therefore, a necessary and significant study concerning stability analysis should be carried out for determining the practicality and effectiveness of a certain mathematical model (see [1,2,10,16]). On the other hand, stability to a certain mathematical model is quite critical for ensuring whether the computational implementation of the model under consideration is overly sensitive to possible round-off errors (see [3,4,13–15,17,19,21,23–28]). From the view-point of numerical approximations, the stability is particularly important since a numerical solution is meaningful only if the certain mathematical model being solved is stable with respect to the data (see [11, 12, 22]). Based on these motivations, this paper is devoted to explore the stability of recovered fracturing fluid model, Problem 1.1. More precisely, the main aim of this paper is twofold. The first one is to introduce a family of perturbated problems (see Problem 3.1) corresponding to Problem 1.1, and to obtain the existence of weak solutions for perturbated problems. However, the second goal is to apply Mosco convergent approach and the theory of nonsmooth for establishing a stability result to Problem 1.1, which reveals that the solution set of Problem 1.1 can be approached in the sense of Kuratowski by the perturbated problem, Problem 3.1, when perturbated parameter tends to zero.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary preliminary materials, and review the variational formula as well as the existence of weak solutions to Problem 1.1. Section 3 is devoted to introduce a family of perturbated problems (see Problem 3.1) corresponding to Problem 1.1, and to impose the mild assumptions to perturbated operators in Problem 3.1. Finally, in Section 4, a stability result to recovered fracturing fluid model, Problem 1.1, is established.

2. Preliminaries and hypotheses

In this section, we shall recall some necessary notations, basic definitions and a result on solvability to Problem 1.1.

Given a normed space X, we denote by $\|\cdot\|_X$ and X^* the norm and topological dual of X, respectively. In what follows, the symbol $\langle\cdot,\cdot\rangle_{X^*\times X}$ stands for the duality pairing between X^* and X. However, if no confusion arises, we often skip the subscript. The weak and the strong convergences in X are denoted by "$\rightharpoonup$" and "$\to$", respectively. Furthermore, we denote by $\mathcal{L}(X_1,X_2)$ the space of linear and bounded operators from a normed space X_1 to a normed space X_2 endowed with the operator norm $\|\cdot\|_{\mathcal{L}(X_1,X_2)}$.

Let us recall the definitions concerning the generalized directional derivative and generalized gradient in the sense of Clarke for locally Lipschitz functions, see [6–9].

Definition 2.1. Let $h\colon X\to\mathbb{R}$ be a locally Lipschitz function defined on a Banach space X. The generalized Clarke directional derivative of h at the point $u\in X$ in the direction $v\in X$, denoted by $h^0(u;v)$, is defined by

$$h^0(u;v)=\limsup_{\lambda\to 0^+,\,w\to u}\frac{h(w+\lambda v)-h(w)}{\lambda}.$$

The generalized Clarke subgradient of h at $u\in X$, denoted by $\partial h(u)$, is a subset in the dual space X^* given by

$$\partial h(u)=\left\{\,\xi\in X^*\mid h^0(u;v)\geq\langle\xi,v\rangle\ \text{ for all } v\in X\,\right\}.$$

Some critical properties for generalized directional derivative and generalized subgradient in the sense of Clarke are selected by the following proposition, see [18, Proposition 3.23].

Proposition 2.2. Assume that $h\colon X\to\mathbb{R}$ is a locally Lipschitz function. Then the following assertions hold:

(i) for every $u\in X$, the function $X\ni v\mapsto h^0(u;v)\in\mathbb{R}$ is positively homogeneous and subadditive, i.e.,

$$h^0(u;\lambda v)=\lambda h^0(u;v)\text{ and } h^0(u;v_1+v_2)\leq h^0(u;v_1)+h^0(u;v_2)$$

for all $\lambda\geq 0$ and $v,u,v_1,v_2\in X$.

(ii) for each $v\in X$ fixed, there exists an element $\xi_v\in\partial h(u)$ satisfying $h^0(u;v)=\langle\xi_v,v\rangle$, so, by the definition of generalized Clarke subgradient, we have $h^0(u;v)=\max\{\,\langle\xi,v\rangle\mid\xi\in\partial h(u)\,\}$.

(iii) the function $X\times X\ni(u,v)\mapsto h^0(u;v)\in\mathbb{R}$ is upper semicontinuous.

Also, we review the definition of Mosco convergence, see e.g. [8, Chapter 4.7] and [20].

Definition 2.3. Let X be a Banach space and $\{K_\rho,K\}_{\rho>0}\subset 2^X\setminus\{\emptyset\}$. We say that K_ρ converges to K in the sense of Mosco as $\rho\to 0$, denoted by $K_\rho\xrightarrow{M}K$, if and only if the conditions hold:

(i) for each $u \in K$, there exists a sequence $\{u_\rho\}_{\rho>0}$ such that $u_\rho \in K_\rho$ for every $\rho > 0$ and $u_\rho \to u$ in X;
(ii) for each sequence $\{u_\rho\}_{\rho>0}$ such that $u_\rho \in K_\rho$ for every $\rho > 0$ and $u_\rho \rightharpoonup u$ in X, we have $u \in K$.

In order to give the definition of weak solutions to Problem 1.1, we need the following function spaces and admissible set for velocity fields:

$$\widetilde{V} = \left\{ \boldsymbol{u} \in C^\infty(\overline{\Omega}; \mathbb{R}^d) |\ \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_D \text{ and } u_\nu = 0 \text{ on } \Gamma_{C_1} \right\}, \quad (2.1)$$

$$V = \overline{\widetilde{V}}^{H^1(\Omega;\mathbb{R}^d)}, \text{ i.e., } V \text{ is the closure of } \widetilde{V} \text{ in } H^1(\Omega; \mathbb{R}^d), \quad (2.2)$$

$$K = \{ \boldsymbol{u} \in V \ |\ u_\nu \le 0 \text{ on } \Gamma_{C_2} \text{ and } u_\nu \ge 0 \text{ on } \Gamma_{C_3} \}. \quad (2.3)$$

Because Γ_D has positive measure, it is not difficult to apply Korn inequality to conclude that the norm $\|\cdot\|_V$ defined by

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{D}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)} \quad \text{for all } \boldsymbol{v} \in V,$$

is an equivalent norm of V. In the meanwhile, by the Rellich-Kondrachov theorem (see e.g. [18, Theorem 2.16]), we can see that the embeddings

$$H^1(\Omega; \mathbb{R}^3) \subset L^4(\Omega; \mathbb{R}^3), \quad \text{and} \quad H^1(\Omega; \mathbb{R}^2) \subset L^q(\Omega; \mathbb{R}^2) \text{ for any } q \ge 1$$

are continuous and compact.

Moreover, we make the following hypotheses on the data of Problem 1.1.

$\underline{H(\boldsymbol{f})}$: $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^d)$.

$\underline{H(h)}$: $h \in L^2(\Gamma_{C_2})$.

$\underline{H(\phi)}$: $\phi \in L^2(\Gamma_{C_3})$.

$\underline{H(j)}$: $j \colon \Gamma_{C_1} \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following conditions:

(i) $j(\cdot, \boldsymbol{\xi})$ is measurable on Γ_{C_1} for all $\boldsymbol{\xi} \in \mathbb{R}^d$, and $j(\cdot, \boldsymbol{0}) \in L^1(\Gamma_{C_1})$;
(ii) $j(\boldsymbol{x}, \cdot)$ is locally Lipschitz for a.e. $\boldsymbol{x} \in \Gamma_{C_1}$;
(iii) there exist $c_0 \in L^2(\Gamma_{C_1}; \mathbb{R}_+)$ and $c_1 \ge 0$ such that

$$\|\partial j(\boldsymbol{x}, \boldsymbol{\xi})\|_{\mathbb{R}^d} \le c_0(\boldsymbol{x}) + c_1 \|\boldsymbol{\xi}\|_{\mathbb{R}^d} \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d \text{ and a.e. } \boldsymbol{x} \in \Gamma_{C_1},$$

where ∂j is the generalized Clarke subdifferential operator of j with respect to its second variable;
(iv) there exists a constant $\alpha_j \ge 0$ such that

$$j^0(\boldsymbol{x}, \boldsymbol{r_1}; \boldsymbol{r_2} - \boldsymbol{r_1}) + j^0(\boldsymbol{x}, \boldsymbol{r_2}; \boldsymbol{r_1} - \boldsymbol{r_2}) \le \alpha_j \|\boldsymbol{r_1} - \boldsymbol{r_2}\|^2_{\mathbb{R}^d} \quad \text{for all } \boldsymbol{r_1}, \boldsymbol{r_2} \in \mathbb{R}^d \text{ and a.e. } \boldsymbol{x} \in \Gamma_{C_1}.$$

$\underline{H(\delta)}$: $\delta \colon \Gamma_{C_1} \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $\delta(\cdot, y)$ is measurable on Γ_{C_1} for all $y \in \mathbb{R}$;
(ii) $\delta(\boldsymbol{x}, \cdot)$ is continuous in $\mathbb{R}$ for a.e. $\boldsymbol{x} \in \Gamma_{C_1}$;
(iii) there exist constants $\delta_0, \delta_1 > 0$ such that

$$0 < \delta_0 \le \delta(\boldsymbol{x}, z) \le \delta_1 \text{ for all } z \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Gamma_{C_1}.$$

$\underline{H(\beta)}$: $\beta \colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ enjoys the following properties

(i) $\beta(\cdot, \boldsymbol{u})$ is measurable in Ω for all $\boldsymbol{u} \in \mathbb{R}^d$;
(ii) $\beta(\boldsymbol{x}, \cdot)$ is continuous in $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Omega$;
(iii) there exist constants $\beta_0, \beta_1 > 0$ such that

$$0 < \beta_0 \le \beta(\boldsymbol{x}, z) \le \beta_1 \text{ for all } z \in \mathbb{R}^d \text{ and a.e. } \boldsymbol{x} \in \Omega.$$

$\underline{H(g)}$: $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodoary function such that

(i) there exist a function $\alpha_g \in L^4(\Omega;\mathbb{R})$ and a constant $c_g > 0$ such that

$$|g(\boldsymbol{x}, r)| \leq \alpha_g(\boldsymbol{x}) + c_g|r|^4 \text{ for all } r \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Omega;$$

(ii) there exists a constant $d_g > 0$ such that

$$(g(\boldsymbol{x}, r_1) - g(\boldsymbol{x}, r_2))(r_1 - r_2) \geq d_g|r_1 - r_2|^2 \text{ for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Omega.$$

From [5], we have the weak variational formulation of Problem 1.1 as follows.

PROBLEM 2.4. Find a concentration field $y \in H^1(\Omega)$ and a velocity filed $\boldsymbol{u} \in K$ such that

$$\begin{aligned} &2\mu \int_\Omega \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v} - \boldsymbol{u})\, d\boldsymbol{x} + \int_{\Gamma_{C_1}} \delta(y) j^0(\boldsymbol{x}, \boldsymbol{u}_\tau; \boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, d\Gamma \\ &\geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u})\, d\boldsymbol{x} - \int_{\Gamma_{C_3}} \phi(v_\nu - u_\nu)\, d\Gamma \end{aligned} \tag{2.4}$$

for all $\boldsymbol{v} \in K$, and

$$\int_\Omega \beta(\boldsymbol{u}) \nabla y \cdot \nabla z\, d\boldsymbol{x} + \int_\Omega g(\boldsymbol{x}, y) z\, d\boldsymbol{x} - \int_{\Gamma_{C_2} \cup \Gamma_{C_3}} hz\, d\Gamma = 0 \quad \text{for all} \quad z \in H^1(\Omega). \tag{2.5}$$

It should be pointed out that a pair of functions $(\boldsymbol{u}, y) \in K \times H^1(\Omega)$ that satisfies Problem 2.4 is called to be a weak solution of Problem 1.1.

Let us consider the functions

$$A\colon V \to V^*,\ \langle A\boldsymbol{u}, \boldsymbol{v}\rangle = 2\mu \int_\Omega \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v})\, d\boldsymbol{x} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V, \tag{2.6}$$

$$\widetilde{\boldsymbol{f}} \in V^*,\ \langle \widetilde{\boldsymbol{f}}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x} - \int_{\Gamma_{C_3}} \phi v_\nu\, d\Gamma \quad \text{for all } \boldsymbol{v} \in V, \tag{2.7}$$

$$\gamma\colon V \to L^2(\Gamma; \mathbb{R}^d),\ \gamma\boldsymbol{v} = \boldsymbol{v}_\tau \quad \text{for all } \boldsymbol{v} \in V. \tag{2.8}$$

Using (2.6)–(2.8), we can observe that Problem 2.4 can be rewritten equivalently to the following one:

PROBLEM 2.5. Find a concentration field $y \in H^1(\Omega)$ and a velocity filed $\boldsymbol{u} \in K$ such that

$$\langle A\boldsymbol{u} - \widetilde{\boldsymbol{f}}, \boldsymbol{v} - \boldsymbol{u}\rangle + \int_{\Gamma_{C_1}} \delta(y) j^0(\boldsymbol{x}, \gamma\boldsymbol{u}; \gamma(\boldsymbol{v} - \boldsymbol{u}))\, d\Gamma \geq 0 \quad \text{for all} \quad \boldsymbol{v} \in K, \tag{2.9}$$

and

$$\int_\Omega \beta(\boldsymbol{u}) \nabla y \cdot \nabla z\, d\boldsymbol{x} + \int_\Omega g(\boldsymbol{x}, y) z\, d\boldsymbol{x} - \int_{\Gamma_{C_2} \cup \Gamma_{C_3}} hz\, d\Gamma = 0 \quad \text{for all} \quad z \in H^1(\Omega).$$

Under the assumptions $H(j)$, $H(\delta)$, $H(\beta)$, $H(\boldsymbol{f})$, $H(h)$, $H(\phi)$ and $H(g)$, the authors in [5, Theorem 5.1] applied a surjective theorem of pseudomonotone operators, monotonicity arguments and Schauder fixed point theorem to establish the following existence theorem for Problem 1.1.

THEOREM 2.6. Assume that $H(j)$, $H(\delta)$, $H(\beta)$, $H(\boldsymbol{f})$, $H(h)$, $H(\phi)$, $H(g)$ and the following smallness condition are satisfied

$$2\mu > \delta_1 \alpha_j \|\gamma\|^2.$$

Then Problem 1.1 has at least one weak solution $(\boldsymbol{u}, y) \in K \times H^1(\Omega)$.

3. Perturbation of recovered fracturing fluid model

In order to explore the stability analysis for the recovered fracturing fluid model, Problem 1.1, this section is devoted to introduce a family of perturbated problems corresponding to Problem 1.1 and to deliver the variational formulations for perturbated problems. Let $\rho > 0$ be a given parameter. Assume that data μ, $\boldsymbol{f}$, ϕ, δ, j, β, g and h are perturbated by parameter ρ, (so, we have perturbated operators and coefficients μ_ρ, $\boldsymbol{f}_\rho$, δ_ρ, j_ρ, ϕ_ρ, β_ρ, g_ρ and h_ρ. Let us consider the following pertubated recovered fracturing fluid model.

Problem 3.1. Find a velocity field $\boldsymbol{u}_\rho\colon \Omega \to \mathbb{R}^d$, a pressure field $p_\rho\colon \Omega \to \mathbb{R}$ and a concentration field $y_\rho\colon \Omega \to \mathbb{R}$ such that

$$-\mu_\rho \Delta \boldsymbol{u}_\rho + \nabla p_\rho = \boldsymbol{f}_\rho \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\nabla \cdot \boldsymbol{u}_\rho = 0 \quad \text{in} \quad \Omega, \tag{3.2}$$

$$\boldsymbol{u}_\rho = \boldsymbol{0} \quad \text{on} \quad \Gamma_D, \tag{3.3}$$

$$\begin{cases} u_{\rho_\nu} = 0, \\ -\boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) \in \delta_\rho(y_\rho)\partial j_\rho(\boldsymbol{x}, \boldsymbol{u}_{\rho_\tau}), \end{cases} \quad \text{on} \quad \Gamma_{C_1}, \tag{3.4}$$

$$\begin{cases} u_{\rho_\nu} \le \rho, \\ \tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho) \le 0, \\ (u_{\rho_\nu} - \rho)\tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho) = 0, \\ \boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) = \boldsymbol{0}, \end{cases} \quad \text{on} \quad \Gamma_{C_2}, \tag{3.5}$$

$$\begin{cases} u_{\rho_\nu} \ge 0, \\ \tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho) = -\phi_\rho, \\ \boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) = \boldsymbol{0}, \end{cases} \quad \text{on} \quad \Gamma_{C_3}, \tag{3.6}$$

and

$$\begin{cases} -\operatorname{div}(\beta_\rho(\boldsymbol{u}_\rho)\nabla y_\rho) + g_\rho(\boldsymbol{x}, y_\rho) = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial y_\rho}{\partial \nu_{\beta_\rho}} := (\beta_\rho(\boldsymbol{u}_\rho)\nabla y_\rho) \cdot \boldsymbol{\nu} = h_\rho \chi_{\Gamma_{C_2} \cup \Gamma_{C_3}} & \text{on} \quad \Gamma. \end{cases} \tag{3.7}$$

It follows from Problem 1.1 that $\boldsymbol{\tau}(\boldsymbol{u}, p) := \boldsymbol{P} \cdot \boldsymbol{\nu}$ stands for the traction vector of total stress tensor $\boldsymbol{P}$ on Γ, and the total stress tensor $\boldsymbol{P}$ on Γ satisfies the following equation (see equation (3.8) in paper [5])

$$\boldsymbol{P} = -p\boldsymbol{I} + \boldsymbol{S} \quad \text{with} \quad \boldsymbol{S} = 2\mu \boldsymbol{D}(\boldsymbol{u}).$$

However, on the boundary $\Gamma_{C_1} \cup \Gamma_{C_2} \cup \Gamma_{C_3}$, we can decompose the extra stress tensor field $\boldsymbol{S}$ into the normal and tangential components, i.e., $S_\nu = (\boldsymbol{S}\boldsymbol{\nu}) \cdot \boldsymbol{\nu} = 2\mu D(\boldsymbol{u})_\nu$ and

$$\begin{aligned} \boldsymbol{S}_\tau = \boldsymbol{S}\boldsymbol{\nu} - (\boldsymbol{S}\boldsymbol{\nu} \cdot \boldsymbol{\nu})\boldsymbol{\nu} &= 2\mu \boldsymbol{D}(\boldsymbol{u})\boldsymbol{\nu} - (2\mu D(\boldsymbol{u})_\nu)\boldsymbol{\nu} \\ &= 2\mu \boldsymbol{D}(\boldsymbol{u})\boldsymbol{\nu} - (2\mu \boldsymbol{D}(\boldsymbol{u})\boldsymbol{\nu} - 2\mu \boldsymbol{D}(\boldsymbol{u})_\tau) \\ &= 2\mu \boldsymbol{D}(\boldsymbol{u})_\tau. \end{aligned}$$

Using these notation above, we have $\tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho) = \boldsymbol{\tau}_\rho(\boldsymbol{u}_\rho, p_\rho) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho, p_\rho) = \boldsymbol{\tau}_\rho(\boldsymbol{u}_\rho, p_\rho) - \tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho)\boldsymbol{\nu}$. Therefore, it holds

$$\tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho) = S_{\rho_\nu}(\boldsymbol{u}_\rho) - p_\rho = 2\mu_\rho D(\boldsymbol{u}_\rho)_\nu - p_\rho \quad \text{and} \quad \boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) = \boldsymbol{S}_{\rho_\tau}(\boldsymbol{u}_\rho) = 2\mu_\rho \boldsymbol{D}(\boldsymbol{u}_\rho)_\tau.$$

Moreover, we assume that μ_ρ, $\boldsymbol{f}_\rho$, ϕ_ρ, δ_ρ, j_ρ, β_ρ, g_ρ and h_ρ satisfy the following conditions.

$\underline{H(\boldsymbol{f}_\rho)}$: $\boldsymbol{f}_\rho \in L^2(\Omega; \mathbb{R}^d)$, and $\boldsymbol{f}_\rho \to \boldsymbol{f}$ in $L^2(\Omega; \mathbb{R}^d)$ as $\rho \to 0$.

$\underline{H(h_\rho)}$: $h_\rho \in L^2(\Gamma_{C_2})$, and $h_\rho \to h$ in $L^2(\Gamma_{C_2})$ as $\rho \to 0$.

$\underline{H(\phi_\rho)}$: $\phi_\rho \in L^2(\Gamma_{C_3})$, and $\phi_\rho \to \phi$ in $L^2(\Gamma_{C_3})$ as $\rho \to 0$.

$\underline{H(\mu_\rho)}$: $\mu_\rho, \mu \in (0, +\infty)$, and $\mu_\rho \to \mu$ as $\rho \to 0$.

$\underline{H(j_\rho)}$: $j_\rho \colon \Gamma_{C_1} \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) j_ρ is such that hypotheses $H(j)$ hold with $c_{0_\rho} \in L^2(\Gamma_{C_1}; \mathbb{R}_+)$, $c_{1_\rho} \geq 0$, $\alpha_{j_\rho} \geq 0$, respectively;

(ii) for all $\{\boldsymbol{u}_\rho\}, \{\boldsymbol{v}_\rho\} \subset \mathbb{R}^d$ with $\boldsymbol{u}_\rho \to \boldsymbol{u}$ and $\boldsymbol{v}_\rho \to \boldsymbol{v}$ in $L^2(\Gamma_{C_1}; \mathbb{R}^d)$ as $\rho \to 0$, we have

$$\limsup_{\rho \to 0} j_\rho^0(\boldsymbol{u}_\rho; \boldsymbol{v}_\rho - \boldsymbol{u}_\rho) \leq j^0(\boldsymbol{u}; \boldsymbol{v} - \boldsymbol{u}) \quad \text{for a.e. } \boldsymbol{x} \in \Gamma_{C_1}.$$

$\underline{H(\delta_\rho)}$: $\delta_\rho \colon \Gamma_{C_1} \times \mathbb{R} \to \mathbb{R}$ is such that

(i) δ_ρ satisfies $H(\delta)$ with $\delta_{0_\rho} > 0$ and $\delta_{1_\rho} > 0$;

(ii) for all $\{z_\rho\} \subset \mathbb{R}$ with $z_\rho \to z$ as $\rho \to 0$, we have

$$\lim_{\rho \to 0} \delta_\rho(\boldsymbol{x}, z_\rho) = \delta(\boldsymbol{x}, z) \quad \text{for a.e. } \boldsymbol{x} \in \Gamma_{C_1}.$$

$\underline{H(\beta_\rho)}$: $\beta_\rho \colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) β_ρ fulfills condition $H(\beta)$ with $\beta_{0_\rho} > 0$, $\beta_{1_\rho} > 0$ and $\inf_{\rho > 0} \beta_{0_\rho} > 0$;

(ii) for all $\{\boldsymbol{u}_\rho\} \subset \mathbb{R}^d$ and $\boldsymbol{u} \in \mathbb{R}^d$ with $\boldsymbol{u}_\rho \to \boldsymbol{u}$ as $\rho \to 0$, we have

$$\lim_{\rho \to 0} \beta_\rho(\boldsymbol{x}, \boldsymbol{u}_\rho) = \beta(\boldsymbol{x}, \boldsymbol{u}) \quad \text{for a.e. } \boldsymbol{x} \in \Omega.$$

$\underline{H(g_\rho)}$: $g_\rho \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

(i) g_ρ reads hypotheses $H(g)$ with $\alpha_{g_\rho} \in L^4(\Omega; \mathbb{R})$, $c_{g_\rho} > 0$, $d_{g_\rho} > 0$ and $\inf_{\rho > 0} d_{g_\rho} > 0$;

(ii) for all $\{r_\rho\} \subset \mathbb{R}$ with $r_\rho \to r$ as $\rho \to 0$, we have

$$\lim_{\rho \to 0} g_\rho(\boldsymbol{x}, r_\rho) = g(\boldsymbol{x}, r) \quad \text{for a.e. } \boldsymbol{x} \in \Omega.$$

Next, let us establish the variational formula of Problem 3.1. We denote by $K_\rho \subset V$ the admissible set to velocity filed given by

$$K_\rho = \{\boldsymbol{u} \in V \mid u_\nu \leq \rho \text{ on } \Gamma_{C_2} \text{ and } u_\nu \geq 0 \text{ on } \Gamma_{C_3}\}. \tag{3.8}$$

Given a smooth tensor $\boldsymbol{\sigma} \colon \Omega \to \mathbb{S}^d$, two smooth vector fields $\boldsymbol{v} \colon \Omega \to \mathbb{R}^d$, $\boldsymbol{w} \colon \Omega \to \mathbb{R}^d$, and a smooth function $\psi \colon \Omega \to \mathbb{R}$, so, the following Green formulas are available (see e.g., [18, Theorems 2.24 and 2.25])

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{D}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega \operatorname{Div} \boldsymbol{\sigma} \cdot \boldsymbol{v}\, d\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{\sigma\nu} \cdot \boldsymbol{v}\, d\Gamma, \tag{3.9}$$

and

$$\int_\Omega \nabla\psi \cdot \boldsymbol{w}\, d\boldsymbol{x} + \int_\Omega \psi \operatorname{div} \boldsymbol{w}\, d\boldsymbol{x} = \int_{\partial\Omega} \psi(\boldsymbol{w} \cdot \boldsymbol{\nu})\, d\Gamma. \tag{3.10}$$

Remark 3.2. If the subsets K and K_ρ are defined by (2.3) and (3.8), respectively, then it is not hard to prove that K and K_ρ are nonempty, closed and convex of V, and $K_\rho \xrightarrow{M} K$ as $\rho \to 0$ (namely, K_ρ converges to K in the sense of Mosco, when ρ tends to 0). Indeed, let sequence $\{\boldsymbol{u}_\rho\}_{\rho > 0}$ be such that $\boldsymbol{u}_\rho \in K_\rho$ for each $\rho > 0$ and $\boldsymbol{u}_\rho \rightharpoonup \boldsymbol{u}$ in V as $\rho \to 0$. Then, from the Sobolev embedding theorem, we have $u_{\rho_\nu} \to u_\nu$ in $L^2(\Gamma)$ as $\rho \to 0$. Since $K_\rho = \{\boldsymbol{v} \in V \mid v_\nu \leq \rho \text{ on } \Gamma_{C_2}\} \cap \{\boldsymbol{v} \in V \mid v_\nu \geq 0 \text{ on } \Gamma_{C_3}\}$, we obtain $\boldsymbol{u}_\rho - \rho \in \{\boldsymbol{v} \in V \mid v_\nu \leq 0 \text{ on } \Gamma_{C_2}\}$ and $\boldsymbol{u}_\rho \in \{\boldsymbol{v} \in V \mid v_\nu \geq 0 \text{ on } \Gamma_{C_3}\}$. Moreover, since the sets $\{\boldsymbol{v} \in V \mid v_\nu \leq 0 \text{ on } \Gamma_{C_2}\}$ and $\{\boldsymbol{v} \in V \mid v_\nu \geq 0 \text{ on } \Gamma_{C_3}\}$ are weakly closed by Mazur's theorem, we deduce that $\boldsymbol{u} \in \{\boldsymbol{v} \in V \mid v_\nu \leq 0 \text{ on } \Gamma_{C_2}\}$ and $\boldsymbol{u} \in \{\boldsymbol{v} \in V \mid v_\nu \geq 0 \text{ on } \Gamma_{C_3}\}$, and hence, $\boldsymbol{u} \in K$. On the other hand, for any $\boldsymbol{u} \in K$, we can observe that $\boldsymbol{u}_\rho = \boldsymbol{u} + \rho\boldsymbol{e} \in K_\rho$ and

$\boldsymbol{u}_\rho \to \boldsymbol{u}$ as $\rho \to 0$ in V, where $\boldsymbol{e} \in V$ is such that $e_\nu \le 1$ on Γ_{C_2} and $e_\nu \ge 0$ on Γ_{C_3}. This means that $K_\rho \xrightarrow{M} K$.

Let $\boldsymbol{u}_\rho \colon \Omega \to \mathbb{R}^d$, $p_\rho \colon \Omega \to \mathbb{R}$ and $y_\rho \colon \Omega \to \mathbb{R}$ be sufficiently smooth such that (3.1)–(3.7) hold, and $\boldsymbol{v}_\rho \in K_\rho$ be arbitrary. Multiplying (3.1) by $\boldsymbol{v}_\rho - \boldsymbol{u}_\rho$ and integrating the resulting equality on Ω, we get

$$-\mu_\rho \int_\Omega \Delta \boldsymbol{u}_\rho \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} + \int_\Omega \nabla p_\rho \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}_\rho \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x}. \tag{3.11}$$

By using the Green formula (3.10), and the conditions $\boldsymbol{v}_\rho = \boldsymbol{u}_\rho = \boldsymbol{0}$ on Γ_D and $v_{\rho_\nu} = u_{\rho_\nu} = 0$ on Γ_{C_1}, as well as the fact that $\boldsymbol{u}_\rho$ and $\boldsymbol{v}_\rho$ satisfy divergence free condition, we obtain

$$\begin{aligned}\int_\Omega \nabla p_\rho \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} &= -\int_\Omega (\nabla \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)) p_\rho\, d\boldsymbol{x} + \int_{\Gamma_D} p_\rho \boldsymbol{\nu} \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\Gamma \\ &\quad + \int_{\Gamma_{C_1}} p_\rho (v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma + \int_{\Gamma_{C_2} \cup \Gamma_{C_3}} p_\rho (v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma \\ &= \int_{\Gamma_{C_2} \cup \Gamma_{C_3}} p_\rho (v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma.\end{aligned} \tag{3.12}$$

But, the divergence free condition $\nabla \cdot \boldsymbol{u}_\rho = 0$ in Ω points out that

$$\frac{\partial(\sum_{j=1}^d \frac{\partial u_{\rho,j}}{\partial x_j})}{\partial x_i} = \sum_{j=1}^d u_{\rho,j,ij} = 0 \quad \text{in } \Omega,$$

where $u_{\rho,j,ij} = \frac{\partial^2 u_{\rho,j}}{\partial x_i \partial x_j}$, $i, j = 1, ..., d$. Then, we have

$$\begin{aligned}\Delta \boldsymbol{u}_\rho \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho) &= \sum_{i=1}^d (\sum_{j=1}^d u_{\rho,i,jj})(v_{\rho,i} - u_{\rho,i}) = \sum_{i=1}^d \big(\sum_{j=1}^d (u_{\rho,i,jj} + u_{\rho,j,ij})\big)(v_{\rho,i} - u_{\rho,i}) \\ &= \sum_{i=1}^d \big(\sum_{j=1}^d (u_{\rho,i,j} + u_{\rho,j,i})_{,j}\big)(v_{\rho,i} - u_{\rho,i}) = 2\,\mathrm{Div}\, \boldsymbol{D}(\boldsymbol{u}_\rho) \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho) \ \text{ in } \Omega.\end{aligned} \tag{3.13}$$

Combining the Green formula (3.9) with equation (3.13), it has

$$\begin{aligned}-\mu_\rho \int_\Omega \Delta \boldsymbol{u}_\rho \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} &= -2\mu_\rho \int_\Omega \mathrm{Div}\, \boldsymbol{D}(\boldsymbol{u}_\rho) \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} \\ &= 2\mu_\rho \int_\Omega \boldsymbol{D}(\boldsymbol{u}_\rho) : \boldsymbol{D}(\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} - 2\mu_\rho \int_\Gamma \boldsymbol{D}(\boldsymbol{u}_\rho)\boldsymbol{\nu} \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\Gamma.\end{aligned} \tag{3.14}$$

Subsequently, we use the conditions $\boldsymbol{v}_\rho = \boldsymbol{u}_\rho = \boldsymbol{0}$ on Γ_D, $v_{\rho_\nu} = u_{\rho_\nu} = 0$ on Γ_{C_1}, $\boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) = \boldsymbol{0}$ on Γ_{C_2}, and $(u_{\rho_\nu} - \rho)\tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho) = 0$, $(v_{\rho_\nu} - \rho)\tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho) \ge 0$ on Γ_{C_2}, and $\tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho) = -\phi_\rho$, $\boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) = \boldsymbol{0}$ on Γ_{C_3}, and

$\tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho) = S_{\rho_\nu}(\boldsymbol{u}_\rho) - p_\rho = 2\mu_\rho D(\boldsymbol{u}_\rho)_\nu - p_\rho$ and $\boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) = \boldsymbol{S}_{\rho_\tau}(\boldsymbol{u}_\rho) = 2\mu_\rho \boldsymbol{D}(\boldsymbol{u}_\rho)_\tau$

on $\Gamma_{C_1} \cup \Gamma_{C_2} \cup \Gamma_{C_3}$, to obtain

$$
\begin{aligned}
&- 2\mu_\rho \int_\Gamma \boldsymbol{D}(\boldsymbol{u}_\rho)\boldsymbol{\nu} \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} \qquad (3.15)\\
&= - 2\mu_\rho \int_{\Gamma_{C_1}\cup\Gamma_{C_2}\cup\Gamma_{C_3}} \Big(D(\boldsymbol{u}_\rho)_\nu (v_{\rho_\nu} - u_{\rho_\nu}) + \boldsymbol{D}(\boldsymbol{u}_\rho)_\tau \cdot (\boldsymbol{v}_{\rho_\tau} - \boldsymbol{u}_{\rho_\tau}) \Big)\, d\Gamma\\
&= - \int_{\Gamma_{C_1}} \boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) \cdot (\boldsymbol{v}_{\rho_\tau} - \boldsymbol{u}_{\rho_\tau})\, d\Gamma - \int_{\Gamma_{C_2}\cup\Gamma_{C_3}} (\tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho) + p_\rho)(v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma\\
&= - \int_{\Gamma_{C_1}} \boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) \cdot (\boldsymbol{v}_{\rho_\tau} - \boldsymbol{u}_{\rho_\tau})\, d\Gamma + \int_{\Gamma_{C_3}} \phi_\rho (v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma - \int_{\Gamma_{C_2}\cup\Gamma_{C_3}} p_\rho (v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma\\
&\quad - \int_{\Gamma_{C_2}} \tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho)(v_{\rho_\nu} - \rho)\, d\Gamma + \int_{\Gamma_{C_2}} \tau_{\rho_\nu}(\boldsymbol{u}_\rho, p_\rho)(u_{\rho_\nu} - \rho)\, d\Gamma\\
&\leq - \int_{\Gamma_{C_1}} \boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) \cdot (\boldsymbol{v}_{\rho_\tau} - \boldsymbol{u}_{\rho_\tau})\, d\Gamma + \int_{\Gamma_{C_3}} \phi_\rho (v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma - \int_{\Gamma_{C_2}\cup\Gamma_{C_3}} p_\rho (v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma.
\end{aligned}
$$

From the definition of the Clarke subgradient and the boundary condition (3.4), it yields $-\boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) = \delta_\rho(y_\rho)\boldsymbol{\xi}_\rho$ and $\boldsymbol{\xi}_\rho \in \partial j_\rho(\boldsymbol{x}, \boldsymbol{u}_{\rho_\tau})$ with $\boldsymbol{\xi}_\rho \cdot (\boldsymbol{v}_{\rho_\tau} - \boldsymbol{u}_{\rho_\tau}) \leq j^0_\rho(\boldsymbol{x}, \boldsymbol{u}_{\rho_\tau}; \boldsymbol{v}_{\rho_\tau} - \boldsymbol{u}_{\rho_\tau})$ on Γ_{C_1}. Hence, we have

$$
- \int_{\Gamma_{C_1}} \boldsymbol{\tau}_{\rho_\tau}(\boldsymbol{u}_\rho) \cdot (\boldsymbol{v}_{\rho_\tau} - \boldsymbol{u}_{\rho_\tau})\, d\Gamma \leq \int_{\Gamma_{C_1}} \delta_\rho(y_\rho) j^0_\rho(\boldsymbol{x}, \boldsymbol{u}_{\rho_\tau}; \boldsymbol{v}_{\rho_\tau} - \boldsymbol{u}_{\rho_\tau})\, d\Gamma. \qquad (3.16)
$$

Combining with (3.14)–(3.16), one has

$$
\begin{aligned}
&- \mu_\rho \int_\Omega \Delta \boldsymbol{u}_\rho \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} \leq 2\mu_\rho \int_\Omega \boldsymbol{D}(\boldsymbol{u}_\rho) : \boldsymbol{D}(\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} + \int_{\Gamma_{C_3}} \phi_\rho (v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma\\
&+ \int_{\Gamma_{C_1}} \delta_\rho(y_\rho) j^0_\rho(\boldsymbol{x}, \boldsymbol{u}_{\rho_\tau}; \boldsymbol{v}_{\rho_\tau} - \boldsymbol{u}_{\rho_\tau})\, d\Gamma - \int_{\Gamma_{C_2}\cup\Gamma_{C_3}} p_\rho (v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma. \qquad (3.17)
\end{aligned}
$$

Inserting (3.12) and (3.17) into (3.11), we deduce

$$
\begin{aligned}
&2\mu_\rho \int_\Omega \boldsymbol{D}(\boldsymbol{u}_\rho) : \boldsymbol{D}(\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} + \int_{\Gamma_{C_1}} \delta_\rho(y_\rho) j^0_\rho(\boldsymbol{x}, \boldsymbol{u}_{\rho_\tau}; \boldsymbol{v}_{\rho_\tau} - \boldsymbol{u}_{\rho_\tau})\, d\Gamma\\
&\geq \int_\Omega \boldsymbol{f}_\rho \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} - \int_{\Gamma_{C_3}} \phi_\rho (v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma \qquad (3.18)
\end{aligned}
$$

for all $\boldsymbol{v}_\rho \in K_\rho$.

On the other hand, for the diffusion system (3.7), we apply the Green formula (3.10) and the boundary condition $(3.7)_2$ to obtain the following variational equation

$$
\int_\Omega \beta_\rho(\boldsymbol{u}_\rho) \nabla y_\rho \cdot \nabla z\, d\boldsymbol{x} + \int_\Omega g_\rho(\boldsymbol{x}, y_\rho) z\, d\boldsymbol{x} - \int_{\Gamma_{C_2}\cup\Gamma_{C_3}} h_\rho z\, d\Gamma = 0 \ \text{ for all } z \in H^1(\Omega).
$$

Therefore, we obtain the weak variational formulation of Problem 3.1 as follows.

Problem 3.3. Find a concentration field $y_\rho \in H^1(\Omega)$ and a velocity filed $\boldsymbol{u}_\rho \in K_\rho$ such that

$$
\begin{aligned}
&2\mu_\rho \int_\Omega \boldsymbol{D}(\boldsymbol{u}_\rho) : \boldsymbol{D}(\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, dx + \int_{\Gamma_{C_1}} \delta_\rho(y_\rho) j^0_\rho(\boldsymbol{x}, \boldsymbol{u}_{\rho_\tau}; \boldsymbol{v}_{\rho_\tau} - \boldsymbol{u}_{\rho_\tau})\, d\Gamma\\
&\geq \int_\Omega \boldsymbol{f}_\rho \cdot (\boldsymbol{v}_\rho - \boldsymbol{u}_\rho)\, d\boldsymbol{x} - \int_{\Gamma_{C_3}} \phi_\rho (v_{\rho_\nu} - u_{\rho_\nu})\, d\Gamma \qquad (3.19)
\end{aligned}
$$

for all $\boldsymbol{v}_\rho \in K_\rho$, and

$$
\int_\Omega \beta_\rho(\boldsymbol{u}_\rho) \nabla y_\rho \cdot \nabla z\, d\boldsymbol{x} + \int_\Omega g_\rho(\boldsymbol{x}, y_\rho) z\, d\boldsymbol{x} - \int_{\Gamma_{C_2}\cup\Gamma_{C_3}} h_\rho z\, d\Gamma = 0 \ \text{ for all } \ z \in H^1(\Omega) \qquad (3.20)
$$

It is not difficult to see that Problem 3.3 can be equivalently rewritten to the following one.

PROBLEM 3.4. Find a concentration field $y_\rho \in H^1(\Omega)$ and a velocity filed $\boldsymbol{u}_\rho \in K_\rho$ such that

$$\langle A_\rho \boldsymbol{u}_\rho - \widetilde{\boldsymbol{f}}_\rho, \boldsymbol{v}_\rho - \boldsymbol{u}_\rho\rangle + \int_{\Gamma_{C_1}} \delta_\rho(y_\rho) j_\rho^0(\boldsymbol{x}, \gamma\boldsymbol{u}_\rho; \gamma(\boldsymbol{v}_\rho - \boldsymbol{u}_\rho))\, d\Gamma \geq 0 \quad \text{for all} \quad \boldsymbol{v}_\rho \in K_\rho, \tag{3.21}$$

and

$$\int_\Omega \beta_\rho(\boldsymbol{u}_\rho)\nabla y_\rho \cdot \nabla z\, d\boldsymbol{x} + \int_\Omega g_\rho(\boldsymbol{x}, y_\rho) z\, d\boldsymbol{x} - \int_{\Gamma_{C_2}\cup\Gamma_{C_3}} h_\rho z\, d\Gamma = 0 \quad \text{for all} \quad z \in H^1(\Omega),$$

where $A_\rho \colon V \to V^*$ and $\widetilde{\boldsymbol{f}}_\rho \in V^*$ are defined by

$$\langle A_\rho \boldsymbol{u}, \boldsymbol{v}\rangle = 2\mu_\rho \int_\Omega \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v})\, d\boldsymbol{x} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V, \tag{3.22}$$

$$\langle \widetilde{\boldsymbol{f}}_\rho, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}_\rho \cdot \boldsymbol{v}\, d\boldsymbol{x} - \int_{\Gamma_{C_3}} \phi_\rho v_\nu\, d\Gamma \quad \text{for all } \boldsymbol{v} \in V. \tag{3.23}$$

4. Stability analysis

In this section, we provide a stability result to the Problem 1.1 which reveals that the solution set of Problem 1.1 can be approached in the sense of Kuratowski by the perturbated problem, Problem 3.1, when perturbated parameter tends to zero.

The main theorem of this section is stated as follows.

Theorem 4.1. *Let $\{\rho_n\} \subset (0, +\infty)$ be such that $\rho_n \to 0$ as $n \to \infty$. Assume that $H(j_\rho)$, $H(\delta_\rho)$, $H(\beta_\rho)$, $H(\boldsymbol{f}_\rho)$, $H(\phi_\rho)$, $H(h_\rho)$, $H(g_\rho)$ and $H(\mu_\rho)$ are satisfied. In addition, the smallness condition $\inf_{\rho>0}(2\mu_\rho - \delta_{1_\rho}\alpha_{j_\rho}\|\gamma\|^2) > 0$ holds. Then, we have*

(i) *for each $\rho > 0$, Problem 3.1 has at least one weak solution $(\boldsymbol{u}_\rho, y_\rho) \in K_\rho \times H^1(\Omega)$;*

(ii) *for each $n \in \mathbb{N}$, if $(\boldsymbol{u}_n, y_n) := (\boldsymbol{u}_{\rho_n}, y_{\rho_n})$ is a solution of Problem 3.1 with $\rho = \rho_n$, then there exists a subsequence of $\{(\boldsymbol{u}_{\rho_n}, y_{\rho_n})\}$, still denoted by the same way, and $(\boldsymbol{u}, y) \in V \times H^1(\Omega)$ such that*

$$\boldsymbol{u}_n \to \boldsymbol{u} \text{ in } V \text{ and } y_n \to y \text{ in } H^1(\Omega),$$

and $(\boldsymbol{u}, y)$ is a weak solution of Problem 1.1.

Proof. (i) Arguing as in the proof of Theorem 2.6, it can be obtained directly that Problem 3.1 is solvable.

(ii) Let $Z = H^1(\Omega)$. Suppose that $(\boldsymbol{u}_n, y_n) := (\boldsymbol{u}_{\rho_n}, y_{\rho_n})$ is a solution of Problem 3.1 with $\rho = \rho_n$ for every $n \in \mathbb{N}$.

Step 1. *The boundedness of* $\cup_{n\in\mathbb{N}}\{(\boldsymbol{u}_n, y_n)\}$.

Set $K_n = K_{\rho_n}$, $A_n = A_{\rho_n}$, $\delta_n = \delta_{\rho_n}$, $j_n = j_{\rho_n}$, $\widetilde{\boldsymbol{f}}_n = \widetilde{\boldsymbol{f}}_{\rho_n}$, $\beta_n = \beta_{\rho_n}$, $g_n = g_{\rho_n}$ and $h_n = h_{\rho_n}$. Then, for each $n \in \mathbb{N}$, it has

$$\langle A_n \boldsymbol{u}_n - \widetilde{\boldsymbol{f}}_n, \boldsymbol{v}_n - \boldsymbol{u}_n\rangle + \int_{\Gamma_{C_1}} \delta_n(y_n) j_n^0(\boldsymbol{x}, \gamma\boldsymbol{u}_n; \gamma(\boldsymbol{v}_n - \boldsymbol{u}_n))\, d\Gamma \geq 0 \quad \text{for all} \quad \boldsymbol{v}_n \in K_n, \tag{4.1}$$

and

$$\int_\Omega \beta_n(\boldsymbol{u}_n)\nabla y_n \cdot \nabla z\, d\boldsymbol{x} + \int_\Omega g_n(\boldsymbol{x}, y_n) z\, d\boldsymbol{x} - \int_{\Gamma_{C_2}\cup\Gamma_{C_3}} h_n z\, d\Gamma = 0 \quad \text{for all} \quad z \in Z. \tag{4.2}$$

We argue by contradiction to assume that $\cup_{n\in\mathbb{N}}\{(\boldsymbol{u}_n, y_n)\}$ is unbounded in $V\times Z$. Then, without loss of generality, we may suppose that

$$\|\boldsymbol{u}_n\|_V \to \infty \quad \text{and} \quad \|y_n\|_Z \to \infty \quad \text{as} \quad n\to\infty.$$

Keeping in mind that $\mathbf{0}_V \in K_n$ (see (3.8)), we insert $\boldsymbol{v}_n = \mathbf{0}_V$ into inequality (4.1) to get that

$$\langle A_n\boldsymbol{u}_n, \boldsymbol{u}_n\rangle - \int_{\Gamma_{C_1}} \delta_n(y_n) j_n^0(\boldsymbol{x}, \gamma\boldsymbol{u}_n; -\gamma\boldsymbol{u}_n)\, d\Gamma \le \langle \widetilde{\boldsymbol{f}}_n, \boldsymbol{u}_n\rangle. \tag{4.3}$$

From the definition of A_n (see (3.22)) and $\mu_n = \mu_{\rho_n}$, the following result holds

$$\langle A_n\boldsymbol{u}_n, \boldsymbol{u}_n\rangle = 2\mu_n \int_\Omega \boldsymbol{D}(\boldsymbol{u}_n) : \boldsymbol{D}(\boldsymbol{u}_n)\, d\boldsymbol{x} = 2\mu_n\|\boldsymbol{u}_n\|_V^2. \tag{4.4}$$

Using hypothesis $H(j_\rho)$(i) implies

$$\begin{aligned} j_n^0(\boldsymbol{x}, \gamma\boldsymbol{u}_n; -\gamma\boldsymbol{u}_n) &= j_n^0(\boldsymbol{x}, \gamma\boldsymbol{u}_n; \gamma(\mathbf{0}_V - \boldsymbol{u}_n)) + j_n^0(\boldsymbol{x}, \gamma\mathbf{0}_V; \gamma\boldsymbol{u}_n) - j_n^0(\boldsymbol{x}, \gamma\mathbf{0}_V; \gamma\boldsymbol{u}_n) \\ &\le \alpha_{j_n}\|\gamma\boldsymbol{u}_n\|_{\mathbb{R}^d}^2 - j_n^0(\boldsymbol{x}, \gamma\mathbf{0}_V; \gamma\boldsymbol{u}_n), \end{aligned}$$

and

$$j_n^0(\boldsymbol{x}, \gamma\mathbf{0}_V; \gamma\boldsymbol{u}_n) \ge -\|\boldsymbol{\xi}_{\gamma\mathbf{0}_V}\|_{\mathbb{R}^d}\|\gamma\boldsymbol{u}_n\|_{\mathbb{R}^d} \ge -c_{0_n}(\boldsymbol{x})\|\gamma\boldsymbol{u}_n\|_{\mathbb{R}^d}$$

for all $\boldsymbol{\xi}_{\gamma\mathbf{0}_V} \in \partial j_n(\boldsymbol{x}, \gamma\mathbf{0}_V)$. The estimates above together with the hypotheses $H(\delta_\rho)$ concludes that

$$\begin{aligned} &- \int_{\Gamma_{C_1}} \delta_n(y_n) j_n^0(\boldsymbol{x}, \gamma\boldsymbol{u}_n; -\gamma\boldsymbol{u}_n)\, d\Gamma \\ \ge &- \int_{\Gamma_{C_1}} \delta_n(y_n)(\alpha_{j_n}\|\gamma\boldsymbol{u}_n\|_{\mathbb{R}^d}^2 - j_n^0(\boldsymbol{x}, \gamma\mathbf{0}_V; \gamma\boldsymbol{u}_n))\, d\Gamma \\ \ge &- \delta_{1_n}\alpha_{j_n}\|\gamma\|^2\|\boldsymbol{u}_n\|_V^2 + \int_{\Gamma_{C_1}} \delta_n(y_n) j_n^0(\boldsymbol{x}, \gamma\mathbf{0}_V; \gamma\boldsymbol{u}_n)\, d\Gamma \\ \ge &- \delta_{1_n}\alpha_{j_n}\|\gamma\|^2\|\boldsymbol{u}_n\|_V^2 + \delta_{0_n}\int_{\Gamma_{C_1}} j_n^0(\boldsymbol{x}, \gamma\mathbf{0}_V; \gamma\boldsymbol{u}_n)\, d\Gamma \\ \ge &- \delta_{1_n}\alpha_{j_n}\|\gamma\|^2\|\boldsymbol{u}_n\|_V^2 - \delta_{0_n}\|c_{0_n}\|_{L^2(\Gamma_{C_1})}\|\gamma\|\|\boldsymbol{u}_n\|_V. \end{aligned} \tag{4.5}$$

By the definition of $\widetilde{\boldsymbol{f}}_\rho$ (see (3.23)), hypotheses $H(\boldsymbol{f}_\rho)$ and $H(\phi_\rho)$, we have

$$\begin{aligned} \langle \widetilde{\boldsymbol{f}}_n, \boldsymbol{u}_n\rangle &= \int_\Omega \boldsymbol{f}_n \cdot \boldsymbol{u}_n\, d\boldsymbol{x} - \int_{\Gamma_{C_3}} \phi_n u_{n_\nu}\, d\Gamma \\ &\le \|\boldsymbol{f}_n\|_{L^2(\Omega;\mathbb{R}^d)}\|\boldsymbol{u}_n\|_{L^2(\Omega;\mathbb{R}^d)} + \|\phi_n\|_{L^2(\Gamma_{C_3})}\|u_{n_\nu}\|_{L^2(\Gamma_{C_3})} \\ &\le C_1\|\boldsymbol{f}_n\|_{L^2(\Omega;\mathbb{R}^d)}\|\boldsymbol{u}_n\|_V + C_2\|\phi_n\|_{L^2(\Gamma_{C_3})}\|\boldsymbol{u}_n\|_V, \end{aligned} \tag{4.6}$$

with some $C_1 > 0, C_2 > 0$. Taking into account (4.3)–(4.6), we have

$$\begin{aligned} &(2\mu_n - \delta_{1_n}\alpha_{j_n}\|\gamma\|^2)\|\boldsymbol{u}_n\|_V^2 - \delta_{0_n}\|c_{0_n}\|_{L^2(\Gamma_{C_1})}\|\gamma\|\|\boldsymbol{u}_n\|_V \\ &\quad \le C_1\|\boldsymbol{f}_n\|_{L^2(\Omega;\mathbb{R}^d)}\|\boldsymbol{u}_n\|_V + C_2\|\phi_n\|_{L^2(\Gamma_{C_3})}\|\boldsymbol{u}_n\|_V, \end{aligned}$$

i.e.,

$$(2\mu_n - \delta_{1_n}\alpha_{j_n}\|\gamma\|^2)\|\boldsymbol{u}_n\|_V \le \delta_{0_n}\|c_{0_n}\|_{L^2(\Gamma_{C_1})}\|\gamma\| + C_1\|\boldsymbol{f}_n\|_{L^2(\Omega;\mathbb{R}^d)} + C_2\|\phi_n\|_{L^2(\Gamma_{C_3})}. \tag{4.7}$$

Because of $\|\boldsymbol{u}_n\|_V \to \infty$ as $n\to\infty$, we use the smallness condition $\inf_{\rho>0}(2\mu_\rho - \delta_{1_\rho}\alpha_{j_\rho}\|\gamma\|^2) > 0$ to obtain

$$+\infty = \lim_{n\to\infty}(2\mu_n - \delta_{1_n}\alpha_{j_n}\|\gamma\|^2)\|\boldsymbol{u}_n\|_V \le C_0$$

with some C_0. This leads to a contradiction, therefore, we conclude that $\{\boldsymbol{u}_n\}$ is bounded in V.

On the other hand, let us take $z = y_n$ in equation (4.2) to get

$$\int_\Omega \beta_n(\boldsymbol{u}_n)\nabla y_n \cdot \nabla y_n \, d\boldsymbol{x} + \int_\Omega g_n(\boldsymbol{x}, y_n) y_n \, d\boldsymbol{x} - \int_{\Gamma_{C_2}\cup\Gamma_{C_3}} h_n y_n \, d\Gamma = 0. \tag{4.8}$$

By virtue of hypothesis $H(\beta_\rho)$(i), one has

$$\int_\Omega \beta_n(\boldsymbol{u}_n)\nabla y_n \cdot \nabla y_n \, d\boldsymbol{x} \geq \beta_{0_n} \|\nabla y_n\|^2_{L^2(\Omega;\mathbb{R}^d)}. \tag{4.9}$$

Also, we use the condition $H(g_\rho)$(i) to obtain

$$\begin{aligned} \int_\Omega g_n(\boldsymbol{x}, y_n) y_n \, d\boldsymbol{x} &= \int_\Omega (g_n(\boldsymbol{x}, y_n) - g_n(\boldsymbol{x}, 0)) y_n \, d\boldsymbol{x} + \int_\Omega g_n(\boldsymbol{x}, 0) y_n \, d\boldsymbol{x} \\ &\geq \int_\Omega d_{g_n} |y_n|^2 \, d\boldsymbol{x} - \int_\Omega \alpha_{g_n}(\boldsymbol{x}) \cdot |y_n| \, d\boldsymbol{x} \\ &\geq d_{g_n} \|y_n\|^2_{L^2(\Omega)} - \|\alpha_{g_n}\|_{L^2(\Omega)} \|y_n\|_{L^2(\Omega)}. \end{aligned} \tag{4.10}$$

Applying Hölder inequality, it gives

$$\begin{aligned} -\int_{\Gamma_{C_2}\cup\Gamma_{C_3}} h_n y_n \, d\Gamma &\geq -\int_{\Gamma_{C_2}} |h_n| \cdot |y_n| \, d\Gamma - \int_{\Gamma_{C_3}} |h_n| \cdot |y_n| \, d\Gamma \\ &\geq -\|h_n\|_{L^2(\Gamma_{C_2})} \|y_n\|_{L^2(\Gamma_{C_2})} - \|h_n\|_{L^2(\Gamma_{C_3})} \|y_n\|_{L^2(\Gamma_{C_3})}. \end{aligned} \tag{4.11}$$

Employing (4.8)–(4.11), it yields

$$\begin{aligned} &\beta_{0_n} \|\nabla y_n\|^2_{L^2(\Omega;\mathbb{R}^d)} + d_{g_n} \|y_n\|^2_{L^2(\Omega)} - \|\alpha_{g_n}\|_{L^2(\Omega)} \|y_n\|_{L^2(\Omega)} \\ &\quad - \|h_n\|_{L^2(\Gamma_{C_2})} \|y_n\|_{L^2(\Gamma_{C_2})} - \|h_n\|_{L^2(\Gamma_{C_3})} \|y_n\|_{L^2(\Gamma_{C_3})} \leq 0. \end{aligned} \tag{4.12}$$

Passing to the limit as $n \to \infty$ in (4.12) and using the assumption, $\|y_n\|_Z \to \infty$ as $n \to \infty$, it yields

$$\begin{aligned} +\infty &= \lim_{n\to\infty} \Big(\beta_{0_n} \|\nabla y_n\|^2_{L^2(\Omega;\mathbb{R}^d)} + d_{g_n} \|y_n\|^2_{L^2(\Omega)} - \|\alpha_{g_n}\|_{L^2(\Omega)} \|y_n\|_{L^2(\Omega)} \\ &\qquad - \|h_n\|_{L^2(\Gamma_{C_2})} \|y_n\|_{L^2(\Gamma_{C_2})} - \|h_n\|_{L^2(\Gamma_{C_3})} \|y_n\|_{L^2(\Gamma_{C_3})} \Big) \\ &\leq 0. \end{aligned}$$

This triggers a contradiction. Consequently, we conclude that $\{y_n\}$ is bounded in Z.

By the reflexivity of $V \times Z$, passing to a subsequence if necessary, we may suppose that there exists a subsequence of $\{(\boldsymbol{u}_n, y_n)\}$, still denoted by the same way, and a pair of functions $(\boldsymbol{u}, y) \in V \times Z$ such that

$$(\boldsymbol{u}_n, y_n) \rightharpoonup (\boldsymbol{u}, y) \quad \text{in } V \times Z \quad \text{as} \quad n \to \infty. \tag{4.13}$$

Step 2. *$(\boldsymbol{u}_n, y_n)$ converges strongly to $(\boldsymbol{u}, y)$ in $V \times Z$.*

Since V is embedded compactly into $L^2(\Omega;\mathbb{R}^d)$ and the trace operator $\widetilde{\gamma}\colon Z \to L^2(\Gamma_{C_1})$ is compact. So, we have

$$(\boldsymbol{u}_n, y_n) \to (\boldsymbol{u}, y) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \times L^2(\Gamma_{C_1}) \quad \text{as} \quad n \to \infty.$$

Remark 3.2 reveals that $K_n \xrightarrow{M} K$ as $n \to \infty$. Exploiting condition (ii) of Definition 2.3 and (4.13), it yields $(\boldsymbol{u}, y) \in K \times Z$.

We next show that $A_n\colon V\to V^*$ is continuous and $A_n\boldsymbol{u}\to A\boldsymbol{u}$ in V^* for any $\boldsymbol{u}\in V$ as $n\to\infty$. Using Hölder inequality, one has

$$\begin{aligned}\langle A_n\boldsymbol{u}-A_n\boldsymbol{v},\boldsymbol{w}\rangle &= 2\mu_n\int_\Omega(\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{v})):\boldsymbol{D}(\boldsymbol{w})\,d\boldsymbol{x}\\ &\le 2\mu_n\|\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)}\|\boldsymbol{D}(\boldsymbol{w})\|_{L^2(\Omega;\mathbb{S}^d)}\\ &\le 2\mu_n\|\boldsymbol{u}-\boldsymbol{v}\|_V\|\boldsymbol{w}\|_V \quad\text{for all } \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V,\end{aligned}\tag{4.14}$$

and

$$\|A_n\boldsymbol{u}\|_{V^*}=\sup_{\boldsymbol{w}\in V,\|\boldsymbol{w}\|_V=1}|\langle A_n\boldsymbol{u},\boldsymbol{w}\rangle|\le 2\mu_n\|\boldsymbol{u}\|_V.\tag{4.15}$$

This implies that A_n is bounded and continuous. Moreover, we use the inequalities (4.14)–(4.15) to obtain

$$\|A_n\boldsymbol{u}-A\boldsymbol{u}\|_{V^*}=\sup_{\boldsymbol{w}\in V,\|\boldsymbol{w}\|_V=1}|\langle A_n\boldsymbol{u}-A\boldsymbol{u},\boldsymbol{w}\rangle|\le 2|\mu_n-\mu|\,\|\boldsymbol{u}\|_V\to 0 \text{ as } n\to\infty.$$

Therefore, we conclude that $A_n\boldsymbol{u}\to A\boldsymbol{u}$ in V^* for all $\boldsymbol{u}\in V$ as $n\to\infty$. Let $\{\boldsymbol{w}_n\}\subset V$ be a bounded sequence. So, we have

$$\|A_n\boldsymbol{w}_n-A\boldsymbol{w}_n\|_{V^*}=\sup_{\boldsymbol{v}\in V,\|\boldsymbol{v}\|_V=1}|\langle A_n\boldsymbol{w}_n-A\boldsymbol{w}_n,\boldsymbol{v}\rangle|\le 2|\mu_n-\mu|\,\|\boldsymbol{w}_n\|_V\to 0 \text{ as } n\to\infty,$$

that is, $A_n\boldsymbol{w}_n-A\boldsymbol{w}_n\to 0$ in V^* as $n\to\infty$.

Recall that $(\boldsymbol{u}_n,y_n)\in K_n\times Z$ is a weak solution of Problem 3.1, then we have

$$\langle A_n\boldsymbol{u}_n-\widetilde{\boldsymbol{f}}_n,\boldsymbol{v}_n-\boldsymbol{u}_n\rangle+\int_{\Gamma_{C_1}}\delta_n(y_n)j_n^0(\boldsymbol{x},\gamma\boldsymbol{u}_n;\gamma(\boldsymbol{v}_n-\boldsymbol{u}_n))\,d\Gamma\ge 0 \text{ for all } \boldsymbol{v}_n\in K_n,\tag{4.16}$$

and

$$\int_\Omega\beta_n(\boldsymbol{u}_n)\nabla y_n\cdot\nabla z\,d\boldsymbol{x}+\int_\Omega g_n(\boldsymbol{x},y_n)z\,d\boldsymbol{x}-\int_{\Gamma_{C_2}\cup\Gamma_{C_3}}h_nz\,d\Gamma=0 \quad\text{for all}\quad z\in Z.\tag{4.17}$$

Because of $(\boldsymbol{u},y)\in K\times Z$ and $K_n\xrightarrow{M}K$ as $n\to\infty$, we can find a sequence $\{\widetilde{\boldsymbol{u}}_n\}$ such that $\widetilde{\boldsymbol{u}}_n\in K_n$ for each $n\in\mathbb{N}$ and $\widetilde{\boldsymbol{u}}_n\to\boldsymbol{u}$ in V (see condition (i) of Definition 2.3). Putting $\boldsymbol{v}_n=\widetilde{\boldsymbol{u}}_n$ into (4.16), it gives

$$\begin{aligned}&\langle\widetilde{\boldsymbol{f}}_n,\boldsymbol{u}_n-\widetilde{\boldsymbol{u}}_n\rangle+\int_{\Gamma_{C_1}}\delta_n(y_n)j_n^0(\boldsymbol{x},\gamma\boldsymbol{u}_n;\gamma(\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n))\,d\Gamma+\langle A_n\widetilde{\boldsymbol{u}}_n,\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n\rangle\\ &\quad\ge\langle A_n\boldsymbol{u}_n,\boldsymbol{u}_n-\widetilde{\boldsymbol{u}}_n\rangle+\langle A_n\widetilde{\boldsymbol{u}}_n,\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n\rangle\\ &\quad=\langle A_n\boldsymbol{u}_n-A_n\widetilde{\boldsymbol{u}}_n,\boldsymbol{u}_n-\widetilde{\boldsymbol{u}}_n\rangle=2\mu_n\|\boldsymbol{u}_n-\widetilde{\boldsymbol{u}}_n\|_V^2.\end{aligned}\tag{4.18}$$

Hence, we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\langle A_n\widetilde{\boldsymbol{u}}_n,\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n\rangle\\ &=\limsup_{n\to\infty}\langle A_n\widetilde{\boldsymbol{u}}_n-A\boldsymbol{u},\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n\rangle+\limsup_{n\to\infty}\langle A\boldsymbol{u},\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n\rangle\\ &\le\limsup_{n\to\infty}\|A_n\widetilde{\boldsymbol{u}}_n-A\boldsymbol{u}\|_{V^*}\|\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n\|_V+\limsup_{n\to\infty}\langle A\boldsymbol{u},\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n\rangle\\ &\le\limsup_{n\to\infty}\left(\|A_n\widetilde{\boldsymbol{u}}_n-A\widetilde{\boldsymbol{u}}_n\|_{V^*}+\|A\widetilde{\boldsymbol{u}}_n-A\boldsymbol{u}\|_{V^*}\right)\|\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n\|_V+\limsup_{n\to\infty}\langle A\boldsymbol{u},\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n\rangle\\ &=0.\end{aligned}\tag{4.19}$$

The compactness of $\gamma\colon V\to L^2(\Gamma;\mathbb{R}^d)$ indicates that $\gamma\boldsymbol{u}_n\to\gamma\boldsymbol{u}$ in $L^2(\Gamma;\mathbb{R}^d)$. Passing to the upper limit as $n\to\infty$ for the inequality (4.18), and utilizing hypotheses $H(\mu_\rho)$,

$H(\boldsymbol{f}_\rho)$, $H(\phi_\rho)$, $H(j_\rho)$(ii), $H(\delta_\rho)$(ii), and inequality (4.19), it finds

$$\begin{aligned}
0 &\geq \limsup_{n\to\infty}\Big(\int_\Omega \boldsymbol{f}_n\cdot(\boldsymbol{u}_n-\widetilde{\boldsymbol{u}}_n)\,d\boldsymbol{x}+\int_{\Gamma_{C_3}}\phi_n(\widetilde{u}_{n_\nu}-u_{n_\nu})\,d\Gamma\Big)+\limsup_{n\to\infty}\langle A_n\widetilde{\boldsymbol{u}}_n,\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n\rangle\\
&\quad+\limsup_{n\to\infty}\int_{\Gamma_{C_1}}\delta_n(y_n)j_n^0(\boldsymbol{x},\gamma\boldsymbol{u}_n;\gamma(\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n))\,d\Gamma\\
&\geq \limsup_{n\to\infty}\Big(\langle\widetilde{\boldsymbol{f}}_n,\boldsymbol{u}_n-\widetilde{\boldsymbol{u}}_n\rangle+\int_{\Gamma_{C_1}}\delta_n(y_n)j_n^0(\boldsymbol{x},\gamma\boldsymbol{u}_n;\gamma(\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n))\,d\Gamma+\langle A_n\widetilde{\boldsymbol{u}}_n,\widetilde{\boldsymbol{u}}_n-\boldsymbol{u}_n\rangle\Big)\\
&\geq \limsup_{n\to\infty}2\mu_n\|\boldsymbol{u}_n-\widetilde{\boldsymbol{u}}_n\|_V^2\\
&= \limsup_{n\to\infty}2\mu\|\boldsymbol{u}_n-\widetilde{\boldsymbol{u}}_n\|_V^2.
\end{aligned}$$

Hence, we obtain $\boldsymbol{u}_n-\widetilde{\boldsymbol{u}}_n\to 0$ in V. This implies that $\boldsymbol{u}_n\to\boldsymbol{u}$ in V as $n\to\infty$.

Besides, we insert $z=y-y_n$ into inequality (4.17) for deriving

$$\begin{aligned}
&\int_\Omega\beta_n(\boldsymbol{u}_n)\nabla y_n\cdot\nabla(y-y_n)\,d\boldsymbol{x}+\int_\Omega g_n(\boldsymbol{x},y_n)\cdot(y-y_n)\,d\boldsymbol{x}\\
&\quad-\int_{\Gamma_{C_2}\cup\Gamma_{C_3}}h_n(y-y_n)\,d\Gamma=0.
\end{aligned}\tag{4.20}$$

Passing to limit as $n\to\infty$ in inequality (4.20), we have

$$\begin{aligned}
0&=\lim_{n\to\infty}\int_{\Gamma_{C_2}\cup\Gamma_{C_3}}h_n(y_n-y)\,d\Gamma+\lim_{n\to\infty}\int_\Omega g_n(\boldsymbol{x},y)(y-y_n)\,d\boldsymbol{x}\\
&=\lim_{n\to\infty}\int_\Omega\beta_n(\boldsymbol{u}_n)\nabla y_n\cdot\nabla(y_n-y)\,d\boldsymbol{x}+\lim_{n\to\infty}\int_\Omega(g_n(\boldsymbol{x},y_n)-g_n(\boldsymbol{x},y))\cdot(y_n-y)\,d\boldsymbol{x}\\
&\geq\lim_{n\to\infty}\big(\beta_{0_n}\|\nabla y_n-\nabla y\|^2_{L^2(\Omega;\mathbb{R}^d)}+d_{g_n}\|y_n-y\|^2_{L^2(\Omega)}\big).
\end{aligned}$$

The latter combined with hypotheses $H(\beta_\rho)$, $H(g_\rho)$ implies that $y_n\to y$ in Z as $n\to\infty$. Therefore, we have that $\{(\boldsymbol{u}_n,y_n)\}$ converges strongly to $(\boldsymbol{u},y)$ in $V\times Z$.

Step 3. $(\boldsymbol{u},y)\in K\times Z$ *is also a weak solution of Problem 1.1.*

Since $(\boldsymbol{u}_n,y_n)\in K_n\times Z$ is a weak solution of Problem 3.1, namely,

$$\langle A_n\boldsymbol{u}_n-\widetilde{\boldsymbol{f}}_n,\boldsymbol{v}_n-\boldsymbol{u}_n\rangle+\int_{\Gamma_{C_1}}\delta_n(y_n)j_n^0(\boldsymbol{x},\gamma\boldsymbol{u}_n;\gamma(\boldsymbol{v}_n-\boldsymbol{u}_n))\,d\Gamma\geq 0 \text{ for all } \boldsymbol{v}_n\in K_n,\tag{4.21}$$

and

$$\int_\Omega\beta_n(\boldsymbol{u}_n)\nabla y_n\cdot\nabla z\,d\boldsymbol{x}+\int_\Omega g_n(\boldsymbol{x},y_n)z\,d\boldsymbol{x}-\int_{\Gamma_{C_2}\cup\Gamma_{C_3}}h_nz\,d\Gamma=0 \text{ for all } z\in Z.\tag{4.22}$$

Let $\boldsymbol{w}\in K$. Because of $K_n\xrightarrow{M}K$, by condition (i) of Definition 2.3, there exists a sequence $\{\boldsymbol{w}_n\}$ such that $\boldsymbol{w}_n\in K_n$ and $\boldsymbol{w}_n\to\boldsymbol{w}$ in V. Inserting $\boldsymbol{v}_n=\boldsymbol{w}_n$ in (4.21), it gives

$$\langle A_n\boldsymbol{u}_n-\widetilde{\boldsymbol{f}}_n,\boldsymbol{w}_n-\boldsymbol{u}_n\rangle+\int_{\Gamma_{C_1}}\delta_n(y_n)j_n^0(\boldsymbol{x},\gamma\boldsymbol{u}_n;\gamma(\boldsymbol{w}_n-\boldsymbol{u}_n))\,d\Gamma\geq 0.\tag{4.23}$$

We pass to the upper limit as $n \to \infty$ to inequality (4.23) and use conditions $H(\boldsymbol{f}_\rho)$, $H(\phi_\rho)$, $H(j_\rho)$(ii), $H(\delta_\rho)$(ii) with properties of A_n to see that

$$\langle A\boldsymbol{u} - \widetilde{\boldsymbol{f}}, \boldsymbol{w} - \boldsymbol{u}\rangle + \int_{\Gamma_{C_1}} \delta(y) j^0(\boldsymbol{x}, \gamma\boldsymbol{u}; \gamma(\boldsymbol{w} - \boldsymbol{u}))\, d\Gamma \tag{4.24}$$

$$\geq \langle A\boldsymbol{u}, \boldsymbol{w} - \boldsymbol{u}\rangle + \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} - \boldsymbol{w})\, d\boldsymbol{x} + \int_{\Gamma_{C_3}} \phi(w_\nu - u_\nu)\, d\Gamma \tag{4.25}$$

$$+ \int_{\Gamma_{C_1}} \delta(y) j^0(\boldsymbol{x}, \gamma\boldsymbol{u}; \gamma(\boldsymbol{w} - \boldsymbol{u}))\, d\Gamma$$

$$\geq \limsup_{n\to\infty} \langle A_n\boldsymbol{u}_n, \boldsymbol{w}_n - \boldsymbol{u}_n\rangle + \limsup_{n\to\infty} \left(\int_\Omega \boldsymbol{f}_n \cdot (\boldsymbol{u}_n - \boldsymbol{w}_n)\, d\boldsymbol{x} + \int_{\Gamma_{C_3}} \phi_n(w_{n_\nu} - u_{n_\nu})\, d\Gamma \right)$$

$$+ \limsup_{n\to\infty} \int_{\Gamma_{C_1}} \delta_n(y_n) j_n^0(\boldsymbol{x}, \gamma\boldsymbol{u}_n; \gamma(\boldsymbol{w}_n - \boldsymbol{u}_n))\, d\Gamma$$

$$\geq \limsup_{n\to\infty} \langle A_n\boldsymbol{u}_n - \widetilde{\boldsymbol{f}}_n, \boldsymbol{w}_n - \boldsymbol{u}_n\rangle + \limsup_{n\to\infty} \int_{\Gamma_{C_1}} \delta_n(y_n) j_n^0(\boldsymbol{x}, \gamma\boldsymbol{u}_n; \gamma(\boldsymbol{w}_n - \boldsymbol{u}_n))\, d\Gamma$$

$$\geq \limsup_{n\to\infty} \left(\langle A_n\boldsymbol{u}_n - \widetilde{\boldsymbol{f}}_n, \boldsymbol{w}_n - \boldsymbol{u}_n\rangle + \int_{\Gamma_{C_1}} \delta_n(y_n) j_n^0(\boldsymbol{x}, \gamma\boldsymbol{u}_n; \gamma(\boldsymbol{w}_n - \boldsymbol{u}_n))\, d\Gamma \right)$$

$$\geq 0.$$

On the other hand, letting $n \to \infty$ for inequality (4.22) and using the conditions $H(\beta_\rho)$(ii), $H(g_\rho)$(ii), $H(h_\rho)$, we deduce that

$$\int_\Omega \beta(\boldsymbol{u}) \nabla y \cdot \nabla z\, d\boldsymbol{x} + \int_\Omega g(\boldsymbol{x}, y) z\, d\boldsymbol{x} - \int_{\Gamma_{C_2} \cup \Gamma_{C_3}} hz\, d\Gamma = 0 \quad \text{for all } z \in Z.$$

This together with (4.24) and the arbitrariness of $\boldsymbol{w} \in K$ implies that $(\boldsymbol{u}, y) \in K \times Z$ is a weak solution of Problem 1.1.

Consequently, we conclude that for each $n \in \mathbb{N}$, if $(\boldsymbol{u}_{\rho_n}, y_{\rho_n})$ is a weak solution of Problem 3.1 with $\rho = \rho_n$, then there exists a subsequence of $\{(\boldsymbol{u}_{\rho_n}, y_{\rho_n})\}$, still denoted by the same way, and $(\boldsymbol{u}, y) \in V \times H^1(\Omega)$ such that $\boldsymbol{u}_{\rho_n} \to \boldsymbol{u}$ in V and $y_{\rho_n} \to y$ in $H^1(\Omega)$, and $(\boldsymbol{u}, y)$ is a weak solution of Problem 1.1. □

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References

[1] M. A. Abbasi, D. O. Ezulike, H. Dehghanpour and R. V. Hawkes, *A comparative study of flowback rate and pressure transient behavior in multifractured horizontal wells completed in tight gas and oil reservoirs*, J. Nat. Gas Sci. Eng. **17**, 82–93, 2014.

[2] M. Asadi, R. A. Woodroof and R.E. Himes, *Comparative study of flowback analysis using polymer concentrations and fracturing-fluid tracer methods: a field study*, SPE Prod. & Oper. **23** (2), 147–157, 2008.

[3] Y. R. Bai, N. S. Papageorgiou and S. D. Zeng, *A singular eigenvalue problem for the Dirichlet (p,q)-Laplacian*, Math. Z. **300**, 325–345, 2022.

[4] J. X. Cen, A. A. Khan, D. Motreanu and S. D. Zeng, *Inverse problems for generalized quasi-variational inequalities with application to elliptic mixed boundary value systems*, Inverse Problems, **38**, 065006, 28 pp, 2022.

[5] J. X. Cen, S. Migórski, C. Min and J. C. Yao, *Hemivariational inequality for contaminant reaction-diffusion model of recovered fracturing fluid in the wellbore of shale gas reservoir*, Commun. Nonliner Sci. Numer. Simulat. **118**, 107020, 2023.

[6] F. H. Clarke, *Generalized gradients and applications*, Trans. Amer. Math. Soc. **205**, 247–262, 1975.

[7] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, Interscience, New York, 1983.

[8] Z. Denkowski, S. Migórski and N. S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.

[9] Z. Denkowski, S. Migórski and N. S. Papageorgiou, *An Introduction to Nonlinear Analysis: Applications*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.

[10] X. X. Dong, W. J. Li, Q. Liu and H. H. Wang, *Research on convection-reaction-diffusion model of contaminants in fracturing flowback fluid in non-equidistant fractures with arbitrary inclination of shale gas development*, J. Petrol. Sci. Eng. **208**, 109479, 2022.

[11] C. J. Fang and W. M. Han, *Well-posedness and optimal control of a hemivariational inequality for nonstationary Stokes fluid flow*, Discrete Contin. Dyn. Syst. **36** (10), 5369-5386, 2016.

[12] C. J. Fang and W. M. Han, *Stability analysis and optimal control of a stationary Stokes hemivariational inequality*, Evol. Equ. Control The. **9** (4), 995-1008, 2020.

[13] E. Ghanbari, M.A. Abbasi, H. Dehghanpour and D. Bearinger, *Flowback volumetric and chemical analysis for evaluating load recovery and its impact on early-time production*, Presented at the SPE Unconventional Resource Conference Canada, Calgary, Alberta, Canada, November, SPE-167165-MS, 2013.

[14] H. Lin, X. Zhou, Y. L. Chen, B. Yang, X. X. Song, X. Y. Sun and L. F. Dong, *Investigation of the factors influencing the flowback ratio in shale gas reservoirs: a study based on experimental observations and numerical simulations*, J. Energy Resour. Technol. **143** (11), 113201, 2021.

[15] Z. B. Liu, X. X. Dong, L. Chen, C. Min and X. C. Zheng, *Numerical simulation of recovered water flow and contaminants diffusion in the wellbore of shale gas horizontal wells*, Environ. Earth. Sci. **79**, 128, 2020.

[16] Z. B. Liu, X. X. Dong and C. Min, *Transient analysis of contaminant diffusion in the wellbore of shale gas horizontal wells*, Water Air Soil Pullut. **229** (7), 1–15, 2018.

[17] Z. H. Liu, D. Motreanu and S. D. Zeng, *Generalized penalty and regularization method for differential variational-hemivariational inequalities*, SIAM J. Optim. **31** 1158–1183, 2021.

[18] S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics, **26**, Springer, New York, 2013.

[19] S. Migórski and S. D. Zeng, *A class of differential hemivariational inequalities in Banach spaces*, J. Glob. Optim. **72**, 761–779, 2018.

[20] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Adv. Math. **3** (4), 510–585, 1969.

[21] S. D. Zeng, Y. R. Bai, L. Gasiński and P. Winkert, *Existence results for double phase implicit obstacle problems involving multivalued operators*, Calc. Var. PDEs **59**, 18 pages, 2020.

[22] B. Zeng, Z. H. Liu and S. Migórski, *On convergence of solutions to variational-hemivariational inequalities*, Z. Angew. Math. Phys. **69** (3), 1-20, 2018.

[23] S. D. Zeng, S. Migórski and Z. H. Liu, *Well-posedness, optimal control, and sensitivity analysis for a class of differential variational-hemivariational inequalities*, SIAM J. Optim. **31**, 2829–2862, 2021.

[24] S. D. Zeng, S. Migórski and Z. H. Liu, *Nonstationary incompressible Navier-Stokes system governed by a quasilinear reaction-diffusion equation (in Chinese)*, Sci. Sin. Math. **52**, 331–354, 2022.

[25] S. D. Zeng, N. S. Papageorgiou and V. D. Rădulescu, *Nonsmooth dynamical systems: From the existence of solutions to optimal and feedback control*, Bull. Sci. Math. **176**, 103131, 2022.

[26] S. D. Zeng, V. D. Rădulescu and P. Winkert, *Double phase implicit obstacle problems with convection and multivalued mixed boundary value conditions*, SIAM J. Math. Anal. **54**, 1898–1926, 2022.

[27] S. D. Zeng, E. Vilches, *Well-posedness of history/state-dependent implicit sweeping processes*, J. Optim. Theory Appl. **186**, 960–984, 2020.

[28] A. Zolfaghari, H. Dehghanpour, E. Ghanbari and D. Bearinger, *Fracture characterization using flowback salt-concentration transient*, SPE J. **21** (1), 233–244, 2016.