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Quasi-Concircular Curvature Tensor on Generalized Sasakian Space-Forms

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Abstract. The object of the present paper is to study Quasi-concircularly flat and $\phi-$quasi-concircularly flat generalized Sasakian-space-forms. Also, we consider generalized Sasakian-space-forms satisfying the condition $P(\xi, X).\widetilde{V} = 0$, $\widetilde{V}(\xi, X).P = 0$, and $\widetilde{V}(\xi, X).\widetilde{V} = 0$ and we obtain some important results. Finally, we give an example.

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1. Introduction

The curvature tensor of Riemannian in differential geometry plays an important role. As well as the sectional curvatures of a manifold determine the curvature tensor R completely. A Riemannian manifold with constant sectional curvature c is a known as real space-form whose curvature tensor is given by

$$R(X, Y)Z = c[g(Y, Z)X - g(X, Z)Y],$$

$\forall\ X, Y, Z \in TM$.

A Sasakian manifold with constant $\phi-$sectional curvature becomes a Sasakain space-form and it has a specific form of its curvature tensor. In order to, Alegre et al. in 2004 [1] introduced the notion of generalized Sasakian space form.

An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is known as generalized Sasakian space form whose curvature tensor R is given by

$$R(X, Y)Z = f_1R_1 + f_2R_2 + f_3R_3,$$

where f_1, f_2, f_3 are differentiable functions on M and

$$\left.\begin{aligned} R_1(X, Y)Z &= g(Y, Z)X - g(X, Z)Y, \\ R_2(X, Y)Z &= g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z, \\ R_3(X, Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ &\quad - g(Y, Z)\eta(X)\xi, \end{aligned}\right\}$$

$\forall\ X, Y, Z \in TM$. In 2004, the author give several examples of generalized Sasakian space-forms. If $f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}$ and $f_1 = \frac{c-1}{4}$, then a generalized Sasakain space form becomes Sasakian space form. The geometry of generalized Sasakian space form has been developed by several authors such as Alegre and Carriazo [2], Sular and Özgür [22],

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Nagaraja et al. [16], Shah [23], Majhi and De [13], Sarkar and Akbar [24], Hui and Chakraborty [11], Venkatesha and Shanmukha [28], Chaubey and Yadav [6], Singh and Kishor [25], Singh and Lalmalsawma [26] and many others. Recently, De and Majhi [8] studied $\xi - Q-$flat generalized Sasakian space-forms and $\phi - Q-$flat Sasakian space forms under the consideration that characteristic vector field ξ is killing and he classified locally symmetric generalized Sasakian space-forms. Also, he proved some geometric properties of generalized Sasakian space-form which depends on the nature of differentiable functions f_1, f_2 and f_3.

2. Preliminaries

A (2n+1) dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if admits a tensor ϕ of type (1,1), ξ is a vector fields of type (0,1) and 1-form η is a tensor of the type (1,0) satisfying [4, 5]

$$\phi^2 = -I + \eta \otimes \xi, \ or \ \phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta(\phi X) = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(\phi X, X) = 0.$$

Again for a $(2n + 1)$ dimensional generalized Sasakian space-form, the following relation holds [1]:

$$\begin{aligned} R(X, Y)Z = {} & f_1[g(Y, Z)X - g(X, Z)Y] + f_2[g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z] \\ & + f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi], \end{aligned} \tag{2.4}$$

$$\begin{aligned} S(X, Y) &= (2nf_1 + 3f_2 - f_3)g(X, Y) - [3f_2 + (2n - 1)f_3]\eta(X)\eta(Y), \\ Q^*X &= (2nf_1 + 3f_2 - f_3)X - [3f_2 + (2n - 1)f_3]\eta(X)\xi, \\ S(\phi X, \phi Y) &= S(X, Y) - 2n(f_1 - f_3)\eta(X)\eta(Y), \end{aligned} \tag{2.5}$$

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.6}$$

$$R(\xi, X)Y = (f_1 - f_3)[g(X, Y) - \eta(Y)X], \tag{2.7}$$

$$\begin{aligned} S(X, \xi) &= 2n(f_1 - f_3)\eta(X), \\ S(\xi, \xi) &= 2n(f_1 - f_3), \\ Q^*\xi &= 2n(f_1 - f_3)\xi, \\ r &= 2n[(2n + 1)f_1 + 3f_2 - 2f_3], \end{aligned} \tag{2.8}$$

where R, S, Q^* and r are the curvature tensor, Ricci tensor, Ricci operator and scalar curvature tensor of the space-form, respectively.

The conformal curvature tensor C [30], the projective curvature tensor [10], and the concircular curvature tensor V [29] on Riemannian manifold (M^{2n+1}, g) defined by

$$\begin{aligned} C(X, Y)Z = {} & R(X, Y)Z - \frac{1}{2n - 1}[S(Y, Z)X - S(X, Z)Y] + g(Y, Z)Q^*X - g(X, Z)Q^*Y] + \frac{r}{2n(2n - 1)}[g(Y, Z)X - g(X, Z)Y], \\ P(X, Y)Z = {} & R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y], \\ V(X, Y)Z = {} & R(X, Y)Z - \frac{r}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{2.9}$$

$\forall X, Y, Z \in TM$ respectively, where Q^* is the Ricci operator defined by $S(X, Y) = g(Q^*X, Y)$.

Conformal curvature tensor, projective curvature tensor and concircular curvature tensor studied by Kim [12], De and Sarkar [9], and Shukla and Shah [21] and many others.

On the other hand, the $Q-$curvature tensor in a Riemannian manifold (M^{2n+1}, g) defined by Montica and Suh [14]

$$Q(X, Y)Z = R(X, Y)Z - \frac{\psi}{2n}[g(Y, Z)X - g(X, Z)Y],$$

$\forall X, Y, Z \in TM$. $Q-$curvature tensor studied by many worker such as De and Majh [8], Singh and Prasad [27] etc.

In a recent paper, Prasad and Maurya [19] introduced "Quasi-concircular curvature tensor " of type (1,3) in (2n+1) dimensional Riemannian manifold (M^{2n+1}, g) denoted by $\widetilde{V}$ and defined by

$$\widetilde{V}(X, Y)Z = aR(X, Y)Z + \frac{r}{2n + 1}\left(\frac{a}{2n} + 2b\right)[g(Y, Z)X - g(X, Z)Y]. \tag{2.10}$$

If $a = 1,\ b = -\frac{1}{n-1}$, then the quasi-concircular curvature tensor $\widetilde{V}$ reduces to concircular curvature tensor defined by Yano and Kon [30]. Quasi-concircular curvature tensor was studied by many authors such as Narain et al. [15], Prasad and Yadav [20], Ahmad et al. [3] etc.

Let C be the Conformal curvature tensor of M^{2n+1}. The tangent space $T_p(M)$ into direct sumn $T_pM = \phi\left(T_pM\right)\oplus\left\{\xi_p\right\}$, where $\left\{\xi_p\right\}$ is a 1-dimensional linear subspace of $T_p(M)$ generated by ξ_p, we have a map:

$$C : T_p(M) \times T_pM \times T_pM \to \phi(T_pM) \oplus \left\{\xi_p\right\}.$$

It may be natural to consider the following particular cases:

(i) $C : T_p(M) \times T_pM \times T_pM \to L\left\{\xi_p\right\}$, that is, the projection of the image of C in $\phi(T_p(M))$ is zero.

(ii) $C : T_p(M) \times T_pM \times T_pM \to \phi(T_p(M))$, that is, the projection of the image of C in $L\left\{\xi_p\right\}$ is zero. This condition is equivalent to

$$C(X,Y)\xi = 0.$$

(iii) $C : \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \to L\left\{\xi_p\right\}$, that is, when C is restricted to $\phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M))$, the projection of the image of C in $\phi(T_p(M))$ is zero. This condition is equivalent to Cabrerizo et al. [7],

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0 \ if\ and\ only\ if\ g(C(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{2.11}$$

A differentiable manifold satisfying the condition (2.11) is called $\phi-$Conformally flat manifold studied by many authors such as Zhen [31], Zhen et al. [32], Özgür [17], Öztürk [18], Singh and Prasad [27] in K-contact manifold, LP-Sasakain manifold and α-cosymplectic manifolds.

Definition 2.1. A generalized Sasakian space-form $(M^{2n+1}, g), n > 1$ is called $\xi-$quasi-concircularly flat if

$$\widetilde{V}(X,Y)\xi = 0 \ on\ M.$$

Definition 2.2. A generalized Sasakian space-form $(M^{2n+1}, g), n > 1$ is called $\phi-$quasi-concircularly flat if

$$g(\widetilde{V}(\phi X, \phi Y)\phi Z, \phi W) = 0 \ on\ M.$$

After introduction and preliminaries, in section 3 we consider $\xi-$quasi concircularly flat generalized Sasakian space form. Section 4 is devoted to study $\phi-$quasi concircularly flat generalized Sasakian-space-forms. Sections 5, 6 and 7 deal with generalized Sasakian-space-forms satisfying $P(\xi, X).\widetilde{V} = 0$, $P(\xi, X).\widetilde{V} = 0$, $\widetilde{V}(\xi, X).P = 0$ and $\widetilde{V}(\xi, X).\widetilde{V} = 0$ respectively. Finally, the results of sections 3 and 4 are illustrated by examples.

3. $\xi-$Quasi-Concircularly Flat Generalized Sasakian Space-Forms

A generalized Sasakian space-form $(M^{2n+1}, g), n > 1$ is $\xi-$quasi-concircularly flat if

$$\widetilde{V}(X,Y)\xi = 0. \tag{3.1}$$

Putting ξ for Z in eqn. (2.10) and using (2.2), we get

$$\widetilde{V}(X,Y)\xi = aR(X,Y)\xi + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)[\eta(Y)X - \eta(X)Y], \tag{3.2}$$

$\forall X, Y \in TM$. Using (2.6) and (2.8) in (3.2), we get

$$\widetilde{V}(X,Y)\xi = \left[a\,(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right][\eta(Y)X - \eta(X)Y],$$

which implies that

$$\widetilde{V}(X,Y)\xi = \left[a\,(f_1 - f_3) + \left(\frac{2n(2n+1)f_1 + 6nf_2 - 4nf_3}{2n+1}\right)\left(\frac{a}{2n} + 2b\right)\right][\eta(Y)X - \eta(X)Y]. \tag{3.3}$$

In view of (3.1) and (3.3), we can state the following theorem.

Theorem 3.1. *A (2n+1)-dimensional generalized Sasakain space-form $M(f_1, f_2, f_3)$ is $\xi-$quasi-concircularly flat if and only if* $a\,(f_1 - f_3) + \left(\frac{2n(2n+1)f_1+6nf_2-4nf_3}{2n+1}\right)\left(\frac{a}{2n} + 2b\right) = 0$.

Particular case : (i) If $a = 1, b = -\frac{1}{2n}$, then quasi-concircualr curvature tensor becomes concircular curvature tensor.
(ii) If $a = 1, b = -\frac{1}{2n}$, $\frac{r}{2n+1} = \psi$, then quasi-concircualr curvature tensor reduces to Q-curvature tensor.
Hence, Theorem 3.1 can be restated as follows.

Corollary 3.2. *A generalized Saasakian space-form $M(f_1, f_2, f_3)$ is $\xi-$concircularly flat if and only if $f_3 = \frac{3}{1-2n} f_2$.*

Corollary 3.3. *A generalized Saasakian space-form $M(f_1, f_2, f_3)$ is $\xi - Q$ flat if and only if $\psi = 2n(f_1 - f_3)$.*

Corollary 3.2 and 3.3 have been proved by De and Majhi [8] in 2019.
Therefore, Corollary 3.2 and 3.3 are particular case of Theorem 3.1.

4. $\phi-$Quasi-Concircularly Flat Generalized Sasakian Space-Forms

In this section, we assume that (M^{2n+1}, g) be $\phi-$quasi-concircularly flat generalized Sasakian space-form. Then,

$$\phi^2 \widetilde{V}(\phi X, \phi Y)\phi Z = 0,$$

holds if and only if

$$g(\widetilde{V}(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{4.1}$$

From (2.1) and (4.1), we get

$$R(\widetilde{V}(\phi X, \phi Y)\phi Z, \phi W) = -\frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(\phi Y, \phi Z)(\phi X, \phi W) - g(\phi X, \phi Z)(\phi Y, \phi W)]. \tag{4.2}$$

Using (2.1) in (2.4), we get

$$\begin{aligned} R(\widetilde{V}(\phi X, \phi Y)\phi Z, \phi W) = {} & f_1[g(\phi Y, \phi Z)(\phi X, \phi W) - g(\phi X, \phi Z)(\phi Y, \phi W)] + f_2[g(\phi X, Z)(Y, \phi W) \\ & - g(\phi Y, Z)(X, \phi W) + 2g(\phi X, Y)g(Z, \phi W)]. \end{aligned} \tag{4.3}$$

In view of (4.2) and (4.3), we get

$$\begin{aligned} & \left[f_1 + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right][g(\phi Y, \phi Z)(\phi X, \phi W) - g(\phi X, \phi Z)(\phi Y, \phi W)] \\ & + f_2\left[g(\phi X, Z)(Y, \phi W) - g(\phi Y, Z)(X, \phi W) + 2g(\phi X, Y)g(Z, \phi W)\right] = 0. \end{aligned} \tag{4.4}$$

Contracting eqn. (4.4) with respect to Y and Z, we get

$$\left[f_1 + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]\left[2ng(\phi X, \phi W) - g\left(\phi^2 X, \phi^2 W\right)\right] + 3f_2 g(\phi X, \phi W) = 0. \tag{4.5}$$

Using (2.1) in (4.5), we get

$$\left[f_1 + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right][2ng(\phi X, \phi W) - g(X, W) + \eta(X)\eta(W)] + 3f_2 g(\phi X, \phi W) = 0. \tag{4.6}$$

Again using (2.3) in (4.6), we get

$$\left[(2n-1)\left\{f_1 + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right\} + 3f_2\right]g(\phi X, \phi W) = 0,$$

which gives

$$f_1 + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right) = -\frac{3f_2}{2n-1}. \tag{4.7}$$

Using (4.7) in (4.4), we get

$$\begin{aligned} f_2\Bigg[& -\frac{3}{2n-1}\{g(\phi Y, \phi Z)(\phi X, \phi W) - g(\phi X, \phi Z)(\phi Y, \phi W)\} \\ & + \{g(\phi X, Z)(Y, \phi W) - g(\phi Y, Z)(X, \phi W) + 2g(\phi X, Y)g(Z, \phi W)\}\Bigg] = 0, \end{aligned}$$

which gives either $f_2 = 0$, or

$$\begin{aligned} & -\frac{3}{2n-1}[g(\phi Y, \phi Z)(\phi X, \phi W) - g(\phi X, \phi Z)(\phi Y, \phi W)] \\ & + [g(\phi X, Z)(Y, \phi W) - g(\phi Y, Z)(X, \phi W) + 2g(\phi X, Y)g(Z, \phi W)] = 0. \end{aligned} \tag{4.8}$$

Replacing W by ϕW in (4.8) and using (2.1), we have

$$\frac{3}{2n-1}[g(\phi Y, \phi Z)(\phi X, W) - g(\phi X, \phi Z)(\phi Y, W)] \\ - [g(\phi X, Z)(Y, W) - g(\phi Y, Z)(X, W) + 2g(\phi X, Y)g(Z, W)] = 0. \tag{4.9}$$

Putting ξ for Z in (4.9), we obtain

$$g(\phi X, Y)\eta(W) = 0,$$

which gives either $\eta = 0$ or $g(\phi X, Y) = 0$. Both case is not possible.

Hence, $\phi-$quasi concirclarly flat generaized Sasakian space-form implies $f_2 = 0$.

Therefore, we have the following theorem.

Theorem 4.1. *A (2n+1)-dimensional $n > 1$ generalized Sasakian space-form $M(f_1, f_2, f_3)$ is $\phi-$quasi concircularly flat if $f_2 = 0$.*

In 2006, Kim [12] proved that for a (2n+1)–dimensional generalized Sasakian space-form the following hold:

(i) If $n > 1$, then M is conformally flat if and only if $f_2 = 0$.
(ii) If M is conformally flat and ξ is Killing, then M is locally symmetric and has constant ϕ-sectional curvature.

Recently in [8], De and Majhi proved a (2n+1)–dimensional generalized Sasakian space-form the following hold:

(i) If $n > 1$, then M is $\phi - Q-$flat if and only if $f_2 = 0$.
(ii) If M is $\phi - Q-$flat and ξ is Killing, then M is locally symmetric and has constant ϕ-sectional curvature.

In view of first part of above theorem, we have the following.

Corollary 4.2. *In a (2n+ 1)-dimensional $n > 1$ generalized Sasakian-space-form $\phi-$quasi concircularly flat, $\phi-Q-$flat and conformally flat are equivalent.*

Again in view of the second part of the above theorem, we have the following.

Corollary 4.3. *A quasi-concircurly flat (2n+1)-dimensional $n > 1$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with ξ as Killing vector field is locally symmetric and has constant $\phi-$sectional curvature.*

Let us consider $f_2 = 0$. Then, from (4.4), we get

$$g(\widetilde{V}(\phi X, \phi Y)\phi Z, \phi W) = \left[f_1 + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right][g(\phi Y, \phi Z)(\phi X, \phi W) - g(\phi X, \phi Z)(\phi Y, \phi W)]. \tag{4.10}$$

Putting $f_2 = 0$ in (4.7), we get

$$f_1 = -\frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right). \tag{4.11}$$

Combining the eqn. (4.10) and (4.11), we have the following theorem.

Theorem 4.4. *A (2n+ 1)-dimensional $n > 1$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is $\phi-$quasi concircularly flat if and only if $f_2 = 0$.*

5. Generalized Sasakian Space-Forms Satisfying $P(\xi, X).\widetilde{V} = 0$

In this section, we assume that generalized Sasakian space-from $M(f_1, f_2, f_3)$ satisfying the condition

$$P(\xi, X).\widetilde{V} = 0,$$

which implies that

$$\left(P(\xi, X).\widetilde{V}\right)(Y, Z)U = 0, \tag{5.1}$$

$\forall X, Y, Z \in TM$, where P is the projective curvature tensor.
In view of (5.1), we get

$$P(\xi, X)\widetilde{V}(Y, Z)U - \widetilde{V}(P(\xi, X)Y, Z)U - \widetilde{V}(Y, P(\xi, X)Z)U - \widetilde{V}(Y, Z)P(\xi, X)U = 0. \tag{5.2}$$

After using (2.7), (2.9) and (2.10), we obtain

$$
\begin{aligned}
P(\xi,X)\widetilde{V}(Y,Z)U =& a\left[\frac{f_3-3f_2}{2n}-f_3\right][g(X,R(Y,Z)U)\xi-\eta(R(Y,Z)U)\eta(X)\xi\,]\\
&+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\left[\frac{f_3-3f_2}{2n}-f_3\right][g(X,Y)\xi-\eta(X)\eta(Y)\xi\,]g(Z,U)\\
&-\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\left[\frac{f_3-3f_2}{2n}-f_3\right][g(X,Z)\xi-\eta(X)\eta(Z)\xi\,]g(Y,U).
\end{aligned}
\tag{5.3}
$$

After using (2.7), (2.9) and (2.10), we obtain

$$
\widetilde{V}(P(\xi,X)Y,Z)U=\left[\frac{f_3-3f_2}{2n}-f_3\right]\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right][g(X,Y)-\eta(X)\eta(Y)][g(Z,U)\xi-\eta(U)Z]. \tag{5.4}
$$

Further use of (2.7), (2.9) and (2.10), we have

$$
\widetilde{V}(Y,P(\xi,X)Z)U=\left[\frac{f_3-3f_2}{2n}-f_3\right]\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right][g(X,Z)-\eta(X)\eta(Z)][\eta(U)Y-g(Y,U)\xi]. \tag{5.5}
$$

Similarly use of (2.7), (2.9) and (2.10), we have

$$
\widetilde{V}(Y,Z)P(\xi,X)U=\left[\frac{f_3-3f_2}{2n}-f_3\right]\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right][g(X,U)-\eta(X)\eta(U)][\eta(Z)Y-\eta(Y)Z]. \tag{5.6}
$$

Now, substituting (5.3), (5.4), (5.5) and (5.6) in (5.2), we get

$$
\begin{aligned}
& a\left[\frac{f_3-3f_2}{2n}-f_3\right][g(X,R(Y,Z)U)\xi-\eta(R(Y,Z)U)\eta(X)\xi\,]+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\left[\frac{f_3-3f_2}{2n}-f_3\right].\\
& [g(X,Y)\xi-\eta(X)\eta(Y)\xi\,]g(Z,U)-\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\left[\frac{f_3-3f_2}{2n}-f_3\right][g(X,Z)\xi-\eta(X)\eta(Z)\xi\,]g(Y,U)\\
& -\left[\frac{f_3-3f_2}{2n}-f_3\right]\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right][g(X,Y)-\eta(X)\eta(Y)][g(Z,U)\xi-\eta(U)Z]\\
& -\left[\frac{f_3-3f_2}{2n}-f_3\right]\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right][g(X,Z)-\eta(X)\eta(Z)][\eta(U)Y-g(Y,U)\xi]\\
& -\left[\frac{f_3-3f_2}{2n}-f_3\right]\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right][g(X,U)-\eta(X)\eta(U)][\eta(Z)Y-\eta(Y)Z]=0.
\end{aligned}
\tag{5.7}
$$

Taking inner product with ξ in (5.7), we have

$$
\begin{aligned}
& a\left[\frac{f_3-3f_2}{2n}-f_3\right][g(X,R(Y,Z)U)-\eta(R(Y,Z)U)\eta(X)]+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\left[\frac{f_3-3f_2}{2n}-f_3\right].\\
& [g(X,Y)-\eta(X)\eta(Y)]g(Z,U)-\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\left[\frac{f_3-3f_2}{2n}-f_3\right][g(X,Z)-\eta(X)\eta(Z)]g(Y,U)\\
& -\left[\frac{f_3-3f_2}{2n}-f_3\right]\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right][g(X,Y)-\eta(X)\eta(Y)][g(Z,U)-\eta(U)\eta(Z)]\\
& -\left[\frac{f_3-3f_2}{2n}-f_3\right]\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right][g(X,Z)-\eta(X)\eta(Z)][\eta(U)\eta(Y)-g(Y,U)]=0.
\end{aligned}
\tag{5.8}
$$

Putting $X=Y=e_i$ in (5.8), where $\{e_i,\xi\}, 1\le i\le 2n$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$
\left[\frac{f_3-3f_2}{2n}-f_3\right]\left[aS(Y,U)-2na(f_1-f_3)g(Z,U)+2n\left\{a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right\}\eta(U)\eta(Z)\right].
$$

Hence, either $(1-2n)f_3-3f_2=0$ or

$$
aS(Y,U)=2na(f_1-f_3)g(Z,U)-2n\left\{a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right\}\eta(U)\eta(Z). \tag{5.9}
$$

In the second case, comparing eqn. (5.9) with (2.5), for Z and W orthogonal to ξ, we get

$$2n(f_1 - f_3) = 2nf_1 + 3f_2 - f_3,$$

which yeilds

$$f_3 = \frac{3}{1-2n} f_2.$$

Then, as $f_3 = \frac{3}{1-2n} f_2$ from (2.5), $S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y)$ and M is an Einstein manifold. Conversely if $\frac{f_3 - 3f_2}{2n} - f_3 = 0$, that is, $f_3 = \frac{3}{1-2n} f_2$, then from eqn. (5.3), (5.4), (5.5) and (5.6), we have

$$P(\xi, X).\widetilde{V} = 0.$$

Hence, we have the following theorem.

Theorem 5.1. *A (2n+1)-dimensional $n > 1$ generalized Sasakain space-form $M(f_1, f_2, f_3)$ satisfies $P(\xi, X).\widetilde{V} = 0$, if and only if $f_3 = \frac{3}{1-2n} f_2$. In such a case, it is an Einstein manifold provided $a \neq 0$.*

Particular case: If $a = 1, b = -\frac{1}{2n}$ and $\frac{r}{2n+1} = \psi$, then quasi-concircular curvature tensor reduces the $Q-$curvature tensor. Now putting $Z = U = \xi$ in (5.9), where $\{e_i, \xi\}, 1 \leq i \leq 2n$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$r = 2n(2n+1)(f_1 - f_3) - 2n\left[(f_1 - f_3)\frac{\psi}{2n}\right]. \tag{5.10}$$

Using $\psi = \frac{r}{2n+1} \Rightarrow r = (2n+1)\psi$, the equation (5.10) becomes

$$\frac{\psi}{2n} = (f_1 - f_3),$$

and the manifold reduces to an Einstein manifold.

Hence, Theorem 5.1 can be restated as follows.

Corollary 5.2. *A (2n+1)-dimensional $n > 1$ generalized Sasakain space-form $M(f_1, f_2, f_3)$ satisfies $P(\xi, X).Q = 0$ if and only if $f_3 = \frac{3}{1-2n} f_2$. In such a case it is an Einstein manifold.*

This Corollary has been proved by De and Majhi [8] by another way.

Therefore, Corollary 5.2 is particular case of Theorem 5.1.

6. Generalized Sasakian Space-Forms Satisfying $\widetilde{V}(\xi, X).P = 0$

We assume that generalized Sasakian space-from $M(f_1, f_2, f_3)$ satisfying the condition

$$\left(\widetilde{V}(\xi, X).P\right)(Y, Z)U = 0, \tag{6.1}$$

$\forall X, Y, ZU \in$TM. View of (6.1), we get

$$\widetilde{V}(\xi, X)P(Y, Z)U - P(\widetilde{V}(\xi, X)Y, Z)U - P(Y, \widetilde{V}(\xi, X)Z)U - P(Y, Z)\widetilde{V}(\xi, X)U = 0. \tag{6.2}$$

Now, using (2.7), (2.9) and (2.10), we have

$$\begin{aligned}\widetilde{V}(\xi, X)P(Y, Z)U = &\left[a(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]\Big[\{g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)X\} \\ &- \frac{1}{2n}\{g(X, Y)\xi - \eta(Y)X\}S(Z, U) + \frac{1}{2n}\{g(X, Z)\xi - \eta(Z)X\}S(Y, U)\Big].\end{aligned} \tag{6.3}$$

Again using (2.7), (2.9) and (2.10), we have

$$\begin{aligned}P(\widetilde{V}(\xi, X)Y, Z)U = &\left[a(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]\Big[\{g(X, Y)R(\xi, Z)U - \eta(Y)R(X, Z)U\} \\ &- \frac{1}{2n}\{g(X, Y)\xi - \eta(Y)X\}S(Z, U) + \frac{1}{2n}\{S(\xi, U)g(X, Y)Z - S(X, U)\eta(Y)Z\}\Big].\end{aligned} \tag{6.4}$$

Further using (2.7), (2.9) and (2.10), we have

$$P(Y,\widetilde{V}(\xi,X)Z)U=\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\left[\{g(X,Z)R(Y,\xi)U-\eta(Z)R(Y,X)U\}\right.$$
$$\left.+\frac{1}{2n}\{g(X,Z)\xi-\eta(Z)X\}-\frac{1}{2n}\{S(\xi,U)g(X,Z)Y-S(X,U)\eta(Z)Y\}\right]. \tag{6.5}$$

Finally, using (2.7), (2.9) and (2.10), we have

$$P(Y,Z)\widetilde{V}(\xi,X)U=\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\left[\{g(X,U)R(Y,Z)\xi-\eta(U)R(Y,Z)X\}\right.$$
$$\left.-\frac{1}{2n}\{S(Z,\xi)Y-S(Y,\xi)Z\}\,g(X,U)+\frac{1}{2n}\{S(Z,X)Y-S(Y,X)Z\}\,\eta(U)\right]. \tag{6.6}$$

Using (6.3)-(6.6) in eqn. (6.2), we get

$$\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\left[g(X,R(Y,Z)U)\xi-\eta(R(Y,Z)U)X-g(X,Y)R(\xi,Z)U\right.$$
$$+\eta(Y)R(X,Z)U-g(X,Z)R(Y,\xi)U+\eta(Z)R(Y,X)U-g(X,U)R(Y,Z)\xi+\eta(U)R(Y,Z)X$$
$$-\frac{1}{2n}\{S(\xi,U)g(X,Y)Z-S(X,U)\eta(Y)Z\}+\frac{1}{2n}\{S(\xi,U)g(X,Z)Y-S(X,U)\eta(Z)Y\}$$
$$\left.+\frac{1}{2n}\{S(Z,\xi)Y-S(Y,\xi)Z\}\,g(X,U)-\frac{1}{2n}\{S(Z,X)Y-S(Y,X)Z\}\,\eta(U)\right]=0. \tag{6.7}$$

In (6.7), taking inner product with W, we get

$$\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\left[g(X,R(Y,Z)U)\eta(W)-\eta(R(Y,Z)U)g(X,W)-g(X,Y)g(R(\xi,Z)U,W)+\eta(Y).\right.$$
$$g(R(X,Z)U,W)-g(X,Z)g(R(Y,\xi)U,W)+\eta(Z)g(R(Y,X)U,W)-g(X,U)g(R(Y,Z)\xi,W)+\eta(U)g(R(Y,Z)X,W)$$
$$-\frac{1}{2n}\{S(\xi,U)g(X,Y)g(Z,W)-S(X,U)\eta(Y)g(Z,W)\}+\frac{1}{2n}\{S(\xi,U)g(X,Z)g(Y,W)-S(X,U)\eta(Z)g(Y,W)\}$$
$$\left.+\frac{1}{2n}\{S(Z,\xi)g(Y,W)-S(Y,\xi)g(Z,W)\}\,g(X,U)-\frac{1}{2n}\{S(Z,X)g(Y,W)-S(Y,X)g(Z,W)\}\,\eta(U)\right]=0. \tag{6.8}$$

Taking $X=Y=e_i$ in (6.8), where $\{e_i,\xi\}$, $1\leq i\leq 2n$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\left[(f_1-f_3)\{\eta(Z)g(U,W)-\eta(U)g(Z,W)\}\right.$$
$$-2(f_1-f_3)\{\eta(W)g(Z,U)-\eta(U)g(Z,W)\}-(2n+1)(f_1-f_3)\eta(U)g(Z,W)$$
$$\left.+S(Z,U)\eta(W)-\eta(U)S(Z,W)+\frac{1}{2n}\{S(U,W)\eta(Z)-S(Z,W)\eta(U)\}+\frac{r}{2n}g(Z,W)\eta(U)\right]=0. \tag{6.9}$$

Putting ξ for W in (6.9), we obtain

$$\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\left[S(Z,U)-2n(f_1-f_3)g(Z,U)-\left\{(2n+1)(f_1-f_3)+\frac{r}{2n}\right\}\eta(U)\eta(Z)\right]=0.$$

Therefore, either $a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)=0$ or

$$S(Z,U)-2n(f_1-f_3)g(Z,U)-\left\{(2n+1)(f_1-f_3)+\frac{r}{2n}\right\}\eta(U)\eta(Z)=0. \tag{6.10}$$

Putting $Z=U=\xi$ in (6.10), we get

$$2n(f_1-f_3)=\frac{r}{2n}.$$

Using in eqn. (6.10), we get

$$S(Z,U)=2n(f_1-f_3)g(Z,U).$$

In this case, the manifold M^{2n+1} is an Einstien manifold. Conversely if

$$a(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right) = 0,$$

then in view of (6.3)-(6.6), we have

$$\widetilde{V}(\xi, X).P = 0.$$

Thus, we have the following theorem.

Theorem 6.1. *A (2n+1)-dimensional $n > 1$ generalized Sasakain space-form $M(f_1, f_2, f_3)$ satisfies $\widetilde{V}(\xi, X).P = 0$ if and only if $a(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right) = 0$. In such a case, the manifold M^{2n+1} is an Einstein manifold .*

Particular case : (i) If $a = 1, b = -\frac{1}{2n}$, then quasi-concircualr curvature tensor becomes concircular curvature tensor.

(ii) If $a = 1, b = -\frac{1}{2n}$ and $\frac{r}{2n+1} = \psi$, then quasi-concircualr curvature tensor reduces to Q-curvature tensor.

Hence, Theorem 6.1 can be restated as follows.

Corollary 6.2. *A (2n+1)-dimensional $n > 1$ generalized Sasakain space-form $M(f_1, f_2, f_3)$ satisfies $V(\xi, X).P = 0$ if and only if $3f_2 + (2n-1)f_3 = 0$.*

Corollary 6.3. *A (2n+1)-dimensional $n > 1$ generalized Sasakain space-form $M(f_1, f_2, f_3)$ satisfies $Q(\xi, X).P = 0$ if and only if $f_1 - f_3 - \frac{\psi}{2n} = 0$.*

These Corollary 6.2 and 6.3 have been proved by De and Yildiz [10] and De and Majhi [8], respectively.

Therefore, Corollary 6.2 and 6.3 are particular case of Theorem 6.1.

7. Generalized Sasakian Space-Forms Satisfying $\widetilde{V}(\xi, X).\widetilde{V} = 0$

Let a (2n+1) dimensional $n > 1$ generalized Sasakian space-from $M(f_1, f_2, f_3)$ satisfying the condition

$$\left(\widetilde{V}(\xi, X).\widetilde{V}\right)(Y, Z)U = 0, \tag{7.1}$$

$\forall X, Y, Z, U \in$TM. View of (7.1), we get

$$\widetilde{V}(\xi, X)\widetilde{V}(Y, Z)U - \widetilde{V}(\widetilde{V}(\xi, X)Y, Z)U - \widetilde{V}(Y, \widetilde{V}(\xi, X)Z)U - \widetilde{V}(Y, Z)\widetilde{V}(\xi, X)U = 0. \tag{7.2}$$

Now using (2.7) and (2.10), we have

$$\begin{aligned}\widetilde{V}(\xi, X)\widetilde{V}(Y, Z)U = &\left[a(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]\Bigg[\{g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)X\}\\ &+ \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\{g(X, Y)\xi - \eta(Y)X\}\, g(Z, U) - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right).\{g(X, Z)\xi - \eta(Z)X\}\, g(Y, U)\Bigg].\end{aligned} \tag{7.3}$$

Again using (2.7) and (2.10), we have

$$\begin{aligned}\widetilde{V}(\widetilde{V}(\xi, X)Y, Z)U = &\left[a(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]^2 [g(U, Z)\xi - \eta(U)Z]g(X, Y)\\ &- \left[a(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]\eta(Y)\widetilde{V}(X, Z)U.\end{aligned} \tag{7.4}$$

Further using (2.7) and (2.10), we have

$$\begin{aligned}\widetilde{V}(Y, \widetilde{V}(\xi, X)Z)U = &\left[a(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]^2 [\eta(U)Y - g(Y, U)\xi\,]g(X, Z)\\ &- \left[a(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]\eta(Z)\widetilde{V}(Y, X)U.\end{aligned} \tag{7.5}$$

Again using (2.7) and (2.10), we have

$$\begin{aligned}\widetilde{V}(Y, Z)\widetilde{V}(\xi, X)U = &\left[a(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]^2 [\eta(Y)X - \eta(X)Y]g(X, U)\\ &- \left[a(f_1 - f_3) + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right]\eta(U)\widetilde{V}(Y, Z)X.\end{aligned} \tag{7.6}$$

Using (7.3)-(7.6) in (7.2), we get

$$\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\Bigg[\{g(X,R(Y,Z)U)\xi-\eta(R(Y,Z)U)X\}+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right).$$
$$\{g(X,Y)\xi-\eta(Y)X\}\,g(Z,U)-\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\{g(X,Z)\xi-\eta(Z)X\}\,g(Y,U)\Bigg]$$
$$-\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]^2[g(U,Z)\xi-\eta(U)Z]g(X,Y)+\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\eta(Y)\widetilde{V}(X,Z)U$$
$$-\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]^2[\eta(U)Y-g(Y,U)\xi\,]g(X,Z)+\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\eta(Z)\widetilde{V}(Y,X)U$$
$$-\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]^2[\eta(Y)X-\eta(X)Y]g(X,U)+\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\eta(U)\widetilde{V}(Y,Z)X. \tag{7.7}$$

Taking inner product with ξ in (7.7), we obtain

$$\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\Bigg[\{g(X,R(Y,Z)U)-\eta(R(Y,Z)U)\eta(X)\}+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right).$$
$$\{g(X,Y)-\eta(X)\eta(X)\}\,g(Z,U)-\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\{g(X,Z)-\eta(Z)\eta(X)\}\,g(Y,U)\Bigg]$$
$$-\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]^2[g(U,Z)-\eta(U)\eta(Z)]g(X,Y)+\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\eta(Y)\eta(\widetilde{V}(X,Z)U)$$
$$-\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]^2[\eta(U)\eta(Y)-g(Y,U)]g(X,Z)+\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\eta(Z)\eta(\widetilde{V}(Y,X)U)$$
$$+\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\eta(U)\eta(\widetilde{V}(Y,Z)X)=0. \tag{7.8}$$

Putting $X=Y=e_i$ in (7.8), where $\{e_i,\xi\}$, $1\le i\le 2n$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$\left[a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\Bigg[S(Z,U)-\left\{2n.a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right\}g(Z,U)$$
$$+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\eta(U)\eta(Z)\Bigg]=0.$$

Hence, either

$$a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)=0,$$

or

$$S(Z,U)=\left[2n.a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]g(Z,U)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\eta(U)\eta(Z). \tag{7.9}$$

When

$$a(f_1-f_3)+\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)=0\Rightarrow\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)=-a(f_1-f_3).$$

Then, equation (7.9) will be

$$S(Z,U)=a(2n-1)(f_1-f_3)g(Z,U)-a(f_1-f_3)\eta(U)\eta(Z)$$
$$\Rightarrow S(Z,U)=Ag(Z,U)+B\eta(U)\eta(Z)$$

where

$$A=a(2n-1)(f_1-f_3),\quad B=-a(f_1-f_3).$$

Hence we have the following theorem.

Theorem 7.1. *A (2n+1)-dimensional $n>1$ generalized Sasakain space-form $M(f_1,f_2,f_3)$ satisfies $\widetilde{V}(\xi,X).\widetilde{V}=0$, then the manifold M^{2n+1} is $\eta-$Einstein manifold.*

8. Examples

Example 8.1. Let $N(a, b)$ be a generalized complex space-form, then the warped product $\mathrm{M} = \mathbb{R}f \times_f N$ endowed with the almost contact metric structure (ϕ, ξ, η, g_f) is a generalized Sasakian-space-form with $M(f_1, f_2, f_3)$ [1] with

$$f_1 = \frac{\overline{a} - (f')^2}{f^2}, \quad f_2 = \frac{\overline{b}}{f^2}, \quad f_3 = \frac{\overline{a} - (f')^2}{f^2} + \frac{f''}{f^2},$$

where $f(t) = t, t \in \mathbb{R}$ and f' denotes the derivstive of f with respect to t. If we choose $\overline{a} = 2, \overline{b} = 0$ and $f(t) = t$ with $t \neq 0$, then $f_1 = \frac{1}{t^2}, f_2 = 0$ and $f_3 = \frac{1}{t^2}$,

$$R(X, Y)Z = \frac{1}{t^2}[g(Y, Z)X - g(X, Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi]. \tag{8.1}$$

In view of (8.1), we have

$$R(X, Y)\xi = 0.$$

Moreover, in this case

$$\left(\frac{2n(2n + 1)f_1 + 6nf_2 - 4nf_3}{2n + 1}\right)\left(\frac{a}{2n} + 2b\right) = -a(f_1 - f_3)$$

will be

$$\left(\frac{2n(2n + 1)f_1 + 6nf_2 - 4nf_3}{2n + 1}\right)\left(\frac{a}{2n} + 2b\right) = -\left(\frac{1}{t^2} - \frac{1}{t^2} = 0\right).$$

Thus, from (3.3), we get $\widetilde{V}(X, Y)\xi = 0$.
Thus, generalized Sasakian space-form is $\xi-$quasi concircularly flat if and only if

$$\left(\frac{2n(2n + 1)f_1 + 6nf_2 - 4nf_3}{2n + 1}\right)\left(\frac{a}{2n} + 2b\right) = -\left(\frac{1}{t^2} - \frac{1}{t^2} = 0\right).$$

Hence, Theorem 3.1 is verified

Example 8.2. In 2004 [1], Alegre et al. showed that warped product $\mathbb{R}f \times_f C^\infty$ is a generalized Sasakian form with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f^2},$$

where $f(t) = t, t \in \mathbb{R}$ and f' denotes the derivstive of f with respect to t. If we choose $m = 4$ and $f(t) = e^t$, then $M(f_1, f_2, f_3)$ is a 5-dimensional conformally flat generalized Sasakian-space-form with $f_1 = -1, f_2 = 0$ and $f_3 = 0$. Hence, the generalized Sasakian-space-form is $\phi-$-quasi concircularly flat. Thus, verify Theorem 4.1 .

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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