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Statistical Analysis of Two-Parameter Bathtub-Shaped Lifetime Distribution under Progressive Censoring with Binomial Removals

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ABSTRACT

The estimation of unknown parameters of two-parameter bathtub-shaped lifetime distribution based on Type-II progressive censoring with binomial removals is studied. Maximum likelihood estimators are evaluated and using Fisher information matrix, asymptotic confidence intervals are provided. By applying Markov Chain Monte Carlo techniques, Bayes estimators, and corresponding highest posterior density confidence intervals of parameters are obtained. The expected time required to complete the life test under this censoring scheme is investigated. Monte Carlo simulations are performed to compare the performances of the different methods, and a real data set is analyzed for illustrative purposes.

Keywords: *Two-parameter bathtub-shaped lifetime distribution, Type-II progressive censoring, Maximum likelihood estimator, Bayes estimator, Binomial removal, Expected experiment time.*

1. INTRODUCTION

Among the different censoring schemes, the progressive Type-II censoring scheme has most widely been used particularly in reliability analysis and survival analysis. Progressive censoring is useful in both industrial life testing applications and clinical settings and it allows the removal of surviving experimental units before the termination of the test. The progressively Type-II censored life test is described as follows. Under this scheme, n units are placed on test at time zero, and m failure are going to be observed. When the first failure is observed, r_1 of surviving units are randomly selected, removed, and so on. This experiment terminates at the time when the m th failures is observed and remaining $r_m = n - m - r_1 - r_2 - \ldots - r_{m-1}$ surviving units are all removed. The statistical inference on the parameters of failure time distribution under progressive Type-II censoring has been studied by several authors [1, 2, 3, 4]. Note that, in this scheme, $r_1, r_2, \ldots, r_m$ are all prefixed. However, in some practical situations, these numbers may occur at random.

There have been several references about the statistical inference on lifetime distributions under progressive censoring with random removals. Recently, [5] studied Weibull distribution under progressive Type-II censoring for competing risk data with binomial removals. [6] studied generalized exponential distribution under progressive censoring with binomial removals. They also obtained the expected termination point under this censoring scheme. [7] studied Fréchet distribution under progressive Type-II censoring with random removals. [8] studied Bayesian estimation based on Rayleigh progressive Type-II censored data with binomial

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removals. [9] studied statistical inference for the Rayleigh distribution under progressively Type-II censoring with binomial removal. [10] studied the statistical inference for generalized Pareto distribution based on progressive Type-II censored data with random removals. [11] studied the inferences using Type-II progressively censored data with binomial removals.

In the area of lifetime analysis, many parametric probability distributions have been introduced to analyze sets of real data with bathtub-shaped failure rates. The bathtub-shape hazard function provides an appropriate conceptual model for some electronic and mechanical products as well as the lifetime of humans. The previous work in detail on parametric probability distributions with bathtub-shaped failure rate function can be referred to many different authors' papers. [12, 13, 14] considered statistical methods of parameter estimation for a two-parameter bathtub-shaped lifetime distribution. [15, 16] studied parameter estimation of this distribution based on hybrid censored scheme and progressive type-II censored scheme, respectively. Also, [17] discussed Bayesian estimation based on progressive Type-II censoring from two-parameter bathtub-shaped lifetime model. [18] considered stress-strength reliability of a two-parameter bathtub-shaped lifetime distribution based on progressively censored samples.

In this paper, we study the two-parameter lifetime distribution with the bathtub shape or increasing hazard function which is proposed by [19]. Its probability density function is given by

$$f(x) = \lambda\theta x^{\theta-1} e^{\lambda(1-e^{x^{\theta}})+x^{\theta}}, \qquad x > 0, \quad (1)$$

and the corresponding cumulative distribution function is $F(x) = 1 - e^{\lambda(1-e^{x^{\theta}})}$, where $x > 0$ and $\lambda, \theta > 0$ are the parameters.

The layout of this paper is as follows: In Section 2, we discuss the maximum likelihood estimation of the unknown parameters. It is observed that the MLE can be obtained by using an iterative procedure. The asymptotic confidence interval are presented in Section 3. Bayes estimate and the associated credible interval are discussed in Section 4. The expected time required to complete the life test under this censoring scheme is investigated in Section 5. Simulation results and data analysis are presented in Sections 6. Finally, we conclude the paper in Section 7.

2. THE STUDY

Let $\mathbf{X} = (X_{1:m,:n}, \ldots, X_{m:m:n})$ be a progressively Type-II censored sample, where $X_1 < \ldots < X_m$ be the ordered failure times out of n randomly selected times and m is pre-determined before the test. At the i th failure, R_i items are removed from the test. With pre-determined number of removals $\mathbf{R} = (R_1 = r_1, \ldots, R_m = r_m)$, the conditional likelihood function can be defined as the following form:

$$L(\mathbf{x}; \lambda, \theta \mid \mathbf{R} = \mathbf{r}) = c\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{r_i} = c(\lambda\theta)^m (\prod_{i=1}^{m} x_i^{\theta-1}) e^{\sum_{i=1}^{m} \lambda(r_i+1)(1-e^{x_i^{\theta}})+x_i^{\theta}}, \quad (2)$$

where $c = n(n-r_1-1)\cdots(n-r_1-\cdots-r_{m-1}-m+1)$. Equation (2) is derived conditional on r_i. Each r_i can be of any integer value between 0 and $n-m-\sum_{j=1}^{i-1} r_j$. It is different from progressive censoring with fixed removal that r_i is a random number and is assumed to follow a binomial distribution with parameter p. It means that each unit leaves with equal probability p and the probability of r_i units leaving after the i th failure occurs is

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1} (1-p)^{n-m-r_1}, \quad 0 \le r_1 \le n-m \quad (3)$$

and

$$P(R_i = r_i \mid R_{i-1} = r_{i-1}, \ldots, R_1 = r_1) = \binom{n-m-\sum_{j=1}^{i-1} r_j}{r_i} p^{r_i} (1-p)^{n-m-\sum_{j=1}^{i} r_j}, \quad (4)$$

where $0 \le r_i \le n-m-\sum_{k=1}^{i-1} r_k$, $i = 2,3,\ldots,m-1$. Also supposing further that R_i is independent of X_i for all i. Therefore, the joint likelihood function of $\mathbf{X} = (X_1,\ldots,X_m)$ and $\mathbf{R} = (R_1,\ldots,R_m)$ can be expressed as

$$L(\mathbf{x},\mathbf{r};\lambda,\theta,p) = L(\mathbf{x};\lambda,\theta \mid \mathbf{R}=\mathbf{r})P(\mathbf{R}=\mathbf{r}), \quad (5)$$

where

$$P(\mathbf{R}=\mathbf{r}) = P(R_{m-1}=r_{m-1} \mid R_{m-2}=r_{m-2},\ldots,R_1=r_1)\ldots P(R_2=r_2 \mid R_1=r_1)P(R_1=r_1). \quad (6)$$

Substituting (3) and (4) into (6), we get

$$P(\mathbf{R}=\mathbf{r}) = Bp^D(1-p)^E, \quad (7)$$

where $B = \dfrac{(n-m)!}{\prod_{i=1}^{m-1} r_i!(n-m-\sum_{i=1}^{m-1} r_i)!}$, $D = \sum_{i=1}^{m-1} r_i$ and $E = (m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i$. [20] is originally who got Equation (7) and some authors used this method (e. g. [10, 11]). Now substituting (2) and (7) into (5), we can write the full likelihood function as

$$L(\mathbf{x},\mathbf{r};\lambda,\theta,p) = Bc(\lambda\theta)^m\left(\prod_{i=1}^{m} x_i^{\theta-1}\right)e^{\sum_{i=1}^{m}\lambda(r_i+1)(1-e^{x_i^\theta})+x_i^\theta} p^D(1-p)^E. \quad (8)$$

The MLEs of p, λ and θ, say $\hat{p}$, $\hat{\lambda}$ and $\hat{\theta}$, respectively, can be obtained as the solution of

$$\frac{\partial \log L}{\partial p} = \frac{D}{p} - \frac{E}{1-p} = 0, \quad (9)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{m}{\lambda} + \sum_{i=1}^{m}(r_i+1)(1-e^{x_i^\theta}) = 0, \quad (10)$$

$$\frac{\partial \log L}{\partial \theta} = \frac{m}{\theta} + \sum_{i=1}^{m}(x_i^\theta+1)\log x_i - \lambda\sum_{i=1}^{m}(r_i+1)e^{x_i^\theta} x_i^\theta \log x_i = 0.$$

From (9) and (10), respectively, we obtain

$$\hat{p} = \frac{D}{D+E},$$

$$\hat{\lambda} = \hat{\lambda}(\theta) = \frac{-m}{\sum_{i=1}^{m}(r_i+1)(1-e^{x_i^\theta})},$$

and $\hat{\theta}$ can be found as the solution of the non-linear equation $k(\theta) = \theta$, where

$$k(\theta) = \frac{-m}{\sum_{i=1}^{m}(1+x_i^\theta)\log x_i - \hat{\lambda}\sum_{i=1}^{m}(r_i+1)(e^{x_i^\theta} x_i^\theta \log x_i)}.$$

Since, $\hat{\theta}$ is a fixed point solution of the above non-linear equation, therefore, it can be resulted using an iterative scheme as $k(\theta_{(j)}) = \theta_{(j+1)}$, where $\theta_{(j)}$ is the j th iterate of $\hat{\theta}$. The iteration procedure should be stopped when $|\theta_{(j)} - \theta_{(j+1)}|$ becomes sufficiently small. Once we obtain $\hat{\theta}$, then $\hat{\lambda}$ can be resulted.

3. CONFIDENCE INTERVAL

The asymptotic variances and covariances of the MLEs, $\hat{\lambda}$, $\hat{\theta}$ and $\hat{p}$, are given by the entries of the inverse of the Fisher information matrix $J_{ij} = E\{-\partial^2 \ell(\mathbf{\Theta})/\partial\theta_i\partial\theta_j\}$, where $i, j = 1,2,3$ and $\mathbf{\Theta} = (\theta_1, \theta_2, \theta_3) = (\lambda, \theta, p)$. Unfortunately, the exact closed forms for the above expectations are difficult to obtain. Therefore, the observed Fisher information matrix $I_{ij} = \{-\partial^2 \ell(\mathbf{\Theta})/\partial\theta_i\partial\theta_j\}_{\mathbf{\Theta}=\hat{\mathbf{\Theta}}}$, which is obtained by dropping the expectation operator E, will be used. The observed Fisher information matrix has second partial derivatives of $\log$-likelihood function as the entries, which can be obtained as follows:

$$I_{11} = -\frac{\partial^2 \log L}{\partial\lambda^2} = \frac{m}{\lambda^2},$$

$$I_{12} = -\frac{\partial^2 \log L}{\partial\lambda\theta} = \sum_{i=1}^{m}(r_i+1)e^{x_i^\theta} x_i^\theta \log x_i = I_{21},$$

$$I_{13} = -\frac{\partial^2 \log L}{\partial\lambda\partial p} = 0 = I_{31},$$

$$I_{22} = -\frac{\partial^2 \log L}{\partial\theta^2} = \frac{m}{\theta^2} - \sum_{i=1}^{m} x_i^\theta (\log^2 x_i) + \lambda\sum_{i=1}^{m}(r_i+1)(\log^2 x_i) x_i^\theta e^{x_i^\theta}(x_i^\theta+1),$$

$$I_{23} = -\frac{\partial^2 \log L}{\partial\theta\partial p} = 0 = I_{32},$$

$$I_{33} = -\frac{\partial^2 \log L}{\partial p^2} = \frac{D}{p^2} + \frac{E}{(1-p)^2}.$$

It is known that the asymptotic distribution of the MLE μ, λ and p is

$$[(\hat{\lambda}-\lambda),(\hat{\theta}-\theta),(\hat{p}-p)] \xrightarrow{D} N_3(0, I^{-1}(\lambda,\theta,p)),$$

where $I^{-1}(\lambda,\theta,p)$, the inverse of the observed Fisher information matrix of the unknown parameters, can be written as

$$I^{-1}(\lambda,\theta,p) = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}^{-1} \Bigg|_{(\lambda,\theta,p)=(\hat{\lambda},\hat{\theta},\hat{p})} = \begin{pmatrix} \sigma_{\hat{\lambda}}^2 & \sigma_{\hat{\lambda},\hat{\theta}} & 0 \\ \sigma_{\hat{\theta},\hat{\lambda}} & \sigma_{\hat{\theta}}^2 & 0 \\ 0 & 0 & \sigma_{\hat{p}}^2 \end{pmatrix}.$$

Using the above matrix, one can derive the approximate $100(1-\gamma)\%$ confidence intervals of the parameters λ, θ and p in the following forms $\hat{\lambda} \pm z_{\frac{\gamma}{2}}\sqrt{\sigma_{\hat{\lambda}}^2}$, $\hat{\theta} \pm z_{\frac{\gamma}{2}}\sqrt{\sigma_{\hat{\theta}}^2}$ and $\hat{p} \pm z_{\frac{\gamma}{2}}\sqrt{\sigma_{\hat{p}}^2}$, where $z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$ th percentile of the standard normal distribution.

4. BAYES ESTIMATION

In this section, we develop the Bayesian inference of the unknown parameters based on progressively Type-II censored data with binomial removals. We mainly discuss the Bayes estimates and the associated credible intervals of the unknown parameters. In our Bayesian analysis, we have assumed only squared error loss function. Note that if three parameters are unknown, the joint conjugate priors do not exist. In such cases, there are several ways to choose the priors. One way is to consider the piecewise independent priors. In this article, we consider the following priors on λ, θ and p which are fairly general:

$$\pi_1(\lambda) \propto \lambda^{a_1-1} e^{-\lambda a_1}, \quad \lambda > 0, a_1, b_1 > 0,$$

$$\pi_2(\theta) \propto \theta^{a_2-1} e^{-\theta b_2}, \quad \theta > 0, a_2, b_2 > 0,$$

$$\pi_3(p) \propto p^{a_3-1}(1-p)^{b_3-1}, \quad 0 < p < 1, a_3, b_3 > 0.$$

Moreover, they are assumed to be independent. Based on the observed sample, the joint posterior density function of λ, θ and p is

$$\pi(\lambda,\theta,p \mid \mathbf{x},\mathbf{r}) = \frac{L(\mathbf{x},\mathbf{r};\lambda,\theta,p)\pi_1(\lambda)\pi_2(\theta)\pi_3(p)}{\int_0^\infty\int_0^\infty\int_0^1 L(\mathbf{x},\mathbf{r};\lambda,\theta,p)\pi_1(\lambda)\pi_2(\theta)\pi_3(p)\,dp\,d\theta\,d\lambda}. \quad (11)$$

From (11), it is obvious that the Bayes estimate will not be analytically obtained. Consequently, we adopt the Gibbs sampling techniques to compute the Bayes estimates and the corresponding credible intervals of the unknown parameters. The conditional posterior pdfs of λ, θ and p are, respectively, as follows:

$$\pi_1(\lambda \mid \theta,\mathbf{x},\mathbf{r}) \propto \lambda^{m+a_1-1} e^{-\lambda\left[b_1+\sum_{i=1}^{m}(r_i+1)(e^{x_i^\theta}-1)\right]},$$

$$\pi_2(\theta \mid \lambda,\mathbf{x},\mathbf{r}) \propto \theta^{m+a_2-1}\left(\prod_{i=1}^{m} x_i^{\theta-1}\right) e^{-b_2\theta+\sum_{i=1}^{\theta} x_i^\theta-\lambda\sum_{i=1}^{m}(r_i+1)(e^{x_i^\theta}-1)},$$

$$\pi_3(p \mid \mathbf{x},\mathbf{r}) \quad \propto p^{D+a_3-1}(1-p)^{E+b_3-1}.$$

It is clear that, direct generation of a pseudo-random number from the posterior pdf of θ, is not easy, instead, we use the Metropolis-Hastings method. Therefore, the algorithm of Gibbs sampling is as follows:

- 1. Take some initial value of λ, θ, such as λ_0, θ_0.
- 2. Set $t = 1$.
- 3. Generate λ_t from $\pi_1(\lambda \mid \theta_{t-1},\mathbf{x},\mathbf{r})$.
- 4. Generate θ_t from $\pi_2(\theta \mid \lambda_{t-1},\mathbf{x},\mathbf{r})$.
- 5. Set $t = t+1$.
- 6. Repeat Steps 2-5, T times.
- 7. Obtain Bayes estimators of λ and θ: $\hat{\lambda}_B = \frac{1}{T}\sum_{t=1}^{T}\lambda_t$ and $\hat{\theta}_B = \frac{1}{T}\sum_{t=1}^{T}\theta_t$.
- Obtain the HPD confidence interval of λ: Order $\lambda_1,\ldots,\lambda_T$ as $\lambda_{(1)},\ldots,\lambda_{(T)}$ and construct all the $100(1-\gamma)\%$ confidence intervals of λ, as $(\lambda_{(1)},\lambda_{([T(1-\gamma)])}),\ldots,(\lambda_{([T\gamma])},\lambda_{(T)})$, where $[M]$ symbolizes the largest integer less than

or equal to M . The HPD confidence interval of λ is the shortest length interval. Similarly, we can construct a $100(1-\gamma)\%$ HPD confidence interval of θ.

Now under squared error loss function, the Bayes estimator of p is given as

$$\hat{p}_B = \int_0^1 p\pi_3(p\,|\,\mathbf{x},\mathbf{r})dp = \frac{D+a_3}{D+E+a_3+b_3}.$$

Also, to obtained the HPD credible interval for p , since $\pi_3(p\,|\,\mathbf{x},\mathbf{r})$ is unimodal, the corresponding $100(1-\gamma)\%$ HPD credible interval (H_L^p, H_U^p) can be obtained from the simultaneous solution of the following equations

$$P[H_L^p < p < H_U^p] = \int_{H_L^p}^{H_U^p} \pi_3(p\,|\,\mathbf{x},\mathbf{r})dp = 1-\gamma \pi_3(H_L^p\,|\,\mathbf{x},\mathbf{r}) = \pi_3(H_U^p\,|\,\mathbf{x},\mathbf{r}).$$

5. EXPECTED EXPRIMENT TIMES

In practical situations, an experimenter may be interested to know whether the test can be completed within a specified time. This information is important for an experimenter to choose an appropriate sampling plan because the time required to complete a test is directly related to cost. Under a progressive Type-II censoring scheme with binomial removals, the expected termination time for the experiment is given by the expectation of the m th order statistic X_m . According to [2], the conditional expectation of X_m , for a fixed set of $\mathbf{R}$ is

$$E[X_m\,|\,\mathbf{R}=\mathbf{r}] = C(\mathbf{r})\sum_{l_1=0}^{r_1}\cdots\sum_{l_m=0}^{r_m}(-1)^A \frac{\binom{r_1}{l_1}\cdots\binom{r_m}{l_m}}{\prod_{i=1}^{m-1}h(l_i)} \int_0^\infty x f(x) F^{h(l_m)-1}(x)dx, \qquad (12)$$

where $A = \sum_{i=1}^m l_i$, $C(\mathbf{r}) = n(n-r_1-1)\ldots(n-\sum_{i=1}^{m-1}(r_i+1))$, $h(l_i) = l_1+\ldots+l_i+i$ and i is the number of live units removed from experimentation (or the number of failure units). Let

$$S = \int_0^\infty x f(x) F^{h(l_m)-1}(x)dx = \int_0^\infty \lambda\theta x^\theta e^{\lambda(1-e^{x^\theta})+x^\theta}\left(1-e^{\lambda(1-e^{x^\theta})}\right)^{h(l_m)-1} dx$$

$$= \lambda\theta \sum_{k=0}^{h(l_m)-1}(-1)^k \binom{h(l_m)-1}{k} \int_0^\infty x^\theta e^{\lambda(k+1)(1-e^{x^\theta})+x^\theta} dx$$

$$= \sum_{k=0}^{h(l_m)-1}(-1)^k \binom{h(l_m)-1}{k} A(k,\lambda,\theta),$$

where $A(k,\lambda,\theta) = \lambda\theta\int_0^\infty x^\theta e^{\lambda(k+1)(1-e^{x^\theta})+x^\theta} dx$.

Substituting this value of S into (12), the expected experimentation time is given by

$$E[X_m\,|\,\mathbf{R}=\mathbf{r}] = C(\mathbf{r})\sum_{l_1=0}^{r_1}\cdots\sum_{l_m=0}^{r_m}(-1)^A \frac{\binom{r_1}{l_1}\cdots\binom{r_m}{l_m}}{\prod_{i=1}^{m-1}h(l_i)} \sum_{k=0}^{h(l_m)-1}(-1)^k \binom{h(l_m)-1}{k} A(k,\lambda,\theta).$$

Furthermore, the expected experiment time of a Type-II censoring test without removal can be found by setting the R terms to 0 in (12). It is given by

$$E[X_m^*] = m\binom{n}{m}\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}A(k,\lambda,\theta). \tag{13}$$

Similarly, the expected experiment time of a complete sampling case with n test units can also be obtained by setting $m = n$ in (13). It is given by

$$E[X_n^{**}] = n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}A(k,\lambda,\theta). \tag{14}$$

For the progressive Type-II censoring with random removals, the expected termination point is given by $E[X_m] = E_{\mathbf{R}}[E(X_m \mid \mathbf{R})]$. For binomial removals, we have

$$E[X_m] = \sum_{r_1=0}^{g(r_1)}\cdots\sum_{r_{m-1}=0}^{g(r_{m-1})} P(\mathbf{R}=\mathbf{r})E[X_m \mid \mathbf{R}=\mathbf{r}], \tag{15}$$

where $g(r_1) = n-m$, $g(r_i) = n-m-r_1-\ldots-r_{i-1}$; $i = 2,\ldots,m-1$, and $P(\mathbf{R}=\mathbf{r})$ is given by (7). The ratio of the expected experiment time (REET) δ_{REET} is computed between progressive Type-II censoring with binomial removals and the complete sampling case using the formula

$$\delta_{REET} = \frac{E[X_m]}{E[X_n^{**}]}.$$

It can be noted that the δ_{REET} provides important information in determining the shortest experiment time significantly if a much larger sample of n test units is used. When δ_{REET} is closer to 1, the termination point will be closer to the complete sample.

The expected experiment times of progressive censoring sample with binomial removals, Type-II censored sample, and complete sample are derived. We will calculate them numerically for various values of n, m and p. Furthermore, we gain some idea about the roles of n, m and p on the experiment according to compare these expected experiment times. A numerical study is conducted and the results are presented in Table 0. From this table, the following general observations can be made. For all the parameters,

- for fixed m and p, as n increases, the δ_{REET} decreases,
- for fixed n and p, as m increases, the δ_{REET} increases,
- for fixed n and m, as p increases, the δ_{REET} increases.

Moreover, for large values of p and m, we see that the δ_{REET} approaches 1 quite sharply.

Table 1: The δ_{REET} for $(\lambda,\theta) = (1,2)$ with varying values of p.

n	m	p								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
6	3	0.2532	0.3924	0.5029	0.6190	0.7051	0.7665	0.8011	0.8195	0.8474
	4	0.5746	0.7075	0.8127	0.8195	0.8627	0.8892	0.9035	0.9204	0.9250
	5	0.8861	0.9213	0.9329	0.9417	0.9517	0.9616	0.9669	0.9677	0.9692
10	5	0.2272	0.4937	0.5674	0.7153	0.8672	0.8726	0.8795	0.8928	0.8985
	8	0.7148	0.9404	0.9466	0.9664	0.9681	0.9705	0.9716	0.9717	0.9727
12	8	0.4856	0.8611	0.9026	0.9457	0.9471	0.9522	0.9533	0.9535	0.9560
	9	0.9343	0.9380	0.9436	0.9617	0.9630	0.9643	0.9663	0.9670	0.9679

	10	0.9632	0.9673	0.9730	0.9775	0.9780	0.9789	0.9790	0.9801	0.9850
14	9	0.7265	0.9000	0.9254	0.9490	0.9493	0.9500	0.9512	0.9527	0.9531
	10	0.8806	0.9082	0.9442	0.9575	0.9605	0.9610	0.9614	0.9638	0.9643
	12	0.9633	0.9732	0.9750	0.9805	0.9830	0.9835			

6. DATA ANALYSIS AND COMPARSION STUDY

In this section, we present some results based on Monte Carlo simulations and real data to compare the performance of the different methods described in the preceding sections.

6.1. Numerical Experiments and Discussions

In this section, we carry out a simulation study to consider the performance of MLEs and Bayes estimators using progressively Type-II censoring under binomial removal scheme. We have studied different sample sizes: $n=20$, 30, 40, different effective sample sizes: $m=10$, 15, 20 with $a_1=b_1=a_2=b_2=a_3=b_3=2$. Without loss of generality, in all cases $\lambda=1.5$, $\theta=2$ and $p=0.5$ are taken. We have generated samples for a given n and m along with a sampling scheme by using binomial removal technique. We estimate the unknown parameters using the MLE and Bayes method. The performance of the estimates are compared based on the average bias (AB) and the corresponding mean squared error (MSE) of the estimates under 1000 replications. In addition, we also computed the 95% confidence intervals and the HPD credible intervals based on the same 1000 replications. Simulation study results are summarized in Tables 2 and 3. Tables show that as sample size increases, the MSEs decrease. Also, for fixed n as m increases, the Biases and MSEs decrease. The performance of the Bayes estimates are better than the MLEs, in terms of both the biases and the MSEs. It should be mentioned that the Bayes estimators are more computationally expensive than those followed by MLEs. Both 95% asymptotic confidence intervals and HPD credible intervals show good coverage of the true value of the parameters being considered. The HPD credible intervals lengths are smaller than the asymptotic confidence intervals lengths.

Table 2: Average bias and mean squared error of the MLE and Bayes estimates of the parameters.

n	m	MLE						Bayes					
		p		θ		λ		p		θ		λ	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	10	0.033	0.013	0.112	0.052	0.111	0.079	0.007	0.008	0.007	0.015	0.021	0.018
30	10	0.029	0.012	0.095	0.034	0.071	0.078	0.007	0.006	0.005	0.015	0.020	0.014
30	15	0.014	0.009	0.070	0.040	0.081	0.077	0.006	0.004	0.001	0.014	0.010	0.013
30	20	0.008	0.007	0.054	0.048	0.080	0.062	0.004	0.003	0.001	0.010	0.008	0.012
40	10	0.010	0.006	0.088	0.030	0.016	0.073	0.007	0.005	0.004	0.014	0.006	0.014
40	15	0.006	0.005	0.067	0.031	0.027	0.070	0.006	0.005	0.005	0.012	0.020	0.013
40	20	0.003	0.005	0.048	0.039	0.024	0.057	0.001	0.004	0.001	0.013	0.002	0.012

Table 3: Average confidence/credible length and coverage percentage for estimates of the parameters.

n	m	MLE						Bayes					
		p		θ		λ		p		θ		λ	
		CL	CP	CL	CP	CL	CP	CL	CP	CL	CP	CL	CP
20	10	0.437	0.965	2.632	1.000	2.820	1.000	0.314	0.968	0.365	0.978	0.365	0.972
30	10	0.435	0.960	2.370	0.995	2.615	0.990	0.289	0.967	0.363	0.965	0.361	0.968
30	15	0.354	0.955	2.068	0.980	2.357	0.987	0.274	0.958	0.364	0.977	0.362	0.970
30	20	0.307	0.935	1.501	0.950	1.696	0.940	0.247	0.956	0.361	0.963	0.354	0.957
40	10	0.308	0.935	2.143	0.990	2.300	0.975	0.247	0.956	0.351	0.953	0.360	0.967
40	15	0.275	0.925	2.044	0.970	2.220	0.960	0.224	0.954	0.356	0.954	0.359	0.962

40	20	0.251	0.915	1.432	0.945	1.611	0.935	0.213	0.951	0.306	0.952	0.350	0.953

6.2. Data Analysis

Here we demonstrate one dataset for illustrative purposes. This data represent the number of revolutions to failure for each of 23 ball bearing in a life test. The data were first used by [21] and is given below:

0.1788, 0.2892, 0.33, 0.4152, 0.4212, 0.4560, 0.4848, 0.5184, 0.5196, 0.5412, 0.5556, 0.6780, 0.6864, 0.6864, 0.6888, 0.8412, 0.9312, 0.9864, 1.0512, 1.0584, 1.2792, 1.2804, 1.7340.

This data has been analyzed previously by [22] and [23]. For illustrative purposes, we have been generated different progressive Type-II censoring using binomial removal scheme from the above data set. We considered this dataset based on $m = 15$ and several values of p to generate several removal schemes. The Kolmogorov-Smirnov (KS) distances between the empirical distribution functions and the fitted distribution functions and corresponding p-values for different sampling schemes are presented in Table 4. The results indicate that the two-parameter bathtub-shaped distribution have a reasonable fitting to all different sampling schemes.

To compare the performances of the different methods, we obtain the MLE and Bayes estimations of the unknown parameters and corresponding confidence or credible interval lengths. For Bayes estimation, we have used the informative priors $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 2$. Table 4 represents the results of data analysis. The HPDs provide the shorter length than the asymptotic confidence lengths for different censoring schemes.

Table 4: Kolmogorov-Smirnov (KS) test statistic and its corresponding p-values for different sampling schemes.

p	Number	Scheme	KS Statistic	p-value
0.2	(1)	(3, 3, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)	0.2697	0.1873
	$(1)'$	(1, 0, 1, 0, 2, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0)	0.2687	0.1905
	$(1)''$	(1, 0, 2, 2, 2, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)	0.2558	0.2363
0.5	(2)	(4, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	0.2256	0.3734
	$(2)'$	(6, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	0.2237	0.3836
	$(2)''$	(1, 4, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	0.2402	0.3015
0.8	(3)	(7, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	0.2209	0.3988
	$(3)'$	(8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	0.2180	0.4146
	$(3)''$	(7, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	0.2219	0.3931

Table 5: Data analysis results of the Ball bearing data.

Scheme	MLE						Bayes					
	θ		λ		p		θ		λ		p	
	$\hat{\theta}$	CL	$\hat{\lambda}$	CL	$\hat{p}$	CL	$\hat{\theta}_B$	CL	$\hat{\lambda}_B$	CL	$\hat{p}_B$	CL
(1)	1.224	0.863	0.704	0.878	0.265	0.303	1.237	0.372	0.725	0.567	0.233	0.147
(2)	1.231	0.884	0.751	0.865	0.572	0.519	1.265	0.341	0.741	0.530	0.556	0.168
(3)	1.200	0.884	0.744	0.876	0.889	0.411	1.241	0.338	0.740	0.671	0.769	0.149

7. CONCLUSION

This paper considered some results of two-parameter bathtub-shaped distribution under progressive Type-II censoring with binomial removals. We investigate the MLEs, interval estimations and Bayes estimations of the unknown parameters. Based on the simulation study, the parameter estimation using Bayesian technique performs better than the MLE approach. We have also computed the experimentation time under Type-II progressive censoring scheme. Using a numerical experiment, we confirm that the role of removal probability is quite significant with respect to the length of the experimentation time.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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