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Hyperbolic Cosine - F Family of Distributions with an Application to Exponential Distribution

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ABSTRACT

A new class of distributions called the hyperbolic cosine – F (HCF) distribution is introduced and its properties are explored. This new class of distributions is obtained by compounding a baseline F distribution with the hyperbolic cosine function. This technique resulted in adding an extra parameter to a family of distributions for more flexibility. A special case with two parameters has been considered in details namely; hyperbolic cosine exponential (HCE) distribution. Various properties of the proposed distribution including explicit expressions for the moments, quantiles, moment generating function, failure rate function, mean residual lifetime, order statistics, stress-strength parameter and expression of the Shannon entropy are derived. Estimations of parameters in HCE distribution for two data sets obtained by eight estimation procedures: maximum likelihood, Bayesian, maximum product of spacings, parametric bootstrap, non-parametric bootstrap, percentile, least-squares and weighted least-squares. Finally, these data sets have been analyzed for illustrative purposes and it is observed that in both cases the proposed model fits better than Weibull, gamma and generalized exponential distributions.

Keywords: *Hyperbolic cosine function, Exponential distribution, Mean residual lifetime, Maximum product of spacings, Maximum likelihood estimation, Bootstrap.*

1. INTRODUCTION

Statistical distributions are commonly applied to describe real world phenomena. Due to the usefulness of statistical distributions, their theory is widely studied and new distributions are developed. Numerous classical distributions have been extensively used over the past decades for modeling data in several areas such as engineering, actuarial, environmental and medical sciences, biological studies, demography, economics, finance and insurance. However, in many applied areas such as lifetime analysis, finance and insurance, there is a clear need for extended forms of these distributions. For that reason, several methods for generating new families of distributions have been studied. The well-known generators are the following: Azzalini's skew family by Azzalini (1985), Marshal-Olkin generated family (MO-G) by Marshall and Olkin (1997), exponentiated family

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(EF) of distributions by Gupta et al. (1998), beta-G by Eugene et al. (2002) and Jones (2004), Kumaraswamy-G (Kw-G) by Cordeiro and de Castro (2011), McDonald-G (Mc-G) by Alexander et al. (2012), gamma-G (type 1) by Zografos and Balakrishnan (2009), gamma-G (type 2) by Ristić and Balakrishnan (2012), gamma-G (type 3) by Torabi and Hedesh (2012), log-gamma-G by Amini et al. (2012), logistic-G by Tahir et al. (2016), exponentiated generalized-G by Cordeiro et al. (2013), geometric exponential-Poisson family Nadarajah et al. (2013a), truncated-exponential skew-symmetric family by Nadarajah et al. (2014), logistic-generated (Lo-G) family by Torabi and Montazari (2014), Transformed-Transformer (T-X) by Alzaatreh et al. (2013), exponentiated (T-X) by Alzaghal et al. (2013), Weibull-G by Bourguignon et al. (2014), Exponentiated half logistic generated family by Cordeiro et al. (2014a), Kumaraswamy Odd log-logistic-G by Alizadeh et al. (2015b), Kumaraswamy Marshall-Olkin by Alizadeh et al. (2015c), Beta Marshall-Olkin by Alizadeh et al. (2015a), Type Half-Logistic family of distributions by Cordeiro et al. (2016) and Odd generalized exponential-G by Tahir et al. (2015b), Another Generalized Transmuted Family of Distributions by Merovci et al. (2016), weighted exponential by Gupta and Kundu (2009), generalized weighted exponential by Kharazmi et al. (2015). These families of distributions have received a great deal of attention in recent years.

The aim of this paper is to propose a new family of continuous distributions, called the hyperbolic cosine – F (HCF) family, and to study some of its mathematical properties. Moreover, we provide general properties, different estimation procedures for the unknown parameters and applications of a special model of the HCF family so-called the hyperbolic cosine –exponential (HCE) distribution.

The rest of the paper is organized as follows. In Section 2, we introduce the HCF model and discuss some general properties of this family of distributions. In Section 3, we consider the HCE distribution and discuss its different properties. We discuss different estimation procedures of the unknown parameters in Section 4. The analysis of two real data sets has been presented in Section 5. Finally in Section 6, we conclude the paper.

2. HYPERBOLIC COSINE – F (HCF) FAMILY OF DISTRIBUTIONS

In this section, we introduce a new class of distributions named hyperbolic cosine –F (HCF**)**. Also, two representations and some basic properties including probability distribution, survival, hazard rate and quantile functions of this model are given here. This new class of distributions is obtained by compounding a baseline probability distribution F with the hyperbolic cosine function. Before introducing the HCF family, we recall the definition of the hyperbolic cosine function. The hyperbolic cosine has similar name to the trigonmetric functions, but it is defined in terms of the exponential function as follows.

$$cosh(x) = \frac{e^{x} + e^{-x}}{2}.$$

The function $\cosh(x)$ is even and has a Taylor series expression with only even exponents for x as follows

$$cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}. \quad (1)$$

Definition 1. Let X be a continuous random variable with cumulative distribution function (CDF) $F(x)$, then model (2) called hyperbolic cosine – F (HCF) distribution and its probability density function (PDF) is as follows

$$g(x,a) = \frac{2a\, e^{a}}{e^{2a} - 1} f(x)\, cosh(a F(x)), \quad (2)$$

where $x > 0$, $a > 0$. Clearly $g(x,a)$ reduces to $f(x)$ when $a \rightarrow 0$.

To motivate the introduction of this new family of distributions, we give two representations as follows.

Representation 1

Suppose that the failure of a device occurs due to the presence of an unknown number, $2N + 1$, of initial defects of some kind. Let $Y_1, \ldots, Y_{2N+1}$ denote the failure times of the initial defects. Let X denote the failure time of the device. Then $X = max(Y_1, \ldots, Y_{2N+1})$. Suppose N is a discrete random variable with a probability mass function as

$$P(N = n) = \begin{cases} \dfrac{2e^{a}}{e^{2a} - 1} \dfrac{a^{2n+1}}{(2n+1)!} & n = 0,1,2,\ldots \\ 0 & o.w. \end{cases}$$

where $0 < a < \infty$. Suppose also that $Y_1, \ldots, Y_{2N+1}$ is a random sample from the baseline distribution with PDF $f(x)$ and CDF $F(x)$, then

$$f_{X|N=n}(x) = (2n+1) f(x) F^{2n}(x),$$

and the marginal probability density function of X is

$$g(x,a) = \frac{2a\, e^{a}}{e^{2a}-1} f(x)\, cosh(a F(x)).$$

Representation 2

Using the series expansion (1), the HCF distribution can be stated as a mixtures of generalized $-F$ ($F^{\alpha}(x)$) distributions as follows.

$$g(x,a) = \frac{2a\, e^{a}}{e^{2a}-1} f(x)\, cosh(a F(x)) = \sum_{n=0}^{\infty} w(a,n) f_U(x),$$

where U ~generalized $-F$ as

$$f_U(x) = (2n+1) f(x) F^{2n}(x)$$

and $w(a,n) = \frac{2a\, e^{a}}{e^{2a}-1} \frac{a^{2n}}{(2n+1)!}$.

Now let us consider some main properties of the HCF distribution.The corresponding CDF associated with (2) is

$$G(x,a) = \frac{2\, e^{a}}{e^{2a}-1} sinh(a F(x)).$$

The survival reliability $\bar{G}(x)$ and the hazard rate function (HRF)

$h(x)$ for HCF distribution are in the following form

$$\bar{G}(x,a) = 1 - \frac{2\, e^{a}}{e^{2a}-1} sinh(a F(x))$$

and

$$h(x,a) = \frac{\frac{2a\, e^{a}}{e^{2a}-1} f(x)\, cosh(a F(x))}{1 - \frac{2\, e^{a}}{e^{2a}-1} sinh(a F(x))},$$

respectively.The $p-th$ quantile x_p of the HCF distribution can be obtained as

$$x_p = F^{-1}\left(\frac{1}{a} arcsinh\left(\frac{e^{2a}-1}{2e^{a}} p\right)\right), \qquad 0 \le p \le 1.$$

$\operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2+1})$, then we get Since

$$x_p = F^{-1}\left(\frac{1}{a}\left(\frac{e^{2a}-1}{2e^{a}} p + \sqrt{\left(\frac{e^{2a}-1}{2e^{a}} p\right)^2 + 1}\right)\right).$$

Hence, If the baseline F distribution is invertable then we can easily generate random samples from the HCF distribution.

3. HCE DISTRIBUTION AND ITS PROPERTIES

In this section we apply the HCF method to a specific class of distribution functions, namely to an exponential distribution and call this new distribution, two-parameter HCE distribution.

Definition 2. A random variable X has hyperbolic cosine - exponential , denoted by $HCE(a,\lambda)$, if its probability density function (PDF) is given by

$$g(x, a, \lambda) = \frac{2a\ e^{a}}{e^{2a} - 1} \lambda e^{-\lambda x} \cosh(a(1 - e^{-\lambda x})), \qquad (3)$$

where $x > 0\ ,\ a > 0\ ,\lambda > 0$. Fig. 1 shows the shapes of $HCE(a, \lambda)$ for different values of a and λ.

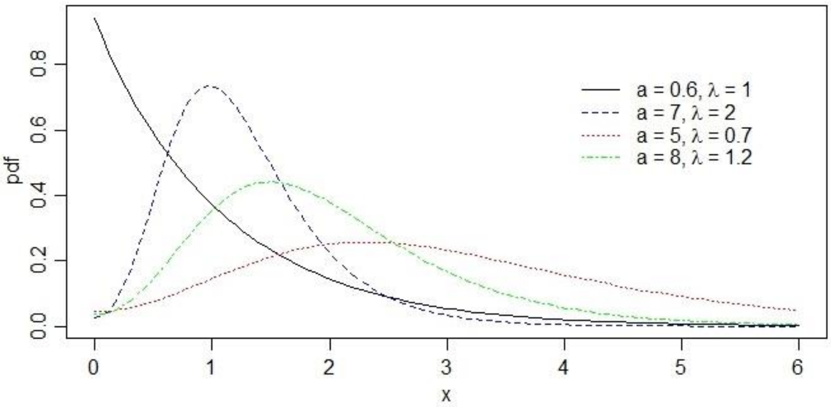

Fig. 1. Plots of the $HCE(a, \lambda)$ for different values of a and λ.

To investigate the effect of parameters a and λ on the skewness of the HCE distribution we plotted the 3D-Plots of $HCE(a, \lambda)$ for different values of a and λ in Fig. 2.

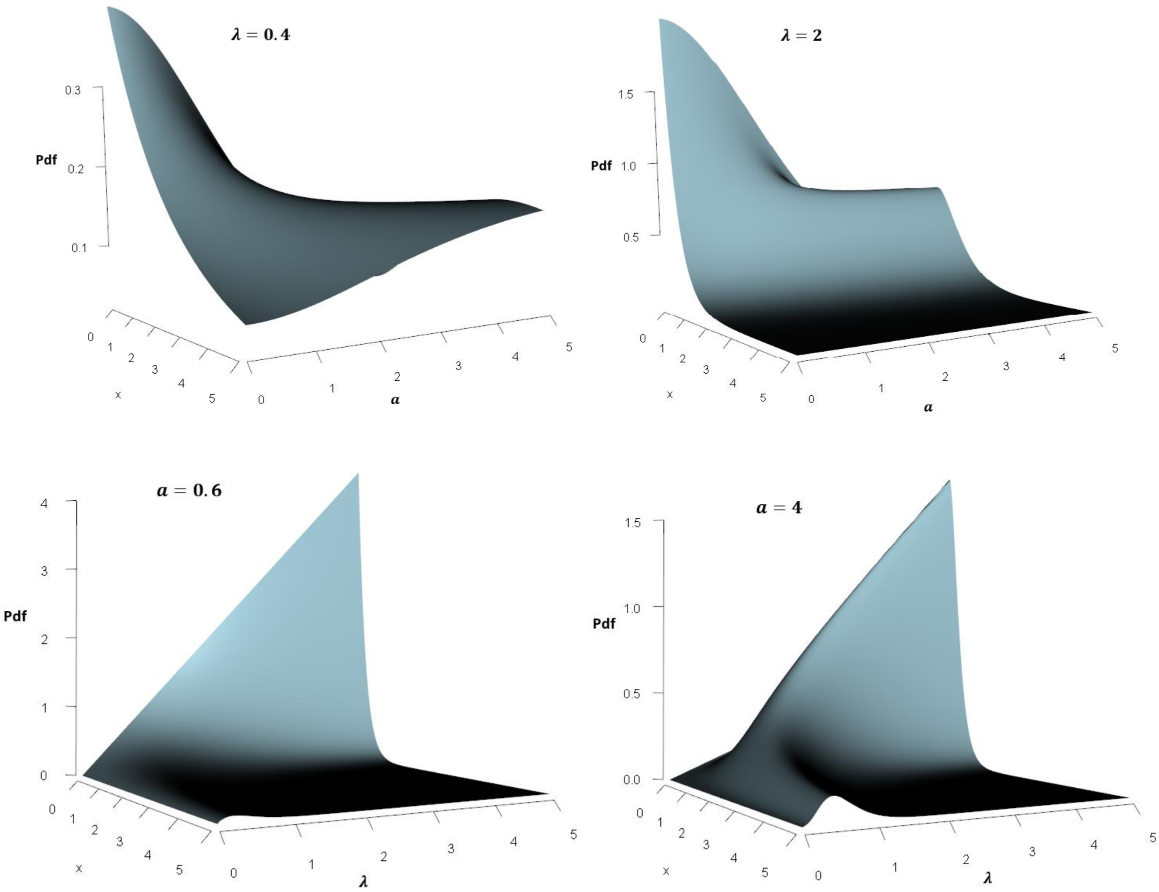

Fig. 2. 3D-Plots of the $HCE(a, \lambda)$ for different values of a and λ.

3.1. Statistical and reliability properties

In this section we study several statistical and reliability properties of the HCE distribution, such as the distribution function (CDF), survival function (SF), conditional survival function (CSF), failure rate (or hazard) function (FR), moment generating function (MGF), mean residual life (MRL) time, jth moment**,** order statistics, stress-strength parameter and Shannon entropy measure.

3.1.1 Distribution, survival, quantile, conditional reliability and failure rate functions

The CDF of (3) can be written as

$$G(x,a,\lambda) = \frac{2\,e^{a}}{e^{2a}-1} sinh(a(1-e^{-\lambda x})),$$

also, survival , quantile and conditional reliability functions are given by

$$\bar{G}(x,a,\lambda) = 1-\frac{2\,e^{a}}{e^{2a}-1} sinh(a(1-e^{-\lambda x})), \tag{4}$$

$$x_p = -\frac{1}{\lambda}\left(log\left(1-\frac{arcsinh\left(\frac{e^{2a}-1}{2e^{a}}p\right)}{a}\right)\right) =$$

$$-\frac{1}{\lambda}(log(1-\frac{log\left(\frac{e^{2a}-1}{2e^{a}}p+\sqrt{\left(\frac{e^{2a}-1}{2e^{a}}p\right)^{2}+1}\right)}{a})), \quad 0 \le p \le 1$$

and

$$\bar{G}(x,a,\lambda|t) = \frac{\bar{G}(x+t,a,\lambda)}{\bar{G}(t,a,\lambda)} = \frac{1-\frac{2\,e^{a}}{e^{2a}-1}\sinh(a(1-e^{-\lambda(x+t)}))}{1-\frac{2\,e^{a}}{e^{2a}-1}\sinh(a(1-e^{-\lambda x}))}\,; x>0\,, t>0\,,$$

respectively. Conditional survival function plays an important role in classifying life time distributions. From (3) and (4) it is easy to verify that the failure rate function is given by

$$h(x,a,\lambda) = \frac{\frac{2a\,e^{a}}{e^{2a}-1}\lambda e^{-\lambda x}\, cosh(a(1-e^{-\lambda x}))}{1-\frac{2\,e^{a}}{e^{2a}-1} sinh\,(a(1-e^{-\lambda x}))}.$$

The failure rate is a key notion in reliability and survival analysis for measuring the ageing process. Understanding the shape of the failure rate is important in reliability theory, risk analysis and other disciplines. The concepts of increasing, decreasing, bathtub shaped(first decreasing and then increasing) and upside-down bathtub shaped(first increasing and then decreasing) failure rates for univariate distributions have been found very useful in reliability theory. The classes of distributions having these ageing properties are designated as the IFR, DFR, BUT and UBT distributions, respectively. For the HCE distribution hazard rate function can be decreasing, increasing, upside-down bathtub-shaped and constant. Figs. 3(a) and 3(b) illustrate some samples of possible shapes of the hazard rate function in IFR and DFR cases for certain values of the vector (a,λ). Although we can not provide an analytic proof in upside-down bathtub-shaped case, using Glaser (1980)'s result, we have confirmed our claim by plotting $\eta'(x) = -\frac{\partial}{\partial x}[\frac{\partial}{\partial x} lng(x)]$ in Fig. 3(c). We observe that $\eta'(x)$ may be having one change of sign from negative to positive.

$$\eta(x) = \lambda - \frac{a\,\lambda\;e^{-\lambda x}\, sinh\, a\left(1-e^{-\lambda x}\right)}{cosh\, a(1-e^{-\lambda x})}$$

$$\eta'(x) = \frac{\partial\eta(x)}{\partial x} = a\,\lambda^{2}e^{-\lambda x}tanh(a\left(1-e^{-\lambda x}\right)) - \left(a\lambda\;e^{-\lambda x}\right)^{2}(1-tanh^{2}(a\left(1-e^{-\lambda x}\right)))$$

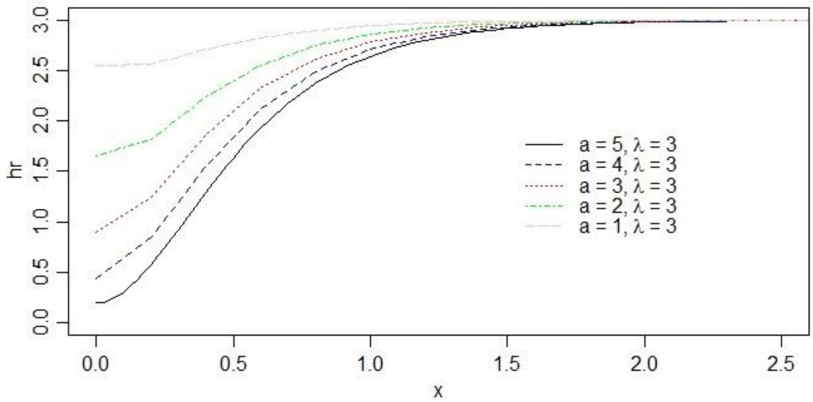

Fig. 3(a). failure rate function shapes for for selected values of the parameters.

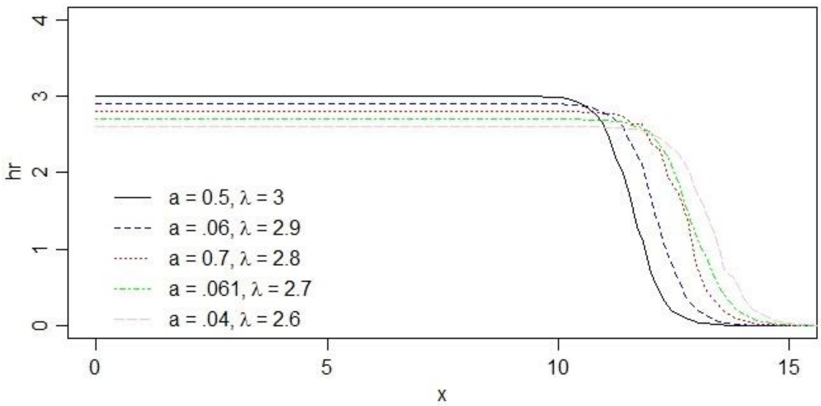

Fig. 3(b). failure rate function shapes for selected values of the parameters.

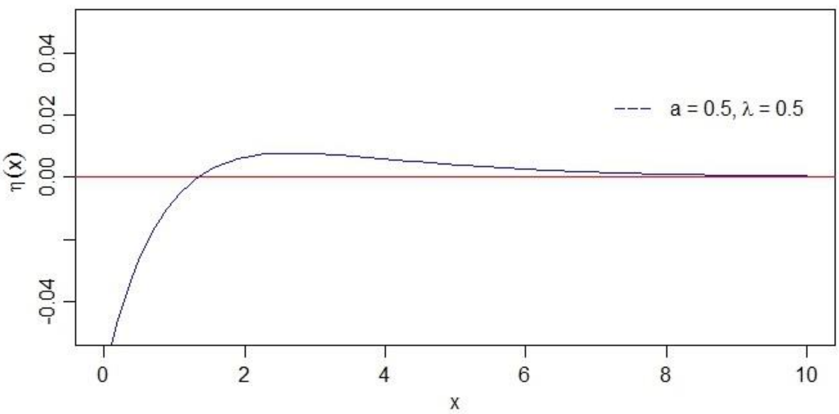

Fig. 3(c). $\eta'(x)$ for selected values of the parameters

3.1.2. Moment generating function and mean residual life time

Now let us consider different moments of the $HCE(a,\lambda)$ distribution. Some of the most important features and characteristics of a distribution can be studied through its moments, such as moment generating function, the jth moment and interested reliability properties such as mean residual life time. The moment generating function of form (3) is immediately written as

$$M_X(t) = \frac{2\,a\,e^{a}}{e^{2a}-1}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{a^{2n}\,(-1)^k\binom{2n}{k}}{(2n)!}\frac{\lambda}{\lambda k+\lambda-t}$$

The jth moment and jth central moment of the HCE distribution can be derived as

$$\mu_j = E(X^j) = \frac{2\,a\,e^{a}}{e^{2a}-1}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{a^{2n}\,(-1)^k\binom{2n}{k}}{(k+1)(2n)!}\,\frac{\Gamma(1+j)}{(\lambda(k+1))^j},$$

in particular, its mean and variance are given, by

$$E(X) = \frac{2\,a\,e^{a}}{e^{2a}-1}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{a^{2n}\,(-1)^k\binom{2n}{k}}{(2n)!}\,\frac{1}{\lambda(k+1)^2}$$

and

$$Var\,(X) = \tau_2 = E(X-\mu_1)^2.$$

One of the well-known properties of the life time distribution is mean residual life time. For the HCE distribution it can be written as

$$m(t) = E(X-t|X>t)$$

$$= \frac{2a\,e^{a}}{e^{2a}-1-2e^{a}\sinh\,a(1-e^{-\lambda t})}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{a^{2n}\,(-1)^k\binom{2n}{k}}{(2n)!}\,\frac{e^{-\lambda(k+1)t}}{(k+1)^2\,\lambda}.$$

3.1.3. Order statistics, stress-strength parameter and Shannon entropy measure

Here we provide an order statistics result. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $HCE(a,\lambda)$, and let $X_{i:n}$ denote the ith order statistic. The PDF of $X_{i:n}$ is given by

$$f_{X_{i:n}}(x) =$$

$$\frac{n!}{(i-1)!\,(n-i)!}\,\frac{2\,a\,\lambda\,e^{a}\,e^{-\lambda x}\,\cosh\,a(1-e^{-\lambda x})}{(e^{2a}-1)}\left(\frac{2\,e^{a}\sinh\,a(1-e^{-\lambda x})}{e^{2a}-1}\right)^{i-1}$$

$$\left(1-\frac{2\,e^{a}\sinh\,a(1-e^{-\lambda x})}{e^{2a}-1}\right)^{n-i}.$$

Now we obtaine the stress-strength parameter. Suppose $X_1 \sim HCE(a_1, \lambda_1)$ and

$X_2 \sim HCE(a_2, \lambda_2)$ are independently distributed, then

$$P(X_1<X_2) = \frac{2a_1e^{a_1}}{e^{2a_1}-1}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{{a_1}^{2n}\,(-1)^k\binom{2n}{k}}{(k+1)(2n)!} -$$

$$\frac{2a_1e^{a_1}}{e^{2a_1}-1}\,\frac{2a_2e^{a_2}}{e^{2a_2}-1}\sum_{n=0}^{\infty}\sum_{t=0}^{\infty}\sum_{k=0}^{2n}\frac{{a_1}^{2n}\,{a_2}^{2n}\,(-1)^t(-1)^{2k}\binom{2n}{k}^2}{(k+1)^2\,(2n)!^2}\left(\frac{\lambda_1}{\lambda_2}\right)^t.$$

The entropy of a random variable measures the variation of the uncertainty. A large value of entropy indicates the greater uncertainty in the data. Shannon entropy Shannon (1948) of $HCE(a,\lambda)$ Can be obtained as

$$H(X) = -\log\left(\frac{2a\,e^{a}}{e^{2a}-1}\right)-\log(\lambda)+\frac{2a\,e^{a}}{e^{2a}-1}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{a^{2n}\,(-1)^k\binom{2n}{k}}{(k+1)^2(2n)!}$$

$$-\log\left(\frac{e^{2a}+1}{2e^{a}}\right)+1-\frac{2\,e^{a}}{e^{2a}-1}\,\arctan\frac{e^{2a}-1}{2e^{a}}.$$

4. DIFFERENT METHODS OF ESTIMATION

In this section, we describe the eight estimation methods considered in this paper for estimating the parameters a and λ of the HCE distribution. For all methods we consider the case when both a and λ are unknown.

4.1. Maximum likelihood estimation

Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with density $f(x, \theta)$. The likelihood function based on observed values $x_1, x_2, \dots, x_n$ is given by

$$L(\theta, \underline{x}) = \prod_{i=1}^{n} f(x_i, \theta)). \qquad (5)$$

The estimator obtained by maximizing (5) is called the MLE estimator of θ. In case of the HCE distribution.The log-likelihood function of the parameter is given as

$$l(a, \lambda, \underline{x}) = log(L) = nln\left(\frac{2a\, e^{a}}{e^{2a} - 1}\right) + nln(\lambda) - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} ln\, cosh\; a\left(1 - e^{-\lambda x_i}\right),$$

so, the MLEs of a and λ, say $\hat{a}$, and $\hat{\lambda}$, respectively, can be obtained as the simultaneous

solutions of

$$\frac{dl}{da} = n\frac{2e^{3a}\,(1-a) - 2e^{a}(1+a)}{2\; a\; e^{a}\;(e^{2a} - 1)} + \sum_{i=1}^{n} \frac{(1 - e^{-\lambda x_i})\;\, sinh\; a(1 - e^{-\lambda x_i})}{cosh\; a(1 - e^{-\lambda x_i})} = 0,$$

$$\frac{dl}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{x_i\;\; a\;\, e^{-\lambda x_i}\;\, sinh\; a\left(1 - e^{-\lambda x_i}\right)}{cosh\; a(1 - e^{-\lambda x_i})} = 0.$$

Due to the non-linearity of these equations the MLEs of parameters can be obtained numerically. In this paper we use statistical software R (R Development Core Team, 2011) to solve these equations. Here we use the Method of Moment Estimation (MME) to specify initial values.

4.2. Maximum product of spacings estimator

Maximum product of spacings (MPS) method was introduced by Cheng and Amin (1983) as an alternative to the MLE method. Ranneby (1984) derived the MPS method from an approximation of the Kullback-Leibler divergence (KLD). Kullback-Leibler divergence between $f\left(x, \underline{\theta}\right)$ and $f\left(x,\; \underline{\theta}_0\right)$ is given by

$$KLD(f\left(x,\; \underline{\theta}_0\right)||f\left(x, \underline{\theta}\right)) = \int f\left(x,\; \underline{\theta}_0\right) \log\left(\frac{f\left(x,\; \underline{\theta}_0\right)}{f\left(x, \underline{\theta}\right)}\right) dx.$$

The KLD is zero if and only if $f\left(x, \underline{\theta}\right) \;=\; f\;\left(x,\; \underline{\theta}_0\;\right)$ for all x.

Let $x_1, \dots, x_n$ be a sample from a CDF $F(x, \theta)$. Let $f(x, \theta)$ denote the corresponding PDF. For estimating $\underline{\theta}$ a perfect method should make the KLD between the model and the true distribution as small as possible. In applications, this can be approximated by estimating

$$\frac{1}{n}\sum_{i=1}^{n} \log\left[\frac{f\;\left(x_i,\; \underline{\theta}_0\;\right)}{f\left(x_i, \underline{\theta}\right)}\right]. \qquad (6)$$

So, by minimizing (6) with respect to $\underline{\theta}$, the estimator of $\underline{\theta}_0$ can be found. Ranneby (1984) suggested another approximation of the KLD, namely

$$\frac{1}{n}\sum_{i=1}^{n+1}\log\left[\frac{F\left(x_{(i)},\underline{\theta}_0\right)-F\left(x_{(i-1)},\underline{\theta}_0\right)}{F\left(x_{(i)},\underline{\theta}\right)-F\left(x_{(i-1)},\underline{\theta}\right)}\right], \tag{7}$$

where $x_{(i)}, i = 1, 2, \ldots, n$ denotes the ordered sample and $F\left(x_{(0)}\right) = 0,\ F\left(x_{(n+1)}\right) = 0$.

The estimator obtained by minimizing (7) is called the MPS estimator of θ_0. It is clear that minimizing (7) is equivalent to maximizing

$$\sum_{i=1}^{n+1}\log[F\left(x_{(i)},\underline{\theta}\right)-F\left(x_{(i-1)},\underline{\theta}\right)].$$

In case of the HCE distribution, the MPSs of a and λ, say $\hat{a}_{MPS}$, and $\hat{\lambda}_{MPS}$, respectively, can be obtained by minimizing

$$\sum_{i=1}^{n+1}\log[G\left(x_{(i)},a,\lambda\right)-G\left(x_{(i-1)},a,\lambda\right)] =$$

$$\sum_{i=1}^{n+1}\log[\frac{2e^{a}}{e^{2a}-1}\sinh(\mathrm{a}(1-e^{-\lambda x_{(i)}}))-\frac{2e^{a}}{e^{2a}-1}\sinh(\mathrm{a}(1-e^{-\lambda x_{(i-1)}}))].$$

with respect to a and λ.

4.3. Estimators based on percentiles

Estimation based on percentiles was originally explored by Kao (1958,1959). In fact the nature of percentiles estimators is based on distribution function. can be obtained by minimizing

$$\sum_{i=1}^{n}\left[x_{(i)}-F^{-1}(p_i,\theta)\right]^2,$$

where $p_i = \frac{i}{n+1}$ and $x_{(i)};\ i = 1, \ldots, n$ denotes the ordered sample. So to obtain the PC estimator of a and λ, we use the same method as for the ML estimator. In case of the HCE distribution, the PCEs of a and λ, say $\hat{a}_{PCE}$, and $\hat{\lambda}_{PCE}$, respectively, can be obtained by minimizing

$$\sum_{i=1}^{n}[x_{(i)}-G^{-1}(p_i,a,\lambda)]^2 =$$

$$\sum_{i=1}^{n}[x_{(i)}+\frac{1}{\lambda}\log(1-\frac{\log(\frac{e^{2a}-1}{2e^{a}}(\frac{i}{n+1})+\sqrt{\left(\frac{e^{2a}-1}{2e^{a}}(\frac{i}{n+1})\right)^2+1})}{a})]^2$$

with respect to a and λ.

4.4. Least squares and weighted least squares estimators

In this section, we derive regression based estimators of the unknown parameter. This method was originally suggested by Swain et al. (1988) to estimate the parameters of beta distributions. It can be used for some other distributions also. Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from a CDF $F(\cdot)$ and suppose $X_{(i)}, i = 1, 2, \ldots, n$ denote the ordered sample in ascending order. The proposed method uses $F(X_{(i)})$. For a sample of size n, we have

$$E[F(X_{(j)})]=\frac{j}{n+1}, \qquad Var[F(X_{(j)})]=\frac{j(n-j+1)}{(n+1)^2(n+2)},$$

$$Cov[F(X_{(j)}),F(X_{(l)})]=\frac{j(n-l+1)}{(n+1)^2(n+2)}, \quad j<l.$$

Using the expectations and the variances, two variants of the least squares method follow.

Method 1: Least squares estimators

The least squares estimators can be obtained by minimizing

$$\sum_{j=1}^{n}\left[F\left(x_{(j)}\right)-\frac{j}{n+1}\right]^2$$

with respect to the unknown parameters. In case of the HCE distribution, the LSEs of a and λ, say $\hat{a}_{LSE}$, and $\hat{\lambda}_{LSE}$, respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}[G\left(x_{(j)},a,\lambda\right)-\frac{j}{n+1}]^2 =$$

$$\sum_{j=1}^{n}\left[\frac{2e^a}{e^{2a}-1}\sinh(\mathrm{a}(1-e^{-\lambda x_{(j)}}))-\frac{j}{n+1}\right]^2$$

with respect to a and λ.

Method 2: Weighted least squares estimators

The weighted least squares estimators can be obtained by minimizing

$$\sum_{j=1}^{n}w_j\left[F\left(x_{(j)}\right)-\frac{j}{n+1}\right]^2$$

with respect to the unknown parameters, where

$$w_j=\frac{1}{Var\left[F\left(X_{(j)}\right)\right]}=\frac{(n+1)^2(n+2)}{j(n-j+1)}$$

In case of the HCE distribution, the WLSEs of a and λ, say $\hat{a}_{WLSE}$, and $\hat{\lambda}_{WLSE}$, respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}w_j[G\left(x_{(j)},a,\lambda\right)-\frac{j}{n+1}]^2 =$$

$$\sum_{j=1}^{n}w_j[\frac{2e^a}{e^{2a}-1}\sinh(\mathrm{a}(1-e^{-\lambda x_{(j)}}))-\frac{j}{n+1}]^2$$

with respect to a and λ.

4.5. Bootstrap estimator (bootstrap confidence intervals)

The uncertainty in the parameters of the fitted distribution can be estimated by parametric (resampling from the fitted distribution) or nonparametric (resampling with replacement from the original data set) bootstraps resampling Efron and Tibshirani (1994). These two parametric and nonparametric bootstrap procedures are described as follows.

Parametric bootstrap procedure

1. Estimate θ (vector of unknown parameters), say $\hat{\theta}$, from sample on the MLE procedure.

2. Generate a bootstrap sample $\{X_1^*,\dots,X_m^*\}$, using $\hat{\theta}$. Obtain the bootstrap estimate of θ, say $\hat{\theta}^*$, from the bootstrap sample based on the MLE procedure.

3. Repeat Step2 NBOOT times.

4. Order $\hat{\theta}^*{}_1, \dots, \hat{\theta}^*{}_{NBOOT}$ as $\hat{\theta}^*{}_{(1)}, \dots, \hat{\theta}^*{}_{(NBOOT)}$. Then obtain γ −quantiles and $100(1-\gamma)\%$ confidence intervals of parameters.

In case of the HCE distribution, the parametric bootstrap estimators (PBs) of α and λ, say $\hat{\alpha}_{PB}$, and $\hat{\lambda}_{PB}$, respectively.

Nonparametric bootstrap procedure

1. Generate a bootstrap sample $\{X_1^*, \dots, X_m^*\}$ with replacment from original data set. Obtain the bootstrap estimate of θ with MLE procedure, say $\hat{\theta}^*$ using the bootstrap sample.

2. Repeat Step2 NBOOT times.

3. Order $\hat{\theta}^*{}_1, \dots, \hat{\theta}^*{}_{NBOOT}$ as $\hat{\theta}^*{}_{(1)}, \dots, \hat{\theta}^*{}_{(NBOOT)}$.Then obtain γ −quantiles and $100(1-\gamma)\%$ confidence intervals of parameters.

In case of the HCE distribution, the nonparametric bootstrap estimators (NPBs) of α and λ , say $\hat{\alpha}_{NPB}$, and $\hat{\lambda}_{NPB}$, respectively.

4.6. Bayesian estimation

In this section, we have a short note on the Bayes estimation of the parameters of HCE distribution.To do this, assume that the vector of unknown parameters (α, λ) have independent prior distributions. Then, by attention to $\alpha > 0$ and $\lambda >$, we consider

$$\alpha \sim Gamma(b,c) \text{ and } \lambda \sim Gamma(d,e)$$

where all of b, c, d and e are positive parameters. Then, the joint posterior probability density function of α and λ given $\underline{x} = (x_1, x_2, \dots, x_n)$ can be written as:

$$\pi^*(\alpha, \lambda | \underline{x}) \propto \pi(\alpha, \lambda) f(\underline{x}, \alpha, \lambda)$$

where $\pi(\alpha, \lambda)$ is the joint prior distribution of the parameters. Since this posterior distribution is cumbersome, we can not provide posterior estimates of the parameters theoretically, but, by using MCMC algorithm in WINBUGS software we will obtain this estimators. The Bayesian estimators **(**Bs) of α and λ, say $\hat{\alpha}_B$, and $\hat{\lambda}_B$, respectively.

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5. DATA ANALYSIS AND APPLICATIONS

In this section, we illustrate the usefulness of the HCE distribution. First, The parameters of HCE distribution are estimated for two data set by eight estimation methods. Second, we fit this distribution to these data sets by ML method and compare the results with the gamma ,Weibull and generalized exponential (GE) with respective densities

$$f_{gamma}(x) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x}, \quad x \geq 0$$

$$f_{Weibull}(x) = \frac{\beta}{\lambda^\beta} x^{\beta-1} e^{-(\frac{x}{\lambda})^\beta}, \quad x \geq 0$$

$$f_{GE}(x) = \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}, \quad x \geq 0 .$$

First data set:Stress-rupture life data

We consider a data set of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, so we have complete data with the exact times of failure. For previous studies with the data sets see Andrews and Herzberg (1985) and Barlow et al. (1984). These data are:

0.0251 0.0886 0.0891 0.2501 0.3113 0.3451 0.4763 0.5650 0.5671 0.6566 0.6748 0.6751 0.6753 0.7696 0.8375
0.8391 0.8425 0.8645 0.8851 0.9113 0.9120 0.9836 1.0483 1.0596 1.0773 1.1733 1.2570 1.2766 1.2985 1.3211
1.3503 1.3551 1.4595 1.4880 1.5728 1.5733 1.7083 1.7263 1.7460 1.7630 1.7746 1.8275 1.8375 1.8503 1.8808
1.8878 1.8881 1.9316 1.9558 2.0048 2.0408 2.0903 2.1093 2.1330 2.2100 2.2460 2.2878 2.3203 2.3470 2.3513

2.4951 2.5260 2.9911 3.0256 3.2678 3.4045 3.4846 3.7433 3.7455 3.9143 4.8073 5.4005 5.4435 5.5295 6.5541 9.0960.

Second data set: Service times of 63 Aircraft Windshield

The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the nonstructural outer ply or failure of the heating system. These failures do not result in damage to the aircraft but do result in replacement of the windshield. We consider the data on service times for a particular model windshield given in Table 16.11 of Murthy et al. (2004). These data were recently studied by Ramos et al. (2013). These data are:

0.046 1.436 2.592 0.140 1.492 2.600 0.150 1.580 2.670 0.248 1.719 2.717 0.280 1.794 2.819 0.313 1.915 2.820 0.389 1.920 2.878 0.487 1.963 2.950 0.622 1.978 3.003 0.900 2.053 3.102 0.952 2.065 3.304 0.996 2.117 3.483 1.003 2.137 3.500 1.010 2.141 3.622 1.085 2.163 3.665 1.092 2.183 3.695 1.152 2.240 4.015 1.183 2.341 4.628 1.244 2.435 4.806 1.249 2.464 4.881 1.262 2.543 5.140.

Before analyzing these data sets, we use the scaled-TTT plot to verifiy our model validity, see Aarset (1987). It allows to identify the shape of hazard function graphically. We provide the empirical scaled-TTT plot of two above data sets. Fig. 4 and Fig. 5 show the scaled-TTT plots are concave. It indicates that the hazard function is increasing; therefore it verifies our model validity.

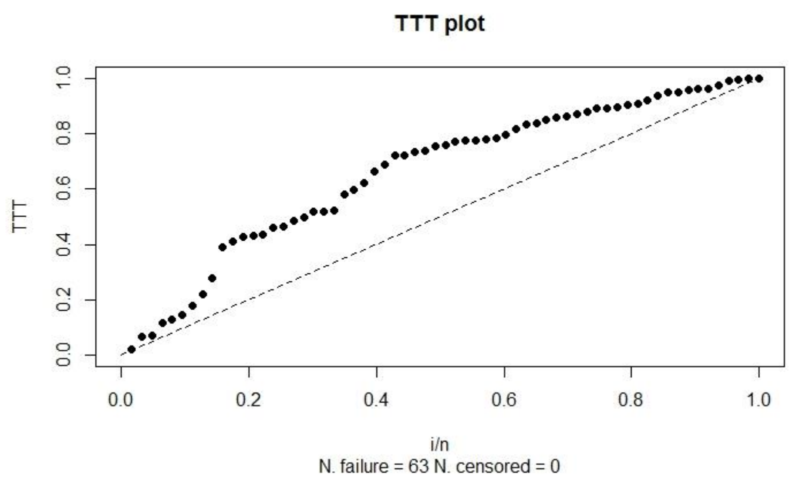

Fig. 4. Scaled-TTT plot of the first data set .

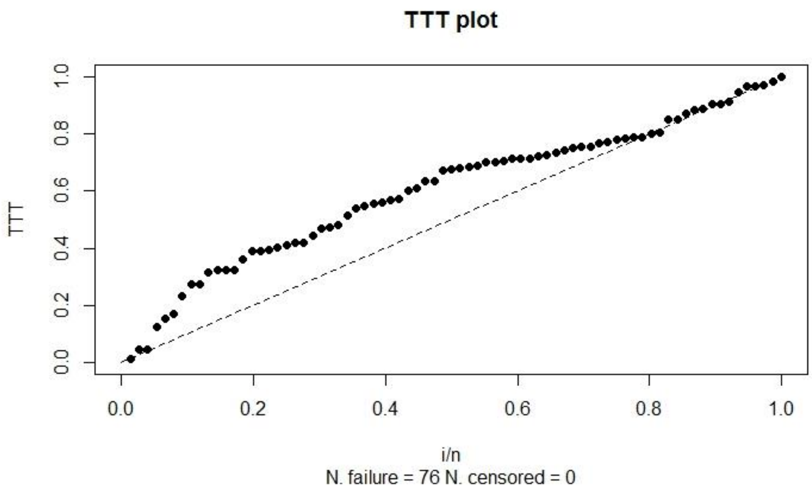

Fig 5. Scaled-TTT plot of the second data set .

Now we apply different estimation methods to estimate parameters of HCE distribution. Also, The performances of estimators can be compared through log-likelihood function. Table 1 shows the estimation of parameters of HCE distribution and corresponding log-likelihood function for the two data sets obtained by eight estimation methods: maximum likelihood, Bayesian, maximum product of spacings, parametric bootstrap, non-parametric bootstrap, percentile, least-squares and weighted least-squares.

Table1. Estimates of the parameters and the corresponding log-likelihood for the two data sets.

Data set	Method	Estimate of α	Estimate of λ	Log-likelihood
First data set	MLE	3.235	0.923	-121.56
	LSE	3.624	1.028	-122.0478
	WLSE	3.501	1.003	-121.8512
	PCE	3.094	0.908	-121.5863
	MPS	2.967	0.868	-121.6907
	PB	3.249	0.924	-121.5603
	NPB	3.306	0.933	-121.5666
	Bayes estimation under quadratic loss function			
	B	3.086	0.897	-121.5925
	Bayes estimation under absolute loss function			
	B	3.11	0.890	- 121.6082
Second data set	MLE	3.694	0.876	-99.817
	LSE	3.811	1.028	-99.919
	WLSE	3.805	0.890	-99.854
	PCE	2.675	0.690	-102.136
	MPS	3.493	0.863	-99.877
	PB	3.805	0.907	-99.830
	NPB	3.776	0.907	-99.827
	Bayes estimation under quadratic loss function			
	B	3.674	0.886	-99.825
	Bayes estimation under absolute loss function			
	B	3.678	0.882	-99.34

Analysis of the first data set

We fit the HCE distribution to the first data set and compare it with the gamma, generalized exponential and Weibull densities. Table 2 shows the MLEs of parameters, log-likelihood, Akaike information criterion (AIC), Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics for the first data set. The selection criterion is that the lowest AIC, A^* and W^*correspond to the best fit model. Thus, the HCE distribution provides the best fit for the data set as it shows the lowest AIC, A^* and W^* than other considered models. The relative histograms, fitted HCE, gamma, generalized exponential and Weibull PDFs for the first data set are plotted in Fig. 6(a). The plots of empirical and fitted survival functions, P-P plots and Q-Q plots for the HCE and other fitted distributions are displayed in Fig. 6(b), Fig. 6(c) and Fig. 6(d), respectively. These plots also support the results in Table 2.

Table 2. The MLEs of parameters for the first data set.

Model	MLEs of parameters	Log-likelihood	AIC	A^*	W^*
HCE	$\hat{a} = 3.239$, $\hat{\lambda} = 0.923$	-121.56	247.12	0.577	0.088
gamma	$\hat{\alpha} = 1.641$, $\hat{\lambda} = 0.838$	-122.249	248.498	0.674	0.113
Weibull	$\hat{\beta} = 1.326$, $\hat{\lambda} = 2.133$	-122.525	249.049	0.788	0.135
GE	$\hat{\alpha} = 1.709$, $\hat{\lambda} = 0.703$	-122.244	248.487	0.671	0.1

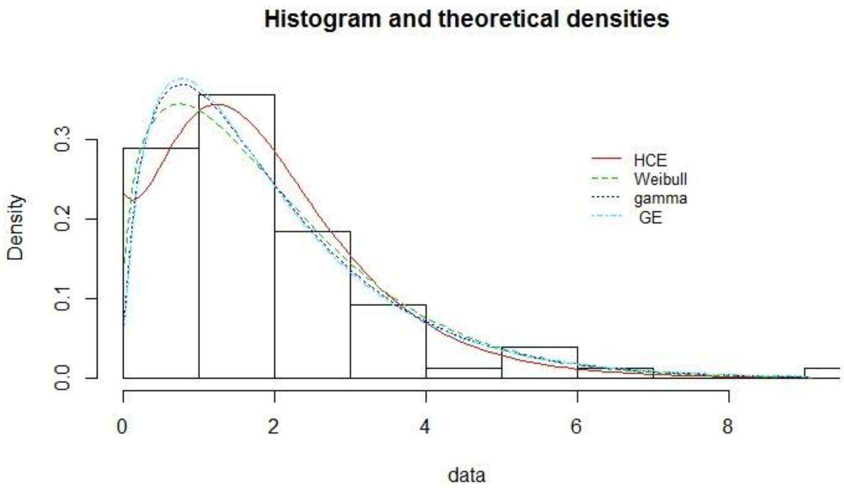

Fig. 6(a). The fitted PDFs and the relative histogram for the first data set.

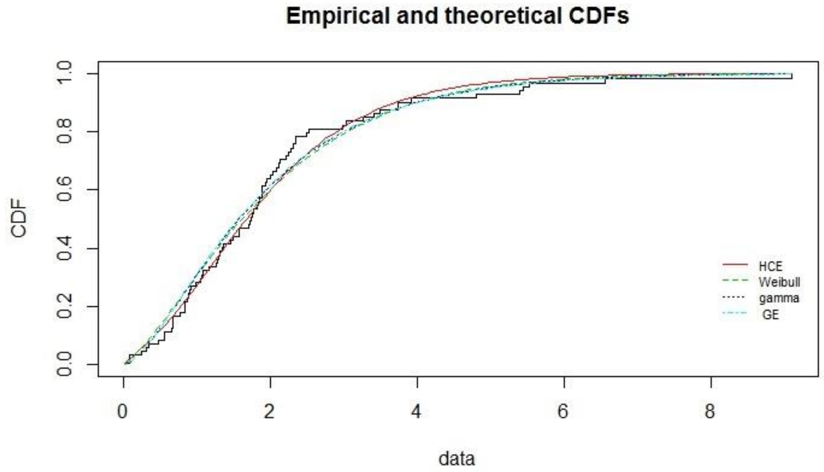

Fig. 6(b). Empirical and fitted survival functions for the first data set.

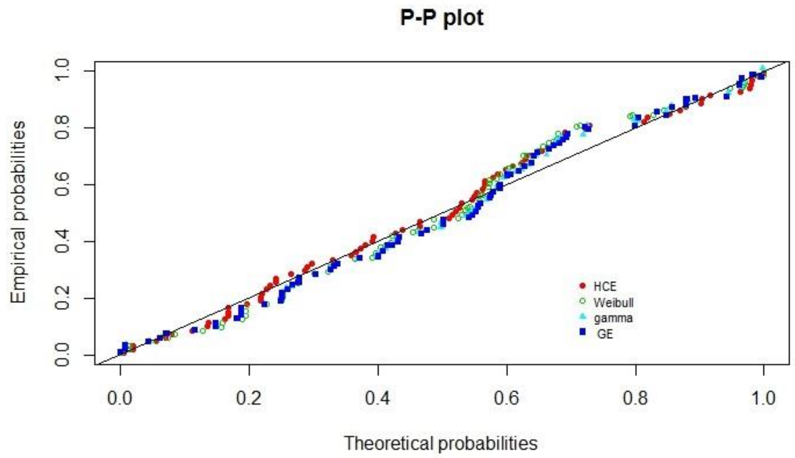

Fig. 6(c). P-P plots of fitted PDFs for the first data set.

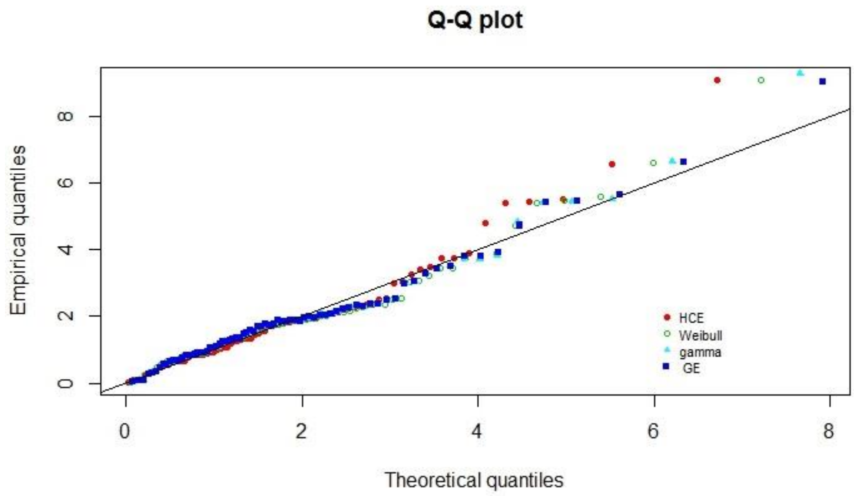

Fig. 6(d). Q-Q plots of fitted PDFs for the first data set.

Analysis of the second data set

We fit the HCE distribution to the second data sets and compare it with the gamma, generalized exponential and Weibull densities. Table 3 shows the MLEs of parameters, log-likelihood, Akaike information criterion (AIC), Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics for the second data set. The HCF distribution provides the best fit for the data set as it shows the lowest AIC, A^* and W^* than other considered models. The relative histograms, fitted HCE, gamma,generalized exponential and Weibull PDFs for second data are plotted in Fig. 7(a). The plots of empirical and fitted survival functions, P-P plots and Q-Q plots for the HCE and other fitted distributions are displayed in Fig. 7(b), Fig. 7(c) and Fig. 7(d), respectively. These plots also support the results in Table 3.

Table 3. The MLEs of parameters for the second data set.

Model	MLEs of parameters	Log-likelihood	AIC	A^*	W^*
HCE	$\hat{\alpha}=3.694$, $\hat{\lambda}=0.896$	**-99.817**	**203.634**	**0.454**	**0.074**
gamma	$\hat{\alpha}=1.908$, $\hat{\lambda}=0.915$	-102.832	209.664	1.162	0.200
Weibull	$\hat{\beta}=1.629$, $\hat{\lambda}=2.310$	-100.318	204.636	0.642	0.093
GE	$\hat{\alpha}=1.897$, $\hat{\lambda}=0.692$	-103.547	211.094	1.315	0.233

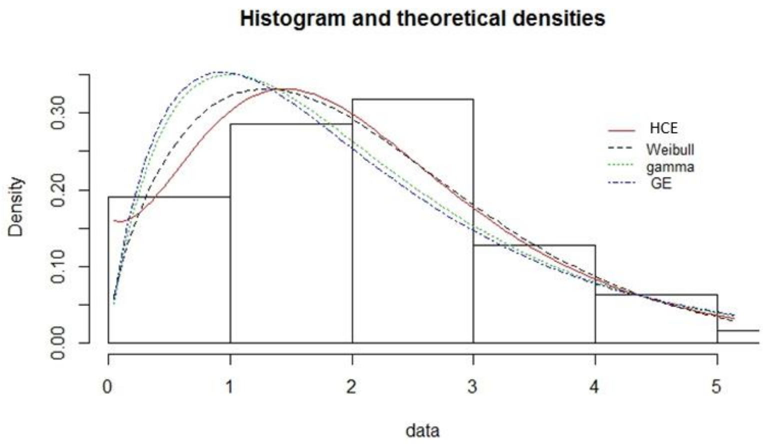

Fig. 7(a). The fitted PDFs and the relative histogram for the second data set.

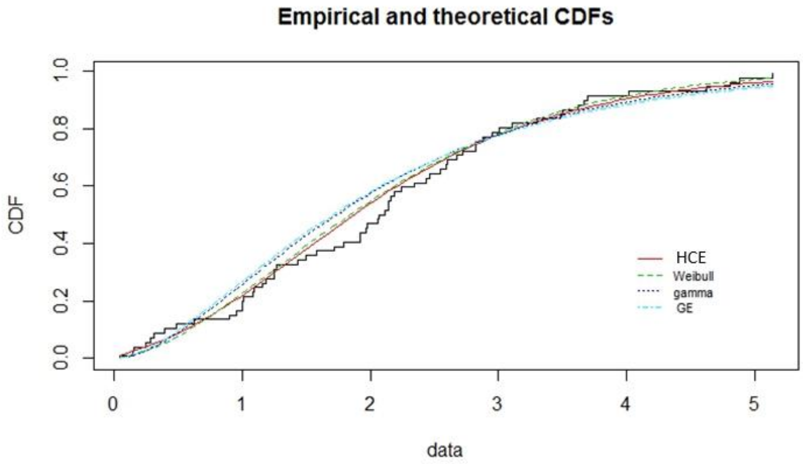

Fig. 7(b). Empirical and fitted survival functions for the second data set.

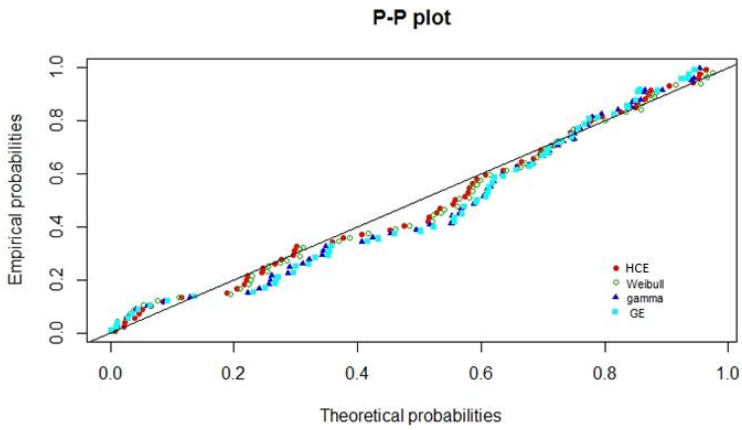

Fig. 7(c). P-P plots of fitted PDFs for the second data set.

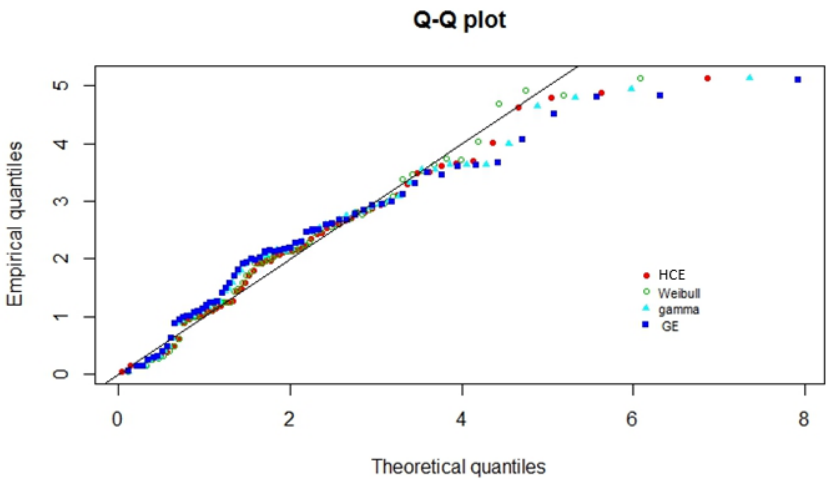

Fig. 7(d). Q-Q plots of fitted PDFs for the second data set.

6. CONCLUSION

In this paper, we have proposed a new class of distributions called the hyperbolic cosine – F (HCF) distribution. The HCF model is constructed by compounding a baseline F distribution with the hyperbolic cosine function. It is expected that this family will be widely applicable in reliability theory, risk analysis and other disciplines. The HCE distribution, as an important special case of this family, is very strong competitor to other well-known distributions commonly used in literature for fitting statistical data. For two real datasets the estimation of parameters is approached by the method of maximum likelihood, Bayesian, maximum product of spacings, parametric bootstrap, non-parametric bootstrap, percentile, least-squares and weighted least-squares. Moreover, two applications of the HCE distribution to real data sets are provided to illustrate that this distribution provides a better fit than Weibull, gamma and generalized exponential distributions.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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