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Best Simultaneous Approximation in Probabilistic Normed Spaces

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ABSTRACT

In the present paper we define the concept of best simultaneous approximation on probabilistic normed spaces and study the existence and uniqueness problem of best simultaneous approximation in these spaces. Firstly, some definitions such as set of p-best simultaneous approximation, simultaneous p-proximinal and simultaneous p-Chebyshev, are generalized. Then some properties related to the p-best simultaneous approximation set is presented and indicated that the simultaneous p-proximinal set is invariant under the addition and multiplication. We also develop the theory of p-best simultaneous approximation in quotient of probabilistic normed spaces and discuss about the relationship between the simultaneous p-proximinal elements of a given space and its quotient space. We show that under what conditions, set of the p-best simultaneous approximation is transferred by the natural map to the quotient space, and conversely. Finally some useful theorems were obtained to characterization for simultaneous p-proximinality and simultaneous p-Chebyshevity of a given space and its quotient space.

Keywords: *probabilistic normed space, p-best simultaneous approximation, simultaneous p-proximinal, simultaneous p-Chebyshev, quotient space.*

1. INTRODUCTION

Firstly, in 1942, Menger [4] introduced the notion of probabilistic metric spaces. The idea of Menger was to use distribution function in stead of nonnegative real numbers as values of the metric. The concept of probabilistic normed spaces (briefly, PN-spaces) was introduced by $\check{S}$erstnev [7] in 1962. It is well known that the theory of probabilistic normed spaces is a new frontier branch between probabilistic theory and functional analysis and has an important background which contains the common metric space as a special case.

Many works on approximation have been done on PN-spaces [2, 3, 6]. Recently, Goudarzi and Vaezpour [1] considered the set of all t-best simultaneous approximation in fuzzy normed spaces. In this paper, we introduce the concept of best simultaneous approximation in probabilistic normed spaces and present some results.

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Now we recall some notations and definitions used in this paper.

A distribution function is a non-decreasing and left-continuous function F from $\mathbb{R}\cup\{-\infty,+\infty\}$ into $[0,1]$ satisfying $F(0)=0$ and $F(+\infty)=1$. The space of distribution functions is denoted by Δ^+ and the set of all F in Δ^+ for which $\lim_{n\to+\infty}F(t)=1$ will be denoted by D^+.

The space Δ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F\le G$ if and only if $F(t)\le G(t)$ for all $t\in\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t)=\begin{cases}0 & \text{if } t\le 0\\ 1 & \text{if } t>0\end{cases}$$

A triangular norm (briefly, t −norm) is a binary operation $T:\Delta^+\times\Delta^+\to\Delta^+$ which is commutative, associative, and non-decreasing in each variable and has ε_0 as the unit element. A continuous t-norm is a continuous binary operation on $[0,1]$ that is commutative, associative, nondecreasing in each variable, and has 1 as identity.

An important examples of t −norms are the Lukasiewicz t –norm $T_L, T_L(a,b)=\max(a+b-1,0)$, the product t –norm $T_P, T_P(a,b)=ab$, and the strongest triangular norm $T_M, T_M(a,b)=\min(a,b)$.

Now, we are ready to recall the definition of a PN-space in the sense of Šerstnev.

Definition 1.1 *A probabilistic normed space (briefly denoted by PN-space) is a triple (X,ν,T) where X is a vector space over the filed K of real or complex numbers, ν is a function from X into Δ^+, T is a continuous triangle function, and for every choice of x and y in X and any $\alpha\ne 0$ in K the following conditions hold:*

(N1) $\nu_x=\varepsilon_0$ if and only if, $x=\theta(\theta$ is a null vector in X),

(N2) $\nu_{\alpha x}(t)=\nu_x(t/|\alpha|)$ for all t in $\mathbb{R}^+$,

(N3) $\nu_{x+y}\ge T(\nu_x,\nu_y)$.

ν is called a probabilistic norm on X (briefly P-norm) and it is called a strong probabilistic norm if for $t>0$, $x\to\nu_x(t)$ is a continuous map on X.

Example 1.2 *[5] Let $(X,\|.\|)$ be a linear normed space. Define*

$$\nu_x(t)=\begin{cases}0 & \text{if } t\le 0\\ \frac{t}{t+\|x\|} & \text{if } t>0.\end{cases}$$

Then (X,ν,T_p) is a PN-space.

Example 1.3 *[5] Let $(X,\|.\|)$ be a linear normed space. Define*

$$\nu_x(t)=\begin{cases}0 & \text{if } t\le 0\\ e^{(-\|x\|/t)} & \text{if } t>0.\end{cases}$$

Then (X,ν,T_p) is a PN-normed space.

Definition 1.4 *[5] The PN-space (X,ν,T) is said to be a probabilistic Banach space whenever X is complete with respect to the probabilistic metric induced by probabilistic norm.*

2. SET OF P-BEST SIMULTANEOUS APPROXIMATION

In this section, we give some characterizations of simultaneous p-proximinal sets in PN-spaces.

Definition 2.1 *Let (X,ν,T) be a PN-space. A subset A of X is said to be R-bounded if there exists $t>0$ and $r\in(0,1)$ such that $\nu_{x-y}(t)>1-r$ for all $x,y\in A$.*

Definition 2.2 *Let (X,ν,T) be a PN-space, W be a subset of X and M be a R-bounded subset in X. For $t>0$, we define,*

$$d(M,W,t)=\sup_{w\in W}\inf_{m\in M}\nu_{m-w}(t).$$

An element $w_0\in W$ is called a p-best simultaneous approximation to M from W if for $t>0$,

$$d(M,W,t)=\inf_{m\in M}\nu_{m-w_0}(t).$$

The set of all p-best simultaneous approximation to M from W will be denoted by $S_W^p(M)$ and we have,

$$S_W^t(M)=\{w\in W: \inf_{m\in M}\nu_{m-w}(t)=d(M,W,t)\}.$$

Definition 2.3 *Let W be a subset of (X,ν,T). It is called a simultaneous p-proximinal subset of X if for each R-bounded set M in X, there exists at least one p-best simultaneous approximation from W to M. Also it is called a simultaneous p-Chebyshev subset of X if for each R-bounded set M in X there exists a unique simultaneous p −best approximation from W to M.*

Theorem 2.4 *Let W be a subset of X and M be a R −bounded subset of X. Then for $t>0$ the following assertions are hold,*

(i) $d(M+x,W+x,t)=d(M,W,t), \forall x\in X$,

(ii) $d(\lambda M,\lambda W,t)=d(M,W,\frac{t}{|\lambda|}), \forall\lambda\in\mathbb{C}$,

(iii) $S_{W+x}^t(M+x)=S_W^t(M)+x, \forall x\in X$,

(iv) $S_{\lambda W}^{|\lambda|t}(\lambda M)=\lambda S_W^t(M), \forall\lambda\in\mathbb{C}$.

Proof. (i) We have

$$\begin{aligned} d(M+x,W+x,t) &= \sup_{w\in W}\inf_{m\in M}\nu_{m+x-(w+x)}(t)\\ &= \sup_{w\in W}\inf_{m\in M}\nu_{m-w}(t)\\ &= d(M,W,t)\end{aligned}$$

(ii) Clearly equality holds for $\lambda=0$, so suppose that $\lambda\ne 0$. Then

$$d(\lambda M,\lambda W,t)=\sup_{w\in W}\inf_{m\in M}\nu_{\lambda(m-w)}(t)$$
$$=\sup_{w\in W}\inf_{m\in M}\nu_{m-w}(\frac{t}{|\lambda|})$$
$$=d(M,W,t)$$

(iii) $x+W\in S^t_{W+x}(M+x)$ if and only if,

$$\inf_{m+x\in M+x}\nu_{m+x-(w+x)}(t)=d(M+x,W+x,t),$$

and by (i), the above equality holds if and only if,

$$\inf_{m\in M}\nu_{m-w}(t)=d(M,W,t),$$

for all $w\in W$ and this shows that $w\in S^t_W(M)$. So $x+W\in S^t_W(M)+x$.

(iv) $y_0\in S^{|\lambda|t}_{\lambda W}(\lambda M)$ if and only if $y_0\in\lambda W$ and,

$$d(\lambda M,\lambda W,|\lambda|t)=\inf_{\lambda m\in\lambda M}\nu_{y_0-\lambda m}(|\lambda|t)$$
$$=\inf_{m\in M}\nu_{\frac{y_0}{\lambda}-m}(t).$$

In view of (ii),

$$d(\lambda M,\lambda W,|\lambda|t)=d(M,W,t).$$

So we have $\frac{y_0}{\lambda}\in W$ and $d(M,W,t)=\inf_{m\in M}\nu_{\frac{y_0}{\lambda}-m}(t)$ or equivalently $\frac{y_0}{\lambda}\in S^t_W(M)$ and the proof is completed.

Corollary 2.5 *Let A be a nonempty subset of PN-space (X,ν,T).* Then the following statements are hold.

(i) A is simultaneous p −proximinal(resp. simultaneous p −Chebyshev) if and only if $A+y$ is simultaneous p −proximinal(resp. simultaneous p −Chebyshev), for each $y\in X$.

(ii) A is simultaneous p −proximinal(resp. simultaneous p −Chebyshev) if and only if αA is simultaneous $|\alpha|p$ −proximinal(resp. simultaneous $|\alpha|p$ −Chebyshev), for each $\alpha\in\mathbb{C}$.

Corollary 2.6 *Let M be a subspace of X and N be a R −bounded subset of X. Then for $t>0$,*

(i) $d(M,N+y,t)=d(M,N,t),\forall y\in M$,

(ii) $d(M,\alpha N,|\alpha|t)=d(M,N,t),\forall 0\neq\alpha\in\mathbb{C}$,

(iii) $S^t_M(N+y)=S^t_M(N)+y,\forall y\in M$,

(iv) $S^{|\alpha|t}_M(\alpha N)=\alpha S^t_M(N),\forall 0\neq\alpha\in\mathbb{C}$.

3. SIMULTANEOUS P-PROXIMINALITY AND SIMULTANEOUS P-CHEBYSHEVITY IN QUOTIENT SPACES

In this section we give characterizations of simultaneous p −proximinality and simultaneous p −Chebyshevity in quotient spaces. First we remind some notations and definitions.

Definition 3.1 *[5] Let (X,ν,T) be an PN-space and M is a subspace in X and $Q:X\to\frac{X}{M}$ is the natural mapping $Q(x)=x+M$. For any $t>0$, we define,*

$$\bar{\nu}_{x+M}(t)=\sup_{y\in M}\nu_{x+y}(t).$$

Theorem 3.2 *[5] Let M be a closed subspace of PN-space (X,ν,T) and $\bar{\nu}$ be given in the above definition. Then*

(i) $\bar{\nu}$ is an p-norm on $\frac{X}{M}$.

(ii) $\bar{\nu}_{Q(x)}(t)\geq\nu_x(t)$.

(iii) If (X,ν,T) is a probabilistic Banach space, then so is $(\frac{X}{M},\bar{\nu},T)$.

Shams and Vaezpour, offered the notion of p-best approximation as follows.

Definition 3.3 *[6] Let A be a nonempty set in PN-space (X,ν,T). For $x\in X$ and $t>0$, we shall denote $P^t_A(x)$ the set of all p −best approximation to x from A, i.e.*

$$P^t_A(x)=\{y\in A:d(A,x,t)=\nu_{y-x}(t)\}$$

where

$$d(A,x,t)=\sup_{y\in M}\nu_{y-x}(t).$$

If each $x\in X$ has at least(resp. exactly) one p −best approximation in A, then A is called a p −proximinal(resp. p −Chebyshev) set.

Lemma 3.4 *Let (X,ν,T) be a PN-space and M be a p −proximinal subspace of X. For each nonempty R −bounded set S in X and $t>0$,*

$$d(S,M,t)=\inf_{s\in S}\sup_{m\in M}\nu_{s-m}(t).$$

Proof. Since M is p −proximinal it follows that for each $s\in S$ there exists $m_s\in P^t_M(S)$ such that for $t>0$,

$$\nu_{s-m_s}(t)=\sup_{m\in M}\nu_{s-m}(t).$$

So,

$$d(S,M,t)=\sup_{m\in M}\inf_{s\in S}\nu_{s-m}(t)$$
$$\geq\inf_{s\in S}\nu_{s-m_s}(t)$$
$$=\inf_{s\in S}\sup_{m\in M}\nu_{s-m}(t)$$
$$\geq\sup_{m\in M}\inf_{s\in S}\nu_{s-m}(t)$$
$$=d(S,M,t).$$

This implies that

$$d(S,M,t)=\inf_{s\in S}\sup_{m\in M}\nu_{s-m}(t).$$

Lemma 3.5 *Let M be a $p-$proximinal subspace of (X,ν,T) and $W \supseteq M$ a subspace of X. Let K be $R-$bounded in X. If $w_0 \in S_W^t(K)$, then $w_0 + M \in S_{\frac{W}{M}}^t(\frac{K}{M})$.*

Proof. Since K is $R-$bounded by theorem 3.2(ii), $\frac{K}{M}$ is also $R-$bounded in $\frac{X}{M}$. Assume that $w_0 \in S_W^t(K)$ and $w_0 + M \notin S_{\frac{W}{M}}^t(\frac{K}{M})$. Thus there exists $w' \in W$ such that for $t > 0$,

$$\inf_{k\in K}\bar{\nu}_{k-(w'+M)}(t) > \inf_{k\in K}\bar{\nu}_{k-(w_0+M)}(t) \geq \inf_{k\in K}\nu_{k-w_0}(t) = d(K,W,t). \quad (1)$$

On the other hand for each $k \in K$ and for $t > 0$,

$$\bar{\nu}_{k-(w'+M)}(t) = \sup_{m\in M}\nu_{k-(w'+m)}(t).$$

Then for each $0 < \varepsilon < 1$ and $k \in K$ there exists $m_k \in M$ such that for $t > 0$,

$$\nu_{k-(w'+m_k)}(t) \geq \bar{\nu}_{k-(w'+M)}(t) - \varepsilon.$$

Since $w' + m_k \in W$ we conclude that

$$d(K,W,t) \geq \inf_{k\in K}\nu_{k-(w'+m_k)}(t) \geq \inf_{k\in K}\bar{\nu}_{k-(w'+M)}(t) - \varepsilon.$$

Thus,

$$d(K,W,t) \geq \inf_{k\in K}\bar{\nu}_{k-(w'+M)}(t). \quad (2)$$

By (1) and (2),

$$d(K,W,t) \geq \inf_{k\in K}\bar{\nu}_{k-(w'+M)}(t) > d(K,W,t),$$

and this is a contradiction. Therefore, $w_0 + M \in S_{\frac{W}{M}}^t(\frac{K}{M})$ and the proof is complete.

Corollary 3.6 *Let M be a $p-$proximinal subspace of (X,ν,T) and $W \supseteq M$ a subspace of X. If W is simultaneous $p-$proximinal then $\frac{W}{M}$ is a simultaneous $p-$proximinal subspaces of $\frac{X}{M}$.*

Corollary 3.7 *Let M be a $p-$proximinal subspace of (X,ν,T) and $W \supseteq M$ a subspace of X. If W is simultaneous $p-$proximinal then for each $R-$bounded set K in X,*

$$Q(S_W^t(K)) \subseteq S_{\frac{W}{M}}^t(\frac{K}{M}).$$

Theorem 3.8 *Let M be a $p-$proximinal subspace of (X,ν,T) and $W \supseteq M$ a subspace of X. If K is a $R-$bounded set in X such that $w_0 + M \in S_{\frac{W}{M}}^t(\frac{K}{M})$ and $m_0 \in S_W^t(K - w_0)$, then $w_0 + m_0 \in S_W^t(K)$.*

Proof. In view of lemma (3.5) for $t > 0$ we have,

$$\begin{aligned}\inf_{k\in K}\nu_{(k-w_0)-m_0}(t) &= \sup_{m\in M}\inf_{k\in K}\nu_{(k-w_0)-m}(t)\\ &= \inf_{k\in K}\sup_{k\in K}\nu_{k-(w_0+m)}(t)\\ &= \inf_{k\in K}\bar{\nu}_{k-(w_0+M)}(t)\\ &\geq \inf_{k\in K}\bar{\nu}_{k-(w+M)}(t), \forall w \in W\\ &\geq \inf_{k\in K}\nu_{k-w}(t), \forall w \in W.\end{aligned}$$

Hence

$$\inf_{k\in K}\nu_{(k-w_0)-m_0}(t) \geq \inf_{k\in K}\nu_{k-w}(t), \forall w \in W.$$

But $w_0 + m_0 \in W$. Then $w_0 + m_0 \in S_W^t(K)$ and so proof is complete.

Theorem 3.9 *Let M be a $p-$proximinal subspace of (X,ν,T) and $W \supseteq M$ a simultaneous $p-$proximinal subspace of X. Then for each $R-$bounded set K in X,*

$$Q(S_W^t(K)) = S_{\frac{W}{M}}^t(\frac{K}{M}).$$

Proof. By corollary (3.7) we obtain,

$$Q(S_W^t(K)) \subseteq S_{\frac{W}{M}}^t(\frac{K}{M}).$$

Also by lemma (3.5) $\frac{W}{M}$ is simultaneous $p-$proximinal in $\frac{X}{M}$. Now let,

$$w_0 + M \in S_{\frac{W}{M}}^t(\frac{K}{M}),$$

where $w_0 \in W$. By simultaneous $p-$proximinality of M there exists $m_0 \in M$ such that $m_0 \in S_W^t(K - w_0)$. Then in view of theorem (3.8) we conclude that $w_0 + m_0 \in S_W^t(K)$. Therefore $w_0 + M \in Q(S_W^t(K))$ and the proof is complete.

Corollary 3.10 *Let W and M be subspaces of (X,ν,T). If M is simultaneous $p-$proximinal then the following assertions are equivalent:*

(i) $\frac{W}{M}$ is simultaneous $p-$proximinal in $\frac{X}{M}$.

(ii) $W + M$ is simultaneous $p-$proximinal in X.

Proof. $(i) \to (ii)$. Let K be an arbitrary $R-$bounded set in X. Then by theorem (3.2(ii)), $\frac{K}{M}$ is a $R-$bounded set in $\frac{X}{M}$. Since $\frac{W+M}{M} = \frac{W}{M}$ and M are simultaneous $p-$proximinal it follows that there exists $w_0 + M \in \frac{W+M}{M}$ and $m_0 \in M$ such that $w_0 + M \in S_{\frac{W+M}{M}}^t(\frac{K}{M})$ and $m_0 \in S_M^t(K - w_0)$. By theorem (3.8), $w_0 + m_0 \in S_{W+M}^t(K)$. This show that $W + M$ is simultaneous $p-$proximinal in X.

$(ii) \to (i)$ Since $W+M$ is simultaneous $p-$proximinal and $W+M \supseteq M$, by corollary (3.6), $\frac{W+M}{M} = \frac{W}{M}$ is simultaneous $p-$proximinal.

Theorem 3.11 *Let W and M be subspaces of (X,ν,T). If M is simultaneous $p-$Chebyshev then the following assertions are equivalent:*

(i) $\frac{W}{M}$ is simultaneous $p-$Chebyshev in $\frac{X}{M}$.

(ii) $W+M$ is simultaneous $p-$Chebyshev in X.

Proof. $(i) \to (ii)$. By hypothesis $\frac{W+M}{M} = \frac{W}{M}$ is simultaneous $p-$Chebyshev. Assume that (ii) is false. Then for some $R-$bounded subset K of X has two distinct simultaneous $p-$best approximations such as l_0 and l_1 in $W+M$. Thus we have,

$$l_0, l_1 \in S^t_{W+M}(K). \tag{3}$$

By lemma (3.5),

$$l_0 + M, l_1 + M \in S^t_{\frac{W+M}{M}}\left(\frac{K}{M}\right) = S^t_{\frac{W}{M}}\left(\frac{K}{M}\right).$$

Since $\frac{W}{M}$ is simultaneous $p-$Chebyshev, $l_0 + M = l_1 + M$. So there exists $0 \neq m_0 \in M$ such that $l_1 = l_0 + m_0$. By (3) for all $t > 0$,

$$\begin{aligned}\inf_{k\in K} \nu_{(k-l_0)-m_0}(t) &= \inf_{k\in K} \nu_{k-l_1}(t)\\ &= \inf_{k\in K} \nu_{k-l_0}(t)\\ &= d(K, W+M, t)\\ &= d(K-l_0, W+M, t)\\ &\geq d(K-l_0, M, t).\end{aligned}$$

This show that both m_0 and zero are simultaneous $p-$best approximation to $S-l_0$ from M and this is a contradiction.

$(ii) \to (i)$. Assume that (i) dose not hold. Then for some $R-$bounded subset K of X, $\frac{K}{M}$ has two distinct simultaneous $p-$approximations such as $w+M$ and $w'+M$ in $\frac{W}{M}$. Thus $w-w'$ is not in M. Since M is simultaneous $p-$proximinal there exists simultaneous $p-$best approximations m and m' to $K-w$ and $K-w'$ from M, respectively. Therefore $m \in S^t_M(K-w)$ and $m' \in S^t_M(K-w')$. Since $W+M \supseteq M$, $w+M$ and $w'+M$ are in $S^t_{\frac{W}{M}}\left(\frac{K}{M}\right) = S^t_{\frac{K+M}{M}}\left(\frac{K}{M}\right)$, by theorem (3.9), $w+m$ and $w'+m'$ are in $S^t_{W+M}(K)$. But $W+M$ is simultaneous $p-$Chebyshev. Thus $w+m = w'+m'$ and so $w-w'$ belongs to M, which is a contradiction.

Corollary 3.12 *Let M be a simultaneous $p-$Chebyshev subspace of (X,ν,T). If $W \supseteq M$ is a simultaneous $p-$Chebyshev subspace in X, then the following assertions are equivalent:*

(i) W is simultaneous $p-$Chebyshev in X.

(ii) $\frac{W}{M}$ is simultaneous $p-$Chebyshev in $\frac{X}{M}$.

4. CONCLUSION

In this paper, we extended the concept of best simultaneous approximation in probabilistic normed spaces. Our results obtained here are generalization of the corresponding results of normed spaces. We indicated that the simultaneous p-proximinal sets were invarianted under the addition and multiplication. Certain results regarding p-best simultaneous approximation in quotient spaces are obtained. We showed that under the condition of p-proximinality of the subspace M, set of the p-best simultaneous approximation is transferred by the natural map to the quotient space, and conversely. Finally, we presented some useful theorems to characterization of simultaneous p-proximinality and simultaneous p-Chebyshevity of a given space and its quotient space.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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