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Some Fixed Point Theorems of R-Weakly Commuting Mappings in Multiplicative Metric Spaces

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ABSTRACT

In this paper, we present a unique common fixed point theorem for pointwise R-weakly commuting maps in complete multiplicative metric space. Another result of R-weakly commuting of type (P) is also established. Our results generalize the results of the main theorem of Xiaoju He, Meimei Song and Danping Chen (Common fixed points for weak commutative mappings ona multiplicative metric space) by using R-weakly commuting maps.

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1.INTRODUCTION AND PRELIMINARIES

Study of the importance of fixed points of mappings satisfying certain contraction condition is important in various research activities. Michael Grossman and Robert Katz [1] introduced multiplicative calculus which is also known asnon-newtonian calculus. Regarding multiplicative calculus, MuttalipOzavsar[2] used multiplicative contraction mappings and proved some fixed point theorems of mappings on complete multiplicative metric space. After this many scholars try to fit the well known mappings which are applicable in different spaces to multiplicative metric space and search its application for otherstreams. We found some of the application about the multiplicative metricspace. Agamieza E. Bashirov, EmineMisirliKurpinar and Ali Ozyapici[3]used multiplicative calculus as a mathematical tool for economics, finance etc.Luc Florac, Hans van Assen[4] used multiplicative calculus in Biomedical Image Analysis, UgurKadak and MuharrenOzluk[5] generalized Runge-Kuttamethod, A. Bashirov, M. Riza [6]used complex multiplicative differentiationand established multiplicative Cauchy-Riemann equation and also complex fourierseries which are expressed in terms of exponents are suitable for multiplicativecalculus. Misirli and Gurefe[7]used multiplicative calculus in numerical

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methods and many more application which we can't find out presently and a lot ofapplication which are at the door step, which we can see in the nearfuture. Resently, Demet et. al. [16] established some results on fixed points ofnon-Newtonian contraction mappings on non-Newtonian metric spaces.

In 1999, R. P. Pant [8,9] used the notion of R-weakly commuting maps.In the year 2006,ImdadMohd. Ali and Javid Ali [10] introduced R-weaklycommuting of type (P) in fuzzy metric space. In this paper, we discuss the common fixed points for R-weak commuting and R - weakly commuting maps of type (P) in complete multiplicative metric space.

2. SOME BASIC PROPERTIES

Definition 1. [3] Let (X,d) be a non empty set. A multiplicative metrices a mapping $d : X \times X \to R^+$ satisfying the following conditions:

(a) $d(x,y) \geq 1$ for all $x, y \in X$ and $d(x,y) = 1$ if and only if $x = y$;

(b) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(c) $d(x,y) \leq d(x,z).d(z,y)$ for all $x, y \in X$ (multiplicative triangle inequality).

Example 1. [2]Let R^+ be the collection of all n-tuples of positive real numbers.Let $d : \mathrm{R}_+^n \times R_+^n \to R$ be defined as follows:

$$d(\mathrm{x},\mathrm{y}) = \left|\frac{x_1}{y_1}\right|\left|\frac{x_2}{y_2}\right|\cdots\left|\frac{x_n}{y_n}\right|,$$

where $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R_+^n$ and $|.| : R_+ \to R_+$ is defined asfollows:

$$|a| = \begin{cases} a, & if \;\; a \geq 1; \\ \frac{1}{a}, & if \;\; a \leq 1. \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric are satisfied.

Definition 2.[2]Let (X,d) be a multiplicative metric space, $\{x_n\}$ bea sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x) = \{y \mid d(x,y) < \varepsilon\}, \varepsilon > 1,$ there exists a natural number N such that $n \geq N,$ then $x_n \in B_\varepsilon(x)$. Here sequence $\{x_n\}$ is said to be multiplicative converging to x ,denoted by $x_n \to x$ as $(n \to \infty)$.

Definition 3.[2] Let (X,d) be a multiplicative metric space and $\{x_n\}$ bea sequence in X . The sequence is called a multiplicative Cauchy sequence ifit holds that for all $\varepsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.

Definition 4. [2]We call a multiplicative metric space complete if everymultiplicative Cauchy sequence in it is multiplicative convergence to $x \in X$.

Definition 5. [2] Let (X,d) be a multiplicative metric space. A mapping $f : X \times X \to X$ is called a multiplicative contraction if there exists a realconstant $\lambda \in [0,1)$ such that $d(f(x_1), f(x_2)) \leq \mathrm{d}(x_1, x_2)^\lambda$ for all $x, y \in X$.

Definition 6. [11]Suppose that S,T are two self-mappings of a multiplicative metric space (X,d), S,T are called commutative mappings if it holdsthat for all $x \in X$, $STx = TSx$.

Definition 7. [11] Suppose that S,T are two self-mappings of a multiplicative metric space (X,d); S,T are called weak commutative mappings ifit holds that for all $x \in X$, $d(STx, TSx) \leq d(Sx, Tx)$.

Remark: Commutative mappings must be weak commutative mappings, but the converse is not true.

Definition 8. [8,9] Let S,T be two self-mappings of multiplicative metricspace (X,d); S,T are called pointwise R-weak commuting on X if there exists $R > 0$ such that

$$d(STx, TSx) \leq Rd(Sx, Tx)$$

for every $x \in X$.

Definition 9.[10] Let S,T are two self-mappings of multiplicative metricspace (X,d); S,T are called R-weak commuting mappings of type (P) if thereexists some $R>0$ such that

$$d(SSx,TTx)\le Rd(Tx,Sx)$$

for every $x\in X$.

Lemma.[9] Let S,T be two self-mappings of a multiplicative metric space (X,d). If S,T are R-weakly commuting maps of type (P) and $\{x_n\}$ be asequence in X such that

$$\lim_{n\to\infty} Sx_n = z = \lim_{n\to\infty} Tx_n$$

for some $z\in X$, then

$$\lim_{n\to\infty} STx_n = Tz$$

if T is continuous at z.

Theorem 10. [2] Let (X,d) be a multiplicative metric space and let $f: X\times X\to X$ be a multiplicative contraction. If (X,d) is complete, then f has a unique fixed point.

3.MAIN RESULTS

Theorem 11. Let A,B,S and T be self mappings from a completemultiplicative metric space into itself satisfying the following conditions:

a) $A(X)\subset T(X)$ and $B(X)\subset S(X)$,

b) $d(Ax,\mathrm{B}y)\le\left\{\max\left\{d(Sx,\mathrm{T}y),d(Ax,\mathrm{S}x),d(By,Ty),d(Ax,\mathrm{Ty}),d(By,\mathrm{Sx})\right\}\right\}^{\lambda}$,

c) $\{A,S\}$ and $\{B,T\}$ are pointwise R-weakly commuting pairs,

d) $\{A,S\}$ and $\{B,T\}$ are compatible pairs of reciprocally continuous mappings.

Then A,B,S and T have a unique common fixed point in X.

Proof. Since $A(X)\subset T(X)$ for an arbitrary point x_0 in X, there exists apoint x_1 in X such that $Tx_1 = Ax_0$ and for x_1 there exists x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n}$$

and $y_{2n+1} = Tx_{2n} = Bx_{2n-1}$ for $n = 0,1,2,...$

From (b), we have

$$d(y_{2n},y_{2n+1}) = d(Ax_{2n-1},Bx_{2n})$$

$$\le\left\{\max\left\{\begin{matrix} d(Sx_{2n-1},Tx_{2n}),d(Ax_{2n-1},Sx_{2n-1}),d(Bx_{2n},Tx_{2n}), \\ d(Ax_{2n-1},Tx_{2n}),d(Bx_{2n},Sx_{2n-1}) \end{matrix}\right\}\right\}^{\lambda}$$

$$= \left\{ \max \left\{ \begin{matrix} d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \\ d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1}) \end{matrix} \right\} \right\}^{\lambda}$$

$$= \left\{ \max \left\{ d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), 1, d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}) \right\} \right\}^{\lambda}$$

$$= d^{\lambda}(y_{2n-1}, y_{2n}).d^{\lambda}(y_{2n}, y_{2n+1}).$$

This implies that

$$d(y_{2n}, y_{2n+1}) \le d^{\frac{\lambda}{1-\lambda}}(y_{2n-1}, y_{2n}).$$

Let $h = \dfrac{\lambda}{1-\lambda}$, then

$$d(y_{2n}, y_{2n+1}) \le d^{h}(y_{2n-1}, y_{2n}). \tag{1}$$

We also obtain

$$d(y_{2n+1}, y_{2n+2}) \le d^{h}(y_{2n}, y_{2n+1}). \tag{2}$$

From (1) and (2), we have

$$d(y_{2n}, y_{2n+1}) \le d^{h}(y_{2n-1}, y_{2n}) \le .... \le d^{h^n}(y_1, y_0) \text{ for all } n \ge 2 .$$

Let m, n be positive integers such that $m \ge n$, then we have

$$d(y_m, y_n) \le d(y_m, y_{m-1}).d(y_{m-1}, y_{m-2})...d(y_{n+1}, y_n)$$

$$\le d^{h^{m-1}}(y_1, y_0).d^{h^{m-2}}(y_1, y_0)...d^{h^n}(y_1, y_0)$$

$$= d^{h^{m-1}+h^{m-2}+...+h^n}(y_1, y_0)$$

$$\le d^{\frac{h^n}{1-h}}(y_1, y_0).$$

This implies that $d(y_m, y_n) \to 1$ as $m, n \to \infty$. Hence $\{y_n\}$ is a multiplicative Cauchy. By the completeness of X, there exists $z \in X$ such that $y_n \to z$ as $n \to \infty$. Consequently, $Ax_{2n}, Bx_{2n-1}, Sx_{2n}, Tx_{2n+1}$ converges to z as $n \to \infty$.

If A and S are compatible, then

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 1$$

that is $Az = Sz$. Also, by reciprocal continuity of A and S, we have

$$\lim_{n\to\infty} ASx_{2n} = Az \text{ and } \lim_{n\to\infty} SAx_{2n} = Sz.$$

Since $A(X) \subset T(X)$, so there exists a point z in X such that $Az = Tw$ .Using (b), we have

$$d(Az, Bw) \le \{\max\{d(Sz, Tw), d(Az, Sz), d(Bw, Tw), d(Az, Tw), d(Bw, Sz)\}\}^{\lambda}$$

$$= \{\max\{d(Az, Az), d(Az, Az), d(Bw, Az), d(Az, Az), d(Bw, Az)\}\}^{\lambda}$$

$$= \{\max\{1, 1, d(Bw, Az)\}\}^{\lambda}$$

$$= d^{\lambda}(Bw, Az)$$

$$\Rightarrow d^{1-\lambda}(Bw, Az) = 1.$$

Therefore, $Az = Bw$.

Thus, $Az = Sz = Bw = Tw$.

As pointwise R-weak commutativity of A and S implies that there exists $R > 0$ such that

$$d(ASz, SAz) \le Rd(Az, Sz)$$

implies $ASz = SAz$ and $SSz = SAz = ASz = AAz$.

Similarly, pointwise R-weak commutativity of B and T implies that

$$BBw = BTw = TBw = TTw.$$

Again from (b), we have

$$d(Az, AAz) = d(Bw, AAz)$$

$$= d(AAz, Bw)$$

$$\le \{\max\{d(SAz, Tw), d(AAz, SAz), d(Bw, Tw), d(AAz, Tw), d(Bw, SAz)\}\}^{\lambda}$$

$$= \{\max\{d(AAz, \mathrm{Az}), d(AAz, AAz), d(Bw, Bw), d(AAz, Az), d(Az, AAz)\}\}^{\lambda}$$

$$= \{\max\{1, 1, d(AAz, Az)\}\}^{\lambda}.$$

Therefore, $AAz = Az$ .Thus $Az = AAz = SAz.$

Thus, Az is a common fixed point of A and S . Again from (b), we have

$$d(Bw, BBw) = d(Az, BBw)$$

$$\le \{\max\{d(Sz, TBw), d(Az, Sz), d(BBw, TBw), d(Az, TBw), d(BBw, Sz)\}\}^{\lambda}$$

$$= \{\max\{d(Bw, BBw), d(Bw, Bw), d(BBw, BBw), d(Bw, BBw), d(BBw, Bw)\}\}^{\lambda}$$

$$= \{\max\{1, 1, d(BBz, Bw)\}\}^{\lambda}.$$

Therefore, $BBw = Bw.$ Thus $Bw = BBw = TBw.$

Therefore, Bw is common fixed point of B and $T.$

If $Bw = Az = u$ then $Au = Su = Bu = Tu = u.$ Hence u is the common fixed point of A, B, S and T .

In order to prove the uniqueness of fixed point, let v be another common fixed point of A, B, S and T . Then from (b), we have

$$
\begin{aligned}
d(u,v) &= d(Au,Bv) \\
&\le \left\{\max\left\{d(Su,Tv), d(Au,Su), d(Bv,Tv), d(Au,Tv), d(Bv,Su)\right\}\right\}^{\lambda} \\
&= \left\{\max\left\{d(u,v), d(u,u), d(v,v), d(u,v), d(v,u)\right\}\right\}^{\lambda} \\
&= \left\{\max\left\{1,1,d(u,v)\right\}\right\}^{\lambda}.
\end{aligned}
$$

Therefore, $d(u,v) \to 1$. Thus, $u = v.$

This shows that fixed point is unique and hence completes the proof.

Theorem 12. Let S, T, A and B be self mappings from a complete multiplicative metric space X into itself satisfying the following conditions:

a) $S(X) \subset B(X)$ and $T(X) \subset A(X)$,

b) $\{A,S\}$ and $\{B,T\}$ are R-weakly commuting of type (P),

c) One of S, T, A and B is continuous,

d) $d(Sx,Ty) \le \left\{\max\left\{\begin{matrix} d(Ax,By), d(Ax,Sx), d(By,Ty), \\ d(Sx,By), d(Ax,Ty) \end{matrix}\right\}\right\}^{\lambda}, \lambda \in \left(0,\frac{1}{2}\right), \forall x,y \in X.$

Then S, T, A and B have a unique common fixed point in X .

Proof. Since $S(X) \subset B(X)$, consider apoint $x_0 \in X$, $\exists\ x_1 \in X$ such that $Sx_0 = Bx_1 = y_0$, $\exists x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$, $\exists\, x_{2n+1} \in X$ such that $Sx_{2n} = Bx_{2n+1} = y_{2n}$, $\exists\, x_{2n+2} \in X$ such that $Tx_{2n+2} = Ax_{2n+2} = y_{2n+1}, \ldots$

Now this we can define a sequence $\{y_n\}$ in $X,$ we obtain

$$
\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\le \left\{\max\left\{\begin{matrix} d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\ d(Sx_{2n}, Bx_{2n+1}), d(Ax_{2n}, Tx_{2n+1}) \end{matrix}\right\}\right\}^{\lambda}
\end{aligned}
$$

$$\leq \left\{ \max \left\{ d\left(y_{2n-1}, y_{2n}\right), d\left(y_{2n-1}, y_{2n}\right), d\left(y_{2n}, y_{2n+1}\right), d\left(y_{2n}, y_{2n}\right), d\left(y_{2n-1}, y_{2n+1}\right) \right\} \right\}^{\lambda}$$

$$\leq \left\{ \max \left\{ d\left(y_{2n-1}, y_{2n}\right), d\left(y_{2n}, y_{2n+1}\right), 1, d\left(y_{2n-1}, y_{2n}\right).d\left(y_{2n}, y_{2n+1}\right) \right\} \right\}^{\lambda}$$

$$= d^{\lambda}\left(y_{2n-1}, y_{2n}\right).d^{\lambda}\left(y_{2n}, y_{2n+1}\right).$$

This implies that

$$d\left(y_{2n}, y_{2n+1}\right) \leq d^{\frac{\lambda}{1-\lambda}}\left(y_{2n-1}, y_{2n}\right).$$

Let $h = \frac{\lambda}{1-\lambda}$, then

$$d\left(y_{2n}, y_{2n+1}\right) \leq d^{h}\left(y_{2n-1}, y_{2n}\right). \tag{3}$$

We also obtain

$$d\left(y_{2n+1}, y_{2n+2}\right) \leq d^{h}\left(y_{2n}, y_{2n+1}\right). \tag{4}$$

From (3) and (4), we know

$$d\left(y_{n}, y_{n+1}\right) \leq d^{h}\left(y_{n-1}, y_{n}\right) \leq .... \leq d^{h^{n}}\left(y_{1}, y_{0}\right) \text{ for all } n \geq 2 .$$

Let $m, n \in \mathbb{N},$ such that $m \geq n$, then we get

$$d\left(y_{m}, y_{n}\right) \leq d\left(y_{m}, y_{m-1}\right).d\left(y_{m-1}, y_{m-2}\right)...d\left(y_{n+1}, y_{n}\right)$$

$$\leq d^{h^{m-1}}\left(y_{1}, y_{0}\right).d^{h^{m-2}}\left(y_{1}, y_{0}\right)...d^{h^{n}}\left(y_{1}, y_{0}\right)$$

$$\leq d^{\frac{h^{n}}{1-h}}\left(y_{1}, y_{0}\right).$$

This implies that $d\left(y_{m}, y_{n}\right) \to 1$ as $m, n \to \infty$. Hence $\left\{y_{n}\right\}$ is a multiplicative Cauchy. By the completeness of X, there exists $z \in X$ such that $y_{n} \to z$ as $n \to \infty$.

Moreover, since $\left\{Sx_{2n-1}\right\} = \left\{Bx_{2n-2}\right\} = \left\{y_{2n-1}\right\}$ and $\left\{Tx_{2n}\right\} = \left\{Ax_{2n-1}\right\} = \left\{y_{2n}\right\}$ are subsequences of $\left\{y_{n}\right\}$, so we obtain

$$\lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Bx_{2n-2} = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Ax_{2n-1} = z.$$

Case 1: Suppose A is continuous, then

$$\lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} AAx_{2n} = Az.$$

Since $\left\{A, S\right\}$ is R-weakly commuting maps of type (P), we have

$$d\left(SSx_{2n}, AAx_{2n}\right) \leq Rd\left(Ax_{2n}, Sx_{2n}\right).$$

Let $n \to \infty$, we get

$$\lim_{n\to\infty} SAx_{2n} = Az.$$

Now, we obtain

$$d\left(SAx_{2n},Tx_{2n+1}\right)\le\left\{\max\left\{\begin{matrix} d\left(AAx_{2n},Bx_{2n+1}\right),d\left(AAx_{2n},SAx_{2n}\right),d\left(Bx_{2n+1},Tx_{2n+1}\right), \\ d\left(SAx_{2n},Bx_{2n+1}\right),d\left(AAx_{2n},Tx_{2n+1}\right) \end{matrix}\right\}\right\}^{\lambda}.$$

Letting $n\to\infty$, we obtain

$$d\left(Az,z\right)\le\left\{\max\left\{d\left(Az,z\right),d\left(Az,Az\right),d\left(z,z\right),d\left(Az,z\right),d\left(Az,z\right)\right\}\right\}^{\lambda}$$

$$=\left\{\max\left\{d\left(Az,z\right),1\right\}\right\}^{\lambda}$$

$$=d^{\lambda}\left(Az,z\right).$$

This implies that $d\left(Az,z\right)=1$ i.e. $Az=z.$

Again, we can obtain

$$d\left(Sz,Tx_{2n+1}\right)\le\left\{\max\left\{\begin{matrix} d\left(Az,Bx_{2n+1}\right),d\left(Az,Sz\right),d\left(Bx_{2n+1},Tx_{2n+1}\right), \\ d\left(Sz,Bx_{2n+1}\right),d\left(Az,Tx_{2n+1}\right) \end{matrix}\right\}\right\}^{\lambda}.$$

Letting $n\to\infty$, we obtain

$$d\left(Sz,z\right)\le\left\{\max\left\{d\left(Az,z\right),d\left(z,Sz\right),d\left(z,z\right),d\left(Sz,z\right),d\left(z,z\right)\right\}\right\}^{\lambda}$$

$$=\left\{\max\left\{d\left(Sz,z\right),1\right\}\right\}^{\lambda}$$

$$=d^{\lambda}\left(Sz,z\right)$$

which implies that $d\left(Sz,z\right)=1$ i.e. $Sz=z.$

Now, $z=Sz\in S\left(X\right)\subseteq B\left(X\right)$, so $\exists z^{*}\in X$ such that $z=Bz^{*}$. Then

$$d\left(z,Tz^{*}\right)=d\left(Sz,Tz^{*}\right)$$

$$\le\left\{\max\left\{\begin{matrix} d\left(Az,Bz^{*}\right),d\left(Az,Sz\right),d\left(Bz^{*},Tz^{*}\right), \\ d\left(Sz,Bz^{*}\right),d\left(Az,Tz^{*}\right) \end{matrix}\right\}\right\}^{\lambda}$$

$$=\left\{\max\left\{d\left(z,Tz^{*}\right),1\right\}\right\}^{\lambda}$$

$$=d^{\lambda}\left(z,Tz^{*}\right)$$

which implies that $d\left(z,Tz^{*}\right)=1$ i.e. $Tz^{*}=z.$

Since $\{B,T\}$ is R-weakly commuting maps of type (P), we have

$$d(Bz,Tz)=d(BBz^*,TTz^*)\le Rd(Tz^*,Bz^*)=Rd(z,z)=R.$$

But $R>0$ and $d(x,y)\ge 1$. Therefore, $d(Bz,Tz)=1$, so $Bz=Tz$. Lastly, we have

$$\begin{aligned} d(z,Tz) &= d(Sz,Tz) \\ &\le \{\max\{d(Az,Bz),d(Az,Sz),d(Bz,Tz),d(Sz,Bz),d(Az,Tz)\}\}^{\lambda} \\ &= \{\max\{d(z,Tz),1\}\}^{\lambda} \\ &= d^{\lambda}(z,Tz) \end{aligned}$$

which implies that $d(z,Tz)=1$ i.e. $Tz=z$.

Case 2: Suppose that B is continuous, we can obtain the same result bythe way of case 1.

Case 3: Suppose that S is continuous, then

$$\lim_{n\to\infty} SAx_{2n}=\lim_{n\to\infty} SSx_{2n}=Sz.$$

Since $\{A,S\}$ is R-weakly commuting maps of type (P), then

$$d(AAx_{2n},SSx_{2n})\le Rd(Sx_{2n},Ax_{2n}).$$

Let $n\to\infty$, we get

$$\lim_{n\to\infty} ASx_{2n}=Sz.$$

Now, we obtain

$$d(SSx_{2n},Tx_{2n+1})\le\left\{\max\left\{\begin{matrix} d(ASx_{2n},Bx_{2n+1}),d(ASx_{2n},SSx_{2n}),d(Bx_{2n+1},Tx_{2n+1}), \\ d(SSx_{2n},Bx_{2n+1}),d(ASx_{2n},Tx_{2n+1}) \end{matrix}\right\}\right\}^{\lambda}.$$

Letting $n\to\infty$, we obtain

$$\begin{aligned} d(Sz,z) &\le \{\max\{d(Sz,z),d(Sz,Sz),d(z,z),d(Sz,z),d(Sz,z)\}\}^{\lambda} \\ &= \{\max\{d(Sz,z),1\}\}^{\lambda} \\ &= d^{\lambda}(Sz,z). \end{aligned}$$

This implies that $d(Sz,z)=1$ i.e. $Sz=z$.

Since, $z=Sz\in S(X)\subseteq B(X)$, so $\exists z^*\in X$ such that $z=Bz^*$. Then

$$d\left(SSx_n, Tz^*\right) \le \left\{\max\left\{\begin{matrix} d\left(ASx_n, Bz^*\right), d\left(ASx_n, SSx_n\right), d\left(Bz^*, Tz^*\right), \\ d\left(SSx_n, Bz^*\right), d\left(ASx_n, Tz^*\right) \end{matrix}\right\}\right\}^{\lambda}.$$

Letting $n \to \infty$, we can obtain

$$d\left(Sz, Tz^*\right) \le \left\{\max\left\{\begin{matrix} d\left(Sz, z\right), d\left(Sz, Sz\right), d\left(z, Tz^*\right), \\ d\left(Sz, z\right), d\left(Sz, Tz^*\right) \end{matrix}\right\}\right\}^{\lambda}$$

$$= \left\{\max\left\{d\left(z, Tz^*\right), 1\right\}\right\}^{\lambda}$$

$$= d^{\lambda}\left(z, Tz^*\right)$$

which implies that $d\left(z, Tz^*\right) = 1$ i.e. $Tz^* = z.$

Since $\{B, T\}$ is R-weakly commuting maps of type (P), we have

$$d\left(Tz, Bz\right) = d\left(TTz^*, BBz^*\right) \le Rd\left(Bz^*, Tz^*\right) = Rd\left(z, z\right) = R.$$

But $R > 0$ and $d\left(x, y\right) \ge 1.$ Therefore, $d\left(Bz, Tz\right) = 1,$ so $Bz = Tz.$

Lastly, we have

$$d\left(Sx_{2n}, Tz\right) \le \left\{\max\left\{\begin{matrix} d\left(Ax_{2n}, Bz\right), d\left(Ax_{2n}, Sx_{2n}\right), d\left(Bz, Tz\right), \\ d\left(Sx_{2n}, Bz\right), d\left(Ax_{2n}, Tz\right) \end{matrix}\right\}\right\}^{\lambda}.$$

Letting $n \to \infty,$ we can obtain

$$d\left(z, Tz\right) \le \left\{\max\left\{d\left(z, Tz\right), d\left(z, z\right), d\left(Tz, Tz\right), d\left(z, Tz\right), d\left(z, Tz\right)\right\}\right\}^{\lambda}$$

$$\Rightarrow d\left(z, Tz\right) \le \left\{\max\left\{d\left(z, Tz\right), 1\right\}\right\}^{\lambda}$$

$$= d^{\lambda}\left(z, Tz\right)$$

which implies that $d\left(z, Tz\right) = 1$ i.e. $Tz = z.$

Now, $z = Tz \in T\left(X\right) \subseteq A\left(X\right),$ so $\exists z^{**} \in X$ such that $z = Az^{**}.$ Then

$$d\left(Sz^{**}, z\right) = d\left(Sz^{**}, z\right)$$

$$\le \left\{\max\left\{\begin{matrix} d\left(Az^{**}, Bz\right), d\left(Az^{**}, Sz^{**}\right), d\left(Bz, Tz\right), \\ d\left(Sz^{**}, Bz\right), d\left(Az^{**}, Tz\right) \end{matrix}\right\}\right\}^{\lambda}$$

$$= \left\{ \max \left\{ \begin{array}{l} d(z,z), d(z, Sz^{**}), d(Bz, Bz), \\ d(Sz^{**}, z), d(z,z) \end{array} \right\} \right\}^{\lambda}$$

$$= \left\{ \max \left\{ d(Sz^{**}, z), 1 \right\} \right\}^{\lambda}$$

$$= d^{\lambda}(Sz^{**}, z)$$

this implies that $d(Sz^{**}, z) = 1$ i.e. $Sz^{**} = z.$

Since $\{S, A\}$ is R-weakly commuting maps of type (P), we have

$$d(Az, Sz) = d(AAz^{**}, SSz^{**}) \leq Rd(Sz^{**}, Az^{**}) = Rd(z,z) = R$$

so $Az = Sz.$

We obtain $Sz = Tz = Az = Bz = z,$ so z is a common fixed point of S, T, A and $B.$

Case 4: Suppose that T is continuous, we can obtain the same result bythe way of case 3.

In addition, we prove that S, T, A and B have a unique common fixed point. Suppose that $w \in X$ is also a common fixed point of S, T, A and B, then

$$d(z,w) = d(Sz, Tw)$$

$$\leq \left\{ \max \left\{ d(Az, Bw), d(Az, Sz), d(Bw, Tw), d(Sz, Bw), d(Az, Tw) \right\} \right\}^{\lambda}$$

$$= \left\{ \max \left\{ d(z,w), 1 \right\} \right\}^{\lambda}$$

$$= d^{\lambda}(z,w)$$

this implies that $d(z,w) = 1$ i.e. $z = w,$ which is a contradiction.

Hence, S, T, A and B have a unique common fixed point.

Example 2. Let $X = R$ be a usual metric space. Define the mappings $d : X \times X \to R^{+}$ by $d(\mathrm{x}, \mathrm{y}) = \mathrm{e}^{|x-y|}$ for all $x, y \in X$. Clearly (X, d) is a completemultiplicative metric space. Consider the following mappings: $Sx = x, Tx = \frac{1}{2}x, Bx = 3x, Ax = 2x$ for all $x \in X$

a) $SX = TX = BX = AX = X$, so $SX \subset BX, TX \subset AX$;

b) $\{A, S\}, \{B, T\}$ are R-weakly commuting maps of type (P);

c) One of S, T, A and B is continuous;

d) Let $\lambda = \frac{1}{3}$, then

$$d\left(Sx,Ty\right) \leq \left\{\max\left\{d\left(Ax,By\right),d\left(Bx,Tx\right),d\left(By,Ty\right),d\left(Sx,B\,\mathrm{y}\right),d\left(Ax,Ty\right)\right\}\right\}^{\lambda}$$

$$\Rightarrow e^{\left|x-\frac{1}{2}y\right|} \leq \left\{\max\left\{e^{|3x-2y|},e^{|2x|},e^{\left|\frac{3}{2}y\right|},e^{|2y-x|},e^{\left|3x-\frac{1}{2}y\right|}\right\}\right\}^{\lambda}$$

$$= \max\left\{e^{|3x-2y|\lambda},e^{|2x|\lambda},e^{\left|\frac{3}{2}y\right|\lambda},e^{|2y-x|\lambda},e^{\left|3x-\frac{1}{2}y\right|\lambda}\right\}.$$

Because $y = \ln x$ is an increasing mapping, so

$$\Leftrightarrow \left|x-\frac{1}{2}y\right| = \max\left\{|3x-2y|\lambda,|2x|\lambda,\left|\frac{3}{2}y\right|\lambda,|2y-x|\lambda,\left|3x-\frac{1}{2}y\right|\lambda\right\}.$$

There are three situations: $(i)\, x \geq \frac{1}{2}y \geq 0$ or $\frac{1}{2}y \geq x \geq 0$; $(ii)\, \frac{1}{2}y < x < 0$ or $x < \frac{1}{2}y < 0$; $(iii)\, x > 0, y < 0$ or $x < 0, y > 0$. No matter what kindof situation, inequality (iii) is true. So all the condition of main theorem aretrue, then we obtain $S0 = T0 = A0 = B0 = 0$, so 0 is the unique common fixed point of S,T,A and B.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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