



Operational Matrices to Solve Nonlinear Volterra-Fredholm Integro-Differential Equations of Multi-Arbitrary Order

Kourosh PARAND^{1,2, *}, Mehdi DELKHOSH²

¹ *Department of Computer Sciences, Shahid Beheshti University, G.C., Tehran, Iran.*

² *Department of Cognitive Modelling, Institute for Cognitive and Brain Sciences, Shahid Beheshti University, G.C, Tehran, Iran.*

Received: 28/06/2016

Accepted: 16/08/2016

ABSTRACT

Fractional calculus has been used for modelling many of physical and engineering processes, that many of them are described by linear and nonlinear Volterra- Fredholm integro- differential equations of multi-arbitrary order. Therefore, an efficient and suitable method for the solution of them is very important. In this paper, the generalized fractional order of the Chebyshev functions (GFCEs) based on the classical Chebyshev polynomials of the first kind is used to obtain the solution of the linear and nonlinear multi-order Volterra-Fredholm integro-differential equations. Also, the operational matrices of the fractional derivative, the product, and the fractional integration to transform the equations to a system of algebraic equations are introduced. Some examples are included to demonstrate the validity and applicability of the technique.

Keywords: Fractional order of Chebyshev functions; Operational matrix; Volterra-Fredholm integro- differential equations; Tau-Collocation method.

1. INTRODUCTION

In this section, some definitions which are useful for our method have been introduced [1].

Definition 1. For any real function $f(t)$, $t > 0$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, is said to be in space C_μ .

*Corresponding author, e-mail: k_parand@sbu.ac.ir

$\mu \in \mathfrak{R}$, and it is in the space C_μ^n if and only if $f^n \in C_\mu, n \in N$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a function $f \in C_\mu, \mu > -1$, is defined as [2]

$$\begin{aligned}
 {}_0I_t^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0, \\
 {}_0I_t^0 f(t) &= f(t).
 \end{aligned}
 \tag{1}$$

Some properties of the operator ${}_0I_t^\alpha$ (for simplicity I^α), which are needed here, are as follows. For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0, \gamma \geq -1$,

$$(i) I^\alpha I^\beta = I^{\alpha+\beta},$$

$$(i) D^\alpha C = 0,$$

$$(ii) D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t), \tag{3}$$

$$(iii) D^\alpha t^\gamma = \begin{cases} 0 & \gamma \in N_0 \text{ and } \gamma < \alpha, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & \text{Otherwise.} \end{cases} \tag{4}$$

$$(iv) D^\alpha (\sum_{i=1}^n c_i f_i(t)) = \sum_{i=1}^n c_i D^\alpha f_i(t), \quad \text{where } c_i \in R. \tag{5}$$

$$(v) (D^\alpha I^\alpha f)(t) = f(t),$$

$$(vi) (I^\alpha D^\alpha f)(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{(t-\alpha)^k}{k!}, \quad t > 0.$$

Definition 4. Suppose that $f(t), g(t) \in C(0, \eta], \eta > 0$ and $w(t)$ is a weight function, then we define:

$$\|f(t)\|_w^2 = \int_0^\eta f^2(t)w(t)dt,$$

$$\langle f(t), g(t) \rangle_w = \int_0^\eta f(t)g(t)w(t)dt.$$

Definition 5 (completeness). A basis set is "complete" for a given class of functions if all functions within the class

$$\sum_{j=1}^{N_1} \mu_j D^{\gamma_j} y(x) + \sum_{j=1}^{N_2} h_j(x) y^{\gamma_j} + \lambda_1 \int_0^x k_1(x,t) y^p dt + \lambda_2 \int_0^1 k_2(x,t) y^q dt = g(x), \tag{6}$$

with these supplementary conditions:

$$y^{(i)}(x_0) = y_0, \quad i = 0, 1, \dots, s-1, \quad \text{s.t. } s-1 < \max\{\gamma_j\} \leq s, \quad s \in N, \tag{7}$$

$$\begin{aligned}
 (ii) I^\alpha I^\beta &= I^\beta I^\alpha, \\
 (iii) (I^\alpha I^\beta f)(t) &= (I^\beta I^\alpha f)(t), \\
 (iv) I^\alpha (t-a)^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\gamma+\alpha}.
 \end{aligned}
 \tag{2}$$

Definition 3. The fractional derivative of $f(t)$ in the Caputo sense is defined as follows [3]

$$\begin{aligned}
 D^\alpha f(t) &= I^{m-\alpha} D^m f(t), \\
 \text{for } m-1 < \alpha &\leq m, m \in N, t > 0 \text{ and } f \in C_{-1}^m.
 \end{aligned}$$

Some properties of the operator D^α , which are needed here, are as follows, for $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0, \gamma \geq -1, N_0 = \{0, 1, 2, \dots\}$ and constant C ,

can be represented to arbitrarily high accuracy as a sum of a sufficiently large number of basis functions [4].

Theorem 1. Suppose that $\{P_i(t)\}$ be a sequence of orthogonal polynomials, $w(t)$ is a weight function for $\{P_i(t)\}$, and $q(t)$ is a polynomial of degree at most $n-1$, then for $p_n(t) \in \{P_i(t)\}$ we have: $\langle p_n(t), q(t) \rangle_w = 0$.

Proof: See the section 2.3 in Ref. [5].

The aim of the paper is to present a numerical method (GFCF Tau-Collocation method) for approximating the solution of a nonlinear Volterra- Fredholm integro-differential equations of multi-arbitrary order as follows:

where $x \in [0, \eta], y_j \geq 0$ and $h_j, g \in L^2([0, \eta]), k_1, k_2 \in L^2([0, \eta]^2)$ are known functions, $y(x)$ is the unknown function, D^{ν_j} is the Caputo fractional differentiation operator, $\mu_j, \lambda_1, \lambda_2$ are real numbers, and N_1, N_2, r_j, p, q are positive integers.

The rest of the paper is as follows: in section 2, the GFCFs, their properties and operational matrices for them are expressed. In Section 3, the numerical method is explained. Applications of the proposed method are shown in section 4. Finally, a conclusion is provided.

2. GENERALIZED FRACTIONAL ORDER OF THE CHEBYSHEV FUNCTIONS

The Chebyshev polynomials of the first kind, $T_n(x)$, can be obtained using the recursive relation as follows [6]:

$$T_0(x) = 1, T_1(x) = x, \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n = 1, 2, \dots$$

The Chebyshev polynomials have many properties, for example, they are orthogonal, are defined recursively, have simple real roots, are complete in the space of polynomials. For these reasons, many authors have used these functions in their works [7, 8, 9, 10].

Using some transformations, some researchers extended Chebyshev polynomials to infinite or semi-infinite domains, for example by using $x = \frac{t-L}{t+L}, L > 0$ the rational Chebyshev functions on semi-infinite domain [11, 12, 13, 14], by using $x = \frac{t}{\sqrt{t^2+L}}, L > 0$ the rational Chebyshev functions on infinite domain [6], and by using $x = 1 - 2(\frac{t}{\eta})^\alpha, \alpha, \eta > 0$ the generalized fractional order of the Chebyshev functions (GFCF) [15] are introduced.

Darani and Nasiri in [16] have introduced the fractional-order Chebyshev functions of the second kind, then have just constructed the derivative operational matrix for them, and have used it to solve linear fractional differential equations.

In the present work, we use the transformation $x = 1 - 2(\frac{t}{\eta})^\alpha, \alpha, \eta > 0$ on the Chebyshev polynomials of the first kind, that was introduced by Parand and Delkhosh [15], and can use to solve nonlinear Volterra- Fredholm integro- differential equations of multi-arbitrary order.

The GFCFs are defined in interval $[0, \eta]$, and are denoted by ${}_\eta FT_n^\alpha(t) = T_n(1 - 2(\frac{t}{\eta})^\alpha)$. The analytical form of ${}_\eta FT_n^\alpha(t)$ of degree $n\alpha$ is given by [15]

$${}_\eta FT_n^\alpha(t) = \sum_{k=0}^n \beta_{n,k,\eta,\alpha} t^{\alpha k}, t \in [0, \eta], \quad (8)$$

where

$$\beta_{n,k,\eta,\alpha} = (-1)^k \frac{n 2^{2k} (n+k-1)!}{(n-k)! (2k)! \eta^{\alpha k}} \text{ and } \beta_{0,k,\eta,\alpha} = 1.$$

Note that ${}_\eta FT_n^\alpha(0) = 1$ and ${}_\eta FT_n^\alpha(\eta) = (-1)^n$.

The GFCFs are orthogonal with respect to the weight function $w(t) = \frac{t^{\frac{\alpha}{2}-1}}{\sqrt{\eta^\alpha - t^\alpha}}$ in the interval $[0, \eta]$:

$$\int_0^\eta {}_\eta FT_n^\alpha(t) {}_\eta FT_m^\alpha(t) w(t) dt = \frac{\pi}{2\alpha} c_n \delta_{mn}. \quad (9)$$

where δ_{mn} is Kronecker delta, $c_0 = 2$, and $c_n = 1$ for $n \geq 1$.

Any function $y(t), t \in [0, \eta]$, can be expanded as follows [4, 6]:

$$y(t) = \sum_{n=0}^\infty a_n {}_\eta FT_n^\alpha(t),$$

and using the property of orthogonality in the GFCFs:

$$a_n = \frac{2\alpha}{\pi c_n} \int_0^\eta {}_\eta FT_n^\alpha(t) y(t) w(t) dt, \quad n = 0, 1, 2, \dots,$$

but in the numerical methods, we have to use first m -terms of the GFCFs and approximate $y(t)$:

$$y(t) \approx y_m(t) = \sum_{n=0}^{m-1} a_n {}_\eta FT_n^\alpha(t) = A^T \Phi(t), \quad (10)$$

with

$$A = [a_0, a_1, \dots, a_{m-1}]^T, \quad (11)$$

$$\Phi(t) = [{}_\eta FT_0^\alpha(t), {}_\eta FT_1^\alpha(t), \dots, {}_\eta FT_{m-1}^\alpha(t)]^T. \quad (12)$$

The following theorem shows that by increasing m , the approximation solution $f_m(t)$ is convergent to $f(t)$ exponentially.

Theorem 2. Suppose that $D^{k\alpha} f(t) \in C[0, \eta]$ for $k = 0, 1, \dots, m$, and ${}_\eta F_m^\alpha$ is the subspace generated by $\{ {}_\eta FT_0^\alpha(t), {}_\eta FT_1^\alpha(t), \dots, {}_\eta FT_{m-1}^\alpha(t) \}$. If $f_m = A^T \Phi$ (in the Eq. (10)) is the best approximation to $f(t)$ from ${}_\eta F_m^\alpha$, then the error bound is presented as follows

$$\| f(t) - f_m(t) \|_w \leq \frac{\eta^{m\alpha} M_\alpha}{2^m \Gamma(m\alpha + 1)} \sqrt{\frac{\pi}{\alpha \cdot m!}}$$

where $M_\alpha \geq |D^{m\alpha} f(t)|, t \in [0, \eta]$.

Proof. See Ref. [15]. *

Theorem 3. The generalized fractional order of the Chebyshev function ${}_\eta FT_n^\alpha(t)$, has precisely n real zeros on interval $(0, \eta)$ in the form

$$t_k = \eta \left(\frac{1 - \cos(\frac{(2k-1)\pi}{2n})}{2} \right)^{\frac{1}{\alpha}}, \quad k = 1, 2, \dots, n.$$

Moreover, $\frac{d}{dt} FT_n^\alpha(t)$ has precisely $n - 1$ real zeros on interval $(0, \eta)$ in the following points:

$$t'_k = \eta \left(\frac{1 - \cos\left(\frac{k\pi}{n}\right)}{2} \right)^{\frac{1}{\alpha}}, \quad k = 1, 2, \dots, n - 1.$$

Proof. See Ref. [15]. *

Now, operational matrices of the fractional derivative, the fractional integration and the product for the GFCFs are constructed. These matrices reduce the computational and storage costs and can be used to solve the linear and nonlinear differential equations.

$$D_{i,j}^{(\alpha)} = \begin{cases} \frac{2}{\sqrt{\pi}c_j} \sum_{k=1}^i \sum_{s=0}^j \beta_{i,k,\eta,\alpha} \beta_{j,s,\eta,\alpha} \frac{\Gamma(\alpha k + 1) \Gamma(s + k - \frac{1}{2}) \eta^{\alpha(k+s-1)}}{\Gamma(\alpha k - \alpha + 1) \Gamma(s + k)}, & i > j \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Proof. See Ref. [15]. *

Remark 1: By theorem 4, we can see that the fractional derivative operational matrix of the GFCFs is a lower triangular matrix, so at least $50(1 + \frac{1}{m})\%$ of the matrix elements are zero, that this reduces the computational and storage costs.

For example, with $\alpha = 0.5, \eta = 10$, and *Digits* = 15, we have:

$$D^{(0.5)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.560499121639792 & 0 & 0 & 0 & 0 \\ -0.81469719363697 & 1.42729929292212 & 0 & 0 & 0 \\ 1.525188431982950 & -1.8378062978560 & 1.6814973649180 & 0 & 0 \\ -1.74727960689570 & 3.28405603947035 & -2.033584576000 & 1.903065723624 & 0 \end{bmatrix}.$$

In the next theorem, the operational matrix of the product of two GFCFs vectors is generalized.

Theorem 5. Let $\Phi(t)$ be GFCFs vector in Eq. (12) and A be a vector, then the elements of \hat{A} , that is an $m \times m$ product operational matrix for the vector $A = \{a_i\}_{i=0}^{m-1}$, are obtained as

$$\Phi(t)\Phi(t)^T A \approx \hat{A}\Phi(t), \tag{15}$$

where

$$\hat{A}_{ij} = \sum_{k=0}^{m-1} a_k \hat{g}_{ijk}, \tag{16}$$

where

$$\hat{g}_{ijk} = \begin{cases} \frac{c_k}{2c_j} & i \neq 0 \text{ and } j \neq 0 \text{ and } (k = i + j \text{ or } k = |i - j|) \\ \frac{c_k}{c_j} & (j = 0 \text{ and } k = i) \text{ or } (i = 0 \text{ and } k = j) \\ 0 & \text{otherwise} \end{cases}$$

Proof. See Ref. [15]. *

Remark 2: By theorem 5, we can be shown that any product operational matrix of GFCFs can be made to the sum of two simple matrices: $\hat{A} = \frac{1}{2}\hat{B} + \frac{1}{2}\hat{C}$ where

1. \hat{B} is a symmetric matrix, that the diagonals of \hat{B} are the elements of vector A , and for $i, j = 0, 1, \dots, m - 1$, matrix elements are calculated as follows:

$$\hat{B}_{i,j} = \begin{cases} 2a_0 & i = j \\ a_k & i \neq j, |i - j| = k. \end{cases}$$

In the next theorem, the operational matrix of the Caputo fractional derivative of order $\alpha > 0$ for the GFCFs is generalized.

Theorem 4. Let $\Phi(t)$ be GFCFs vector in Eq. (12), and $D^{(\alpha)}$ be an $m \times m$ fractional derivative operational matrix of the Caputo fractional derivatives of order $\alpha > 0$ as follows:

$$D^\alpha \Phi(t) = D^{(\alpha)} \Phi(t). \tag{13}$$

where for $i, j = 0, 1, \dots, m - 1$:

2. \hat{C} is a sparse matrix, that at least $50(1 + \frac{1}{m})\%$ of the matrix elements are zero, and for $i, j = 0, 1, \dots, m - 1$, matrix elements are calculated as follows:

$$\hat{C}_{i,j} = \begin{cases} a_{i+j} & j \neq 0, 1 \leq i + j \leq m - 1 \\ 0 & \text{otherwise.} \end{cases}$$

For example, with $m = 5$, at least 60% of the elements in \hat{C} are zero, and the product operational matrix of GFCFs is as follows:

$$\hat{A} = \frac{1}{2} \begin{bmatrix} 2a_0 & 2a_1 & 2a_2 & 2a_3 & 2a_4 \\ a_1 & 2a_0 + a_2 & a_1 + a_3 & a_2 + a_4 & a_3 \\ a_2 & a_1 + a_3 & 2a_0 + a_4 & a_1 & a_2 \\ a_3 & a_2 + a_4 & a_1 & 2a_0 & a_1 \\ a_4 & a_3 & a_2 & a_1 & 2a_0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2a_0 & a_1 & a_2 & a_3 & a_4 \\ a_1 & 2a_0 & a_1 & a_2 & a_3 \\ a_2 & a_1 & 2a_0 & a_1 & a_2 \\ a_3 & a_2 & a_1 & 2a_0 & a_1 \\ a_4 & a_3 & a_2 & a_1 & 2a_0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & a_2 & a_3 & a_4 & 0 \\ 0 & a_3 & a_4 & 0 & 0 \\ 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see that the computational cost of production is very low.

In the following theorem, the operational matrix of fractional integration of GFCFs is generalized.

Theorem 6. Let $\Phi(t)$ be GFCFs vector in Eq. (12), $I^{(\alpha)}$ is the $m \times m$ operational matrix of fractional integration of order $\alpha > 0$:

$$I^\alpha \Phi(t) \approx I^{(\alpha)} \Phi(t). \tag{17}$$

then the elements of $I^{(\alpha)}$ are obtained as

$$I_{i,j}^{(\alpha)} = \begin{cases} \frac{2}{\sqrt{\pi}c_j} \sum_{k=0}^i \sum_{s=0}^j \beta_{i,k,\eta,\alpha} \beta_{j,s,\eta,\alpha} \frac{\Gamma(\alpha k + 1) \Gamma(s + k + \frac{3}{2}) \eta^{\alpha(k+s+1)}}{\Gamma(\alpha k + \alpha + 1) \Gamma(s + k + 2)}, & j \geq i - 1 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

for $i, j = 0, \dots, m - 1$.

Proof. Using Eq. (13)

$$\begin{bmatrix} I_{0,0} & \dots & I_{0,j} & \dots & I_{0,m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I_{i,0} & \dots & I_{i,j} & \dots & I_{i,m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I_{m-1,0} & \dots & I_{m-1,j} & \dots & I_{m-1,m-1} \end{bmatrix} \begin{bmatrix} \Phi_0 \\ \vdots \\ \Phi_j \\ \vdots \\ \Phi_{m-1} \end{bmatrix} \approx \begin{bmatrix} I^\alpha \Phi_0 \\ \vdots \\ I^\alpha \Phi_i \\ \vdots \\ I^\alpha \Phi_{m-1} \end{bmatrix}$$

By orthogonality property of the GFCFs, the Eq. (8), and properties of I^α , for $i, j = 0, 1, \dots, m - 1$, we have

$$I_{i,j}^{(\alpha)} = \frac{2\alpha}{\pi c_j} \int_0^\eta I^\alpha ({}_\eta FT_i^\alpha(t)) ({}_\eta FT_j^\alpha(t)) w(t) dt. \tag{19}$$

If $i < j - 1$ then $deg(I^\alpha ({}_\eta FT_i^\alpha(t))) < deg({}_\eta FT_j^\alpha(t))$, therefore by theorem 2, $I_{i,j}^{(\alpha)} = 0$ for any $i < j - 1$. Now for $i \geq j - 1$ we have:

$$\begin{aligned} I_{i,j}^{(\alpha)} &= \frac{2\alpha}{\pi c_j} \int_0^\eta \sum_{k=0}^i \beta_{i,k,\eta,\alpha} \frac{\Gamma(\alpha k + 1) t^{\alpha k + \alpha}}{\Gamma(\alpha k + \alpha + 1)} \sum_{s=0}^j \beta_{j,s,\eta,\alpha} t^{\alpha s} \frac{t^{\frac{\alpha}{2} - 1}}{\sqrt{\eta^2 - t^\alpha}} dt \\ &= \frac{2\alpha}{\pi c_j} \sum_{k=0}^i \sum_{s=0}^j \beta_{i,k,\eta,\alpha} \beta_{j,s,\eta,\alpha} \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha k + \alpha + 1)} \int_0^\eta \frac{t^{\alpha(k+s+\frac{3}{2})-1}}{\sqrt{\eta^2 - t^\alpha}} dt. \end{aligned}$$

Now, by integration of the above equation, the Eq. (18) can be proved. *

Remark 3: By theorem 6, we can see that the fractional integration operational matrix of the GFCFs is a lower Hessenberg matrix, so at least $50(1 - \frac{3m-2}{m^2})\%$ of the matrix elements are zero, that this reduces the computational and storage costs.

For example, with $\alpha = 0.5, \eta = 10$, and $Digits = 15$, we have:

$$I^{(0.5)} = \begin{bmatrix} 1.78412411615278 & -1.7841241161527 & 0 & 0 & 0 \\ -0.3177475899964 & 1.01837149204616 & -0.70062390204978 & 0 & 0 \\ -0.676282321268230 & 0.50523773587927 & 0.765752624106516 & -0.59470803871789 & 0 \\ 0.158383655059499 & -0.6591832408554 & 0.390772332343744 & 0.63549518000062 & -0.52546792674132 \\ -0.08866456642902 & 0.14207458391164 & -0.45786161413985 & 0.32629713013093 & 0.55392089779850 \end{bmatrix}$$

Remark 4: The fractional integration operational matrix of GFCFs for $\alpha = 1$ and $\eta = 1$ is the same as the integration operational matrix of shifted Chebyshev polynomials [17, 18].

Remark 5: By the Eq. (8), we have

$$I^1 \eta FT_n^\alpha(t) = \int_0^t \eta FT_n^\alpha(s) ds = \sum_{k=0}^n \beta_{n,k,\eta,\alpha} \frac{t^{\alpha k+1}}{\alpha k+1}$$

3. APPLICATION OF THE METHOD

Consider Eq. (6), by previous section, the two variable functions $k(x, t) \in L^2([0,1])^2$ can be approximated as:

$$k(x, t) \approx \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} k_{i,j} \phi(x) \phi(t)$$

or in the matrix form:

$$k(x, t) \approx \Phi^T(x) K \Phi(t) \tag{20}$$

where $K = [k_{i,j}]$ and also can be written $y(x) \approx A^T \Phi(x)$ in Eq. (10) and:

$$g(x) \approx \sum_{n=0}^{m-1} g_n \eta FT_n^\alpha(x) = G^T \Phi(x), \tag{21}$$

where g_n are obtained as

$$g_n = \frac{2\alpha}{\pi c_n} \int_0^1 \eta FT_n^\alpha(t) g(t) w(t) dt.$$

Using the Eqs. (21) and (10), we have $y^2(x) \approx A^T \hat{A} \Phi(x)$ and it is easy to show by induction that for $p = 1, 2, \dots$:

$$y^p(x) \approx A^T (\hat{A})^{p-1} \Phi(x) = \Phi^T(x) ((\hat{A})^{p-1})^T A, \tag{22}$$

by transformation $A_p = ((\hat{A})^{p-1})^T A$, we have $y^p(x) \approx \Phi^T(x) A_p$.

Using the above equations, we have:

$$\begin{aligned} \int_0^x k_1(x, t) y^p(t) dt &= \int_0^x \Phi^T(x) K_1 \Phi(t) \Phi^T(t) A_p dt \\ &= \Phi^T(x) K_1 \int_0^x \Phi(t) \Phi^T(t) A_p dt \\ &= \Phi^T(x) K_1 \int_0^x \widetilde{A}_p \Phi(t) dt \\ &= \Phi^T(x) K_1 \widetilde{A}_p \int_0^x \Phi(t) dt \end{aligned}$$

$$= \Phi^T(x) K_1 \widetilde{A}_p \Psi(x), \tag{23}$$

where \widetilde{A}_p is the product operational matrix of vector A_p and $\Psi(x)$ is vector:

$$\Psi(x) = \{\psi_i(x)\}_{i=0}^{m-1} = \left\{ \int_0^x \eta FT_i^\alpha(t) dt \right\}_{i=0}^{m-1} \tag{24}$$

where the $\psi_i(x)$ can be calculated by using of the remark 5.

Using the Eq. (23), we have:

$$\int_0^1 k_2(x, t) y^q(t) dt = \Phi^T(x) K_2 \widetilde{A}_q \Psi(1). \tag{25}$$

And we have:

$$h_j(x) y^r(x) \approx B_j^T \Phi(x) \Phi^T(x) A_{r_j} = B_j^T \widetilde{A}_{r_j} \Phi(x), \tag{26}$$

where \widetilde{A}_{r_j} is the product operational matrix of vector A_{r_j} and $h_j(x) \approx B_j^T \Phi(x)$.

Now, we must choose the value of α such that values γ_j be multiples of α . Using the properties of the operator D^α and Eq. (13), we can calculate the values $D^{(\gamma_j)}$ and:

$$D^{\gamma_j} y(x) \approx A^T D^{\gamma_j} \Phi(x) = A^T D^{(\gamma_j)} \Phi(x). \tag{27}$$

By substituting the approximations above into Eq. (6) we obtain:

$$\sum_{j=0}^{N_1} \mu_j A^T D^{(y_j)} \Phi + \sum_{j=0}^{N_2} B_j^T \bar{A}_{r_j} \Phi + \lambda_1 \Phi^T K_1 \bar{A}_p \Psi(x) + \lambda_2 \Phi^T K_2 \bar{A}_q \Psi(1) = G^T \Phi. \tag{28}$$

We define the residual function as follows:

$$Res(x) = \left(\sum_{j=0}^{N_1} \mu_j A^T D^{(y_j)} + \sum_{j=0}^{N_2} B_j^T \bar{A}_{r_j} - G^T \right) \Phi(x) + \Phi^T(x) (\lambda_1 K_1 \bar{A}_p \Psi(x) + \lambda_2 K_2 \bar{A}_q \Psi(1)).$$

Now, by choice m arbitrary points $\{x_i\}, i = 1, \dots, m$, in the domain $[0, \eta]$ as collocation points and substituting them in $Res(x)$, and using of the Eq. (7), a set of m nonlinear algebraic equations is generated as follows (Collocation method)

$$Res(x_i) = 0, \quad i = 1, \dots, m.$$

By solving this system, the approximate solution of the Eq. (6) according to the Eq. (10) is obtained.

In this study, the roots of the GFCFs in the interval $[0, \eta]$ (Theorem 3) are used as collocation points, and also consider that all of the computations have been done by Maple 18 on a laptop with CPU Core i7, Windows 8.1 64bit, and 8GB of RAM.

4. ILLUSTRATIVE EXAMPLES

In this section, by using the present method, some well-known examples are solved to show efficiently and applicability GFCFs collocation method based on Spectral method. The present method is applied to solve the linear and nonlinear multi-order Volterra-Fredholm integro-differential equations, and their outputs are compared with the corresponding analytical solutions.

Example 1. Consider the following linear Volterra-Fredholm integral equation:

$$y(x) + \int_0^x x(t+1)y(t)dt + \frac{3}{17} \int_0^1 (tx)y(t)dt = g(x),$$

$$y(0) = 1,$$

where $g(x) = (x + 1)^2 + \frac{1}{4}x(x + 1)^4$ and

$0 \leq x < 1$. The exact solution of this problem is

$y(x) = (x + 1)^2$. By applying the technique described in the last section, the residual function as follows:

$$Res(x) = (A^T - G^T)\Phi + \Phi^T \left(K_1 \bar{A}_p \Psi(x) + \frac{3}{17} K_2 \bar{A}_q \Psi(1) \right),$$

where $\bar{A}_p, \bar{A}_q, K_1, K_2, \Psi(x)$ and G^T are obtained from Eqs. (20) - (27).

For $\alpha = 0.5, m = 10$ and choosing m points $x_i, i = 1, \dots, m$ using theorem 3 as collocation points and using of initial condition, the exact solution is obtained. The absolute error with $m = 10$ is displayed in Fig. 1, we can see the approximate solution is in a good agreement with the exact solution.

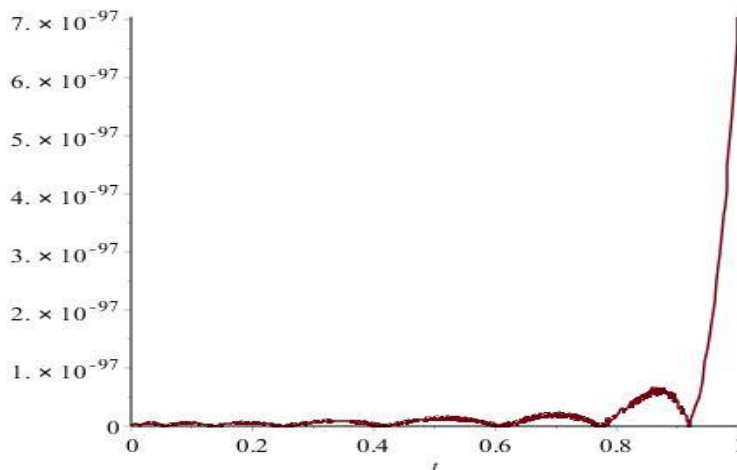


Figure 1: The absolute error function for example 1 with $m = 10$.

Example 2. Consider the following nonlinear Volterra-Fredholm integro-differential equation of fractional order:

$$D^{0.5}y(x) + 12 \int_0^x (2x - 3t)y(t)dt - 60 \int_0^1 (x^3t - x^4t)[y(t)]^2dt = g(x),$$

$$y(0) = 0,$$

where $g(x) = \frac{2}{\sqrt{\pi}}(\frac{4}{3}\sqrt{x^3} - \sqrt{x}) - x^3$ and $0 \leq x < 1$. The exact solution of this problem is $y(x) = x^2 - x$. By applying the technique described in the last section, the residual function as follows:

$$Res(x) = (A^T D^{(0.5)} - G^T)\Phi + \Phi^T(12K_1\widetilde{A}_p\Psi(x) - 60K_2\widetilde{A}_q\Psi(1)).$$

For $\alpha = 0.5, m = 10$ and choosing m points $x_i, i = 1, \dots, m$ using theorem 3 as collocation points and substituting them in $Res(x)$, and using of initial condition, a set of m nonlinear algebraic equations is generated.

The absolute and the residual errors with $m = 10$ are displayed in Fig. 2.

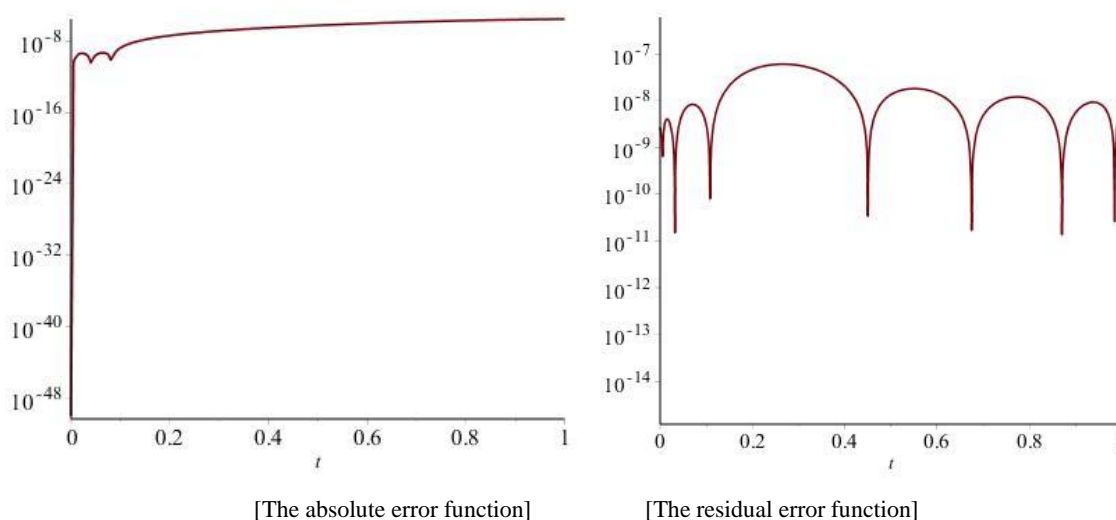


Figure 2: Graphs of the absolute error and the residual error for example 2 with $m = 10$.

Example 3. Next, we consider the nonlinear Volterra-Fredholm integro-differential equations of fractional order as follows:

$$D^{\frac{2}{5}}y(x) + \sqrt[5]{x^4}y(x) - \int_0^x (x - t)y^2(t)dt - \int_0^1 (2xt)y(t)dt = g(x),$$

$$y(0) = 0,$$

where $g(x) = \frac{2}{5}\Gamma(\frac{2}{5}) + \sqrt[5]{x^6} - \frac{25}{126}\sqrt[5]{x^{14}} - \frac{5}{6}x$ and $0 \leq x < 1$. The exact solution of this problem is $y(x) = \sqrt[5]{x^2}$. The residual function as follows:

$$Res(x) = (A^T D^{(\frac{2}{5})} + B^T\widetilde{A}_r - G^T)\Phi - \Phi^T(K_1\widetilde{A}_p\Psi(x) + K_2\widetilde{A}_q\Psi(1)),$$

where $B^T\Phi(x) \approx \sqrt[5]{x^4}$. For $\alpha = 0.4, m = 15$ and choosing m points $x_i, i = 1, \dots, m$ using theorem 3 as collocation points and substituting them in $Res(x)$, and using of initial condition, a set of m nonlinear algebraic equations is generated.

The absolute and the residual errors with $m = 15$ are displayed in Fig. 3.

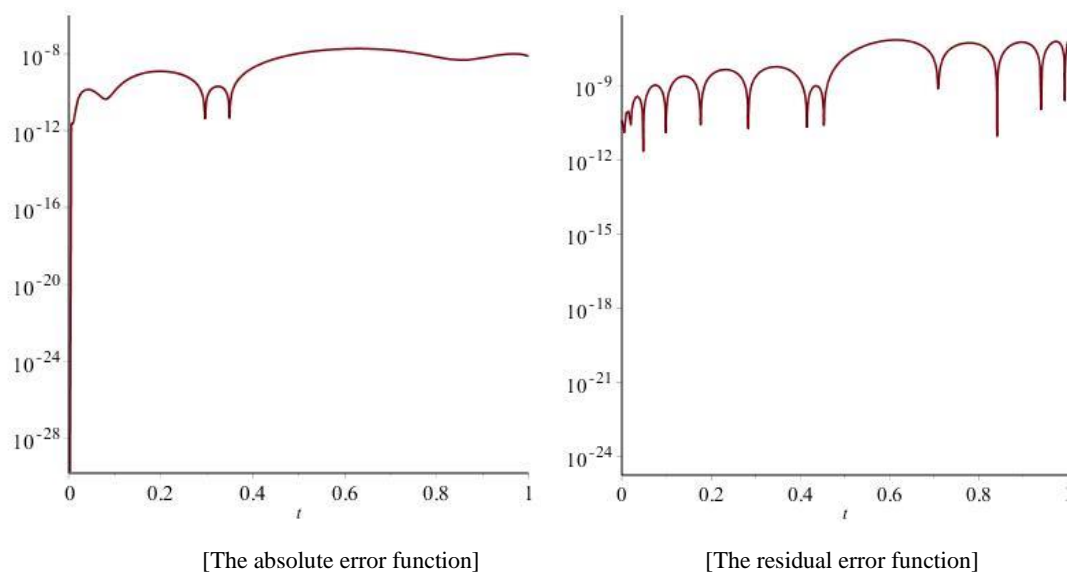


Figure 3: Graphs of the absolute error and the residual error for example 3 with $m = 15$.

Example 4. Consider the following nonlinear Volterra integro-differential equation:

$$y^{(5)}(x) - y''(x) + 3 \int_0^x y^3(t) dt = e^{3x} - 1, \quad 0 \leq x < 1,$$

$$y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(0) = 1.$$

Table 1 shown comparison the approximate solution by the present method, Turkyilmazoglu [19], and Ordokhani and Razzaghi [20] with $m = 10$ and $\alpha = 1$, we can see the approximate solution is in a very good agreement with the exact solution.

Turkyilmazoglu [19] has calculated the approximate solution to this problem as follows:

$$y(x) = 1 + 0.9999999904x + 0.5000002058x^2 + 0.1666648062x^3 + 0.04167543425x^4 + 0.008309686813x^5 + 0.001426582760x^6 + 0.0001636434044x^7 + 0.00004147874967x^8. \tag{29}$$

The absolute error of the present method and the approximate solution of Eq. (29) with $m = 10$ and $\alpha = 1$ are displayed in Fig. 4.

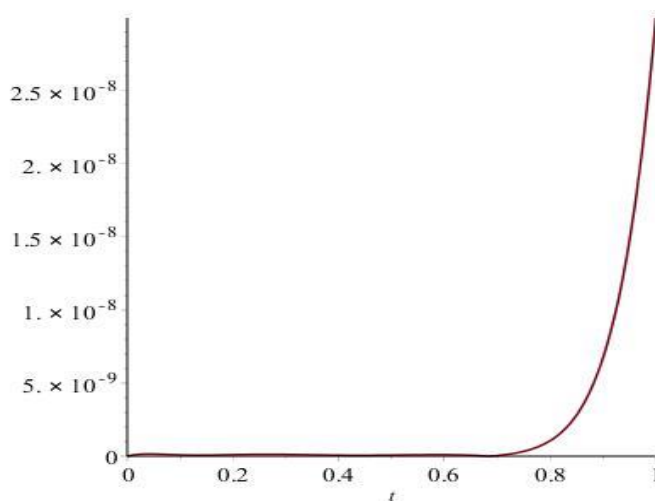


Figure 4: The residual error function for example 4 with $m = 10$.

Table 1: Obtained values of $y(t)$ by Turkyilmazoglu [19], Ordokhani and Razzaghi [20], the present method, and the exact solution for example 4 with $m = 10$

t	Exact solution	Ref. [19]	Ref. [20]	Present solution	Abs. Er.
0.0	1.0000000000	1.0000000000	1.000000	1.0000000000	0.000e-00
0.2	1.2214027581	1.221402758	1.221341	1.2214027581	4.402e-14
0.4	1.4918246976	1.491824698	1.491758	1.4918246976	6.511e-13
0.6	1.8221188003	1.822118800	1.822086	1.8221188003	2.070e-12
0.8	2.2255409284	2.225540929	2.225467	2.2255409273	1.107e-09
1.0	2.7182818284	2.718281828	2.718199	2.7182817983	3.007e-08

Example 5. Now, we consider the nonlinear Volterra-Fredholm integro-differential equations of multi-order as follows:

$$D^{\frac{1}{2}}y(x) - 2y'(x) + 2x^2y(x) - 7 \int_0^x ty(t)dt + \frac{28}{97} \int_0^1 x^2(x+t)y^2(t)dt = g(x),$$

$$y(0) = 0,$$

where $g(x) = \frac{3}{4}\sqrt{\pi}x + (\frac{2}{\sqrt{\pi}} - 3)\sqrt{x} + \frac{1127}{4365}x^2 - 2$ and $0 \leq x < 1$. The exact solution of this problem is $y(x) = \sqrt{x^3} + x$. The residual function as follows:

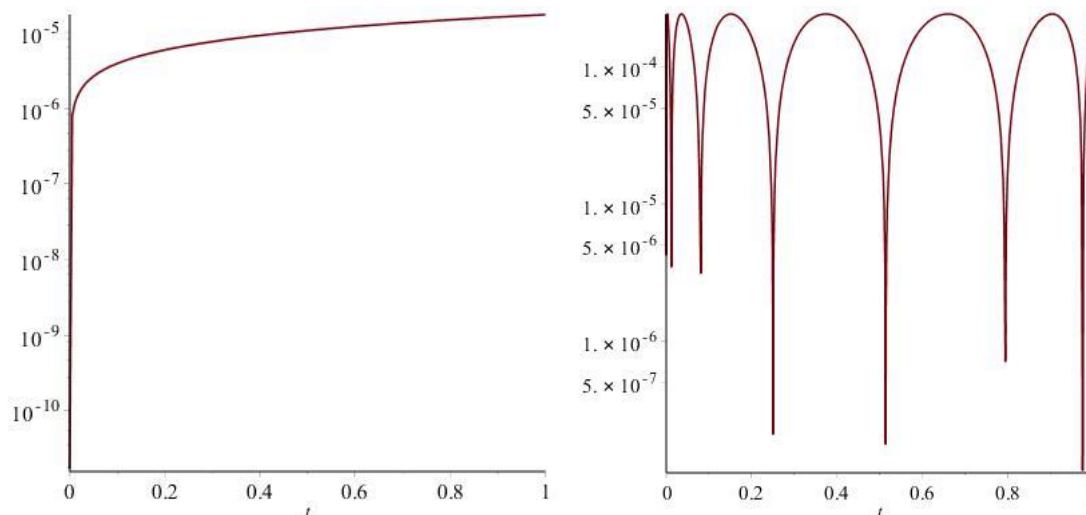
$$Res(x) = (A^T D^{(0.5)} - 2A^T D^{(1)} + B^T \widehat{A}_r - G^T) \Phi - \Phi^T (7K_1 \widehat{A}_p \Psi(x) - \frac{28}{97} K_2 \widehat{A}_q \Psi(1)), \quad (30)$$

where $B^T \Phi(x) = 2x^2$.

Now, we must choose the value of α such that, $\gamma_1 = 0.5$ and $\gamma_2 = 1$ be multiples of α . For example $\alpha = 0.5$, we can calculate the values $D^{(0.5)}$ and $D^{(1)} = D^{(0.5)}D^{(0.5)}$.

For $m = 7$ and choosing m points $x_i, i = 1, \dots, m$ using theorem 3 as collocation points and using of initial condition, we can obtain approximate solutions. The absolute error and the residual error with $m = 7$ are displayed in Fig. 5.

To show the convergence of the present method to solve this example for $\alpha = 0.5$ in the Fig. 6, we have shown that by increasing the m the residual function decreases, to show the convergence of the present method.



[The absolute error function] [The residual error function]

Figure 5: Graphs of the absolute error and the residual error for example 5 with $m = 7$.

Example 6. For the last example, we consider the linear Volterra-Fredholm integro-differential equations of multi-order as follows:

$$D^{\frac{1}{2}}y(x) + D^{\frac{3}{2}}y(x) + x^2y(x) - 4 \int_0^x ty(t)dt + 12 \int_0^1 (x+t)y(t)dt = g(x),$$

$$y(0) = -1, D^{\frac{1}{2}}y(0) = 0,$$

where $g(x) = \frac{8}{3\sqrt{\pi}}\sqrt{x^3} + \frac{4}{\sqrt{\pi}}\sqrt{x} + x^2 - 8x - 3$. The exact solution of this problem is $y(x) = x^2 - 1$. The residual function as follows:

$$Res(x) = (A^T D^{(0.5)} + A^T D^{(1.5)} + B^T \widehat{A}_r - G^T)\Phi - \Phi^T(4K_1 \widehat{A}_p \Psi(x) - 12K_2 \widehat{A}_q \Psi(1)),$$

where $B^T \Phi(x) = x^2$.

Now, we must choose the value of α such that, $\gamma_1 = 0.5$ and $\gamma_2 = 1.5$ be multiples of α . For example $\alpha = 0.5$, we can calculate the values $D^{(0.5)}$ and $D^{(1.5)} = D^{(0.5)}D^{(0.5)}D^{(0.5)}$.

For $m = 10$ and choosing m points $x_i, i = 1, \dots, m$ using theorem 3 as collocation points and using of initial condition, we can obtain the exact solution. The absolute error and the residual error with $m = 10$ are displayed in Fig. 7. We can see the approximate solution is in a very good agreement with the exact solution.

To show the convergence of the present method to solve this example for $\alpha = 0.5$ in the Fig. 8, we have shown that by increasing the m the residual function decreases, to show the convergence of the presented method.

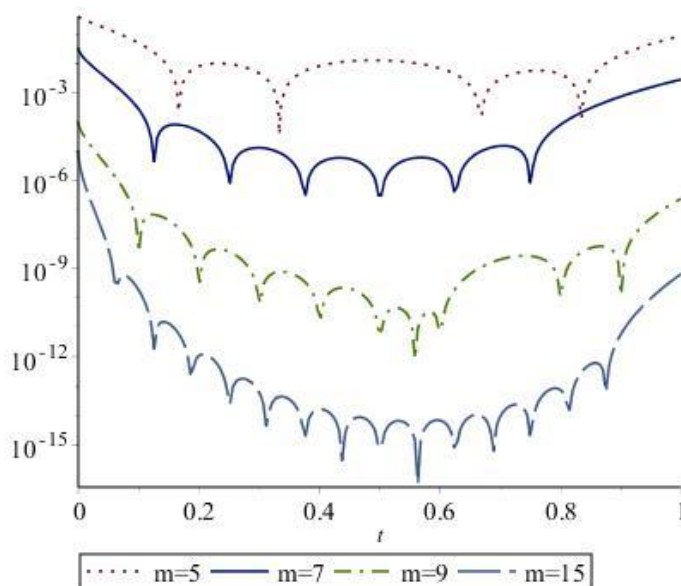


Figure 6: The residual errors for example 5 with $m = 5, 7, 9, 15$ and $\alpha = 0.5$, to show the convergence rate of the GFCF collocation method.

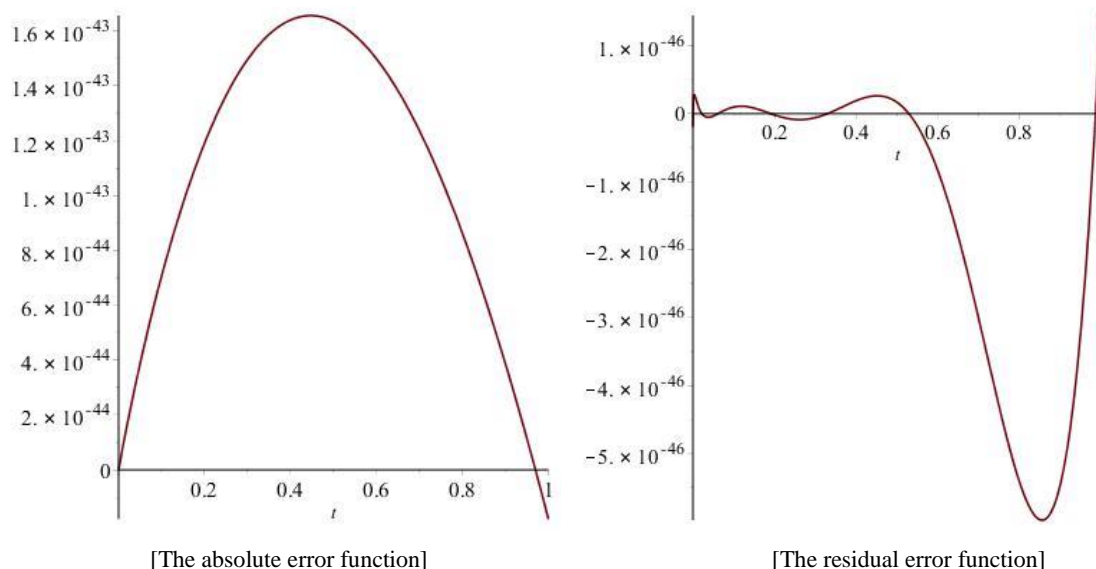


Figure 7: Graphs of the absolute error and the residual error for example 6 with $m = 10$.

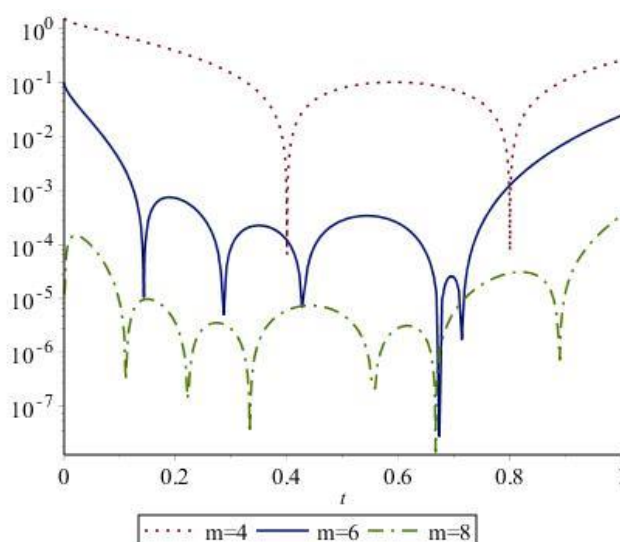


Figure 8: The residual errors for example 6 with $m = 4,6,8$ and $\alpha = 0.5$, to show the convergence rate of the GFCF collocation method.

5. CONCLUSION

In this paper, first, the generalized fractional order of the Chebyshev functions (GFCF) of the first kind have been introduced, next, the operational matrices of the fractional derivative, the product, and the fractional integration of these orthogonal functions are obtained. These matrices can be used to solve the linear and nonlinear Volterra- Fredholm integro- differential equations of multi-arbitrary order and reduce the computational cost. As shown, the method is converging

and has an appropriate accuracy and stability, that the sufficient accuracy is due choosing the basic of fractional. Illustrative examples show that this method has good results.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

- [1] Kazem, S., Abbasbandy, S. and Kumar S., "Fractional-order Legendre functions for solving fractional-order differential equations", *Appl. Math. Modell.*, 37: 5498-5510, (2013).
- [2] Delkhosh, M., "Introduction of Derivatives and Integrals of Fractional order and Its Applications", *Appl. Math. and Phys.*, 1(4): 103-119, (2013).
- [3] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier, San Diego, (2006).
- [4] Craven, B.D., "Stone's Theorem and Completeness of Orthogonal Systems", *J. Australian Math. Soc.*, 12(2): 211-223, (1971).
- [5] Szego, G., *orthogonal polynomials*, American Mathematical Society Providence, Rhode Island, (1975).
- [6] Boyd, J.P., *Chebyshev and Fourier Spectral Methods*, Second Edition, Dover Publications, Mineola, New York, (2000).
- [7] Bhrawy, A.H. and Alofi, A.S., "The operational matrix of fractional integration for shifted Chebyshev polynomials", *Appl. Math. Letters*, 26: 25-31 (2013).
- [8] Parand, K., Abbasbandy, S., Kazem, S. and Rezaei, A.R., "An improved numerical method for a class of astrophysics problems based on radial basis functions", *Phys. Scr.*, 83(1): 015011, 11 pages, (2011).
- [9] Doha, E.H., Bhrawy, A.H. and Ezz-Eldien, S.S., "A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order", *Comput. Math. Appl.*, 62: 2364-2373, (2011).
- [10] Saadatmandi, A. and Dehghan, M., "Numerical solution of hyperbolic telegraph equation using the Chebyshev Tau method", *Num. Meth. Partial Diff. Equ.* 26(1): 239-252, (2010).
- [11] Parand, K., Taghavi, A. and Shahini, M., "Comparison between rational Chebyshev and modified generalized Laguerre functions pseudospectral methods for solving Lane-Emden and unsteady gas equations", *Acta Phys. Polo. B*, 40(12): 1749-1763, (2009).
- [12] Parand, K., Rezaei, A.R. and Taghavi, A., "Numerical approximations for population growth model by rational Chebyshev and Hermite functions collocation approach: a comparison", *Math. Method. Appl. Sci.*, 33(17): 2076-2086, (2010).
- [13] Parand, K. and Khaleqi, S., "The rational Chebyshev of Second Kind Collocation Method for Solving a Class of Astrophysics Problems", *Euro. Phys. J. Plus*, 131: 1-24, (2016).
- [14] Parand, K., Dehghan, M. and Taghavi, A., "Modified generalized Laguerre function Tau method for solving laminar viscous flow: The Blasius equation", *Int. J. Numer. Method. H.*, 20(7): 728-743, (2010).
- [15] Parand, K. and Delkhosh, M., "Solving Volterra's population growth model of arbitrary order using the generalized fractional order of the Chebyshev functions", *Ricerche Mat.*, 65(1): 307-328, (2016).
- [16] Darani, M.A. and Nasiri, M., "A fractional type of the Chebyshev polynomials for approximation of solution of linear fractional differential equations", *Comp. Meth. Diff. Equ.*, 1: 96-107, (2013).
- [17] Butcher, E.A., Ma, H., Bueler, E., Averina, V. and Szabo, Z., "Stability of linear time-periodic delay-differential equations via Chebyshev polynomials", *Int. J. Numer. Meth. Eng.*, 59: 895-922, (2004).
- [18] Mason, J.C. and Handscomb, D.C., *Chebyshev polynomials*, CRC Press Company, ISBN 0-8493-0355-9, (2003).
- [19] Turkyilmazoglu, M., "High-order nonlinear Volterra-Fredholm-Hammerstein integro-differential equations and their effective computation", *Appl. Math. Comput.*, 247: 410-416, (2014).
- [20] Ordokhani, Y. and Razzaghi, M., "Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via a collocation method and rationalized Haar functions", *Appl. Math. Lett.*, 21: 4-9, (2008).