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On Some Classes of Γ-AG-Groupoids

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ABSTRACT

In this paper, we have introduced the notion of Γ-regular, weakly Γ-regular, left Γ-regular, right Γ-regular, Γ-completely regular and Γ- left quasi regular of Γ-AG-groupoids, and we have investigated their properties.

Keywords: *Γ-AG-groupoid, Γ-regular, Γ-intra-regular, weakly Γ-regular, left Γ-regular, right Γ-regular, Γ-completely regular, Γ-left quasi regular.*

1. INTRODUCTION

Kazim, M. A. and Naseeruddin, MD. defined the concept of LA-semigroup as follows a groupoid S is called a left almost semigroup, abbreviated as LA-semigroup if $(ab)c=(cb)a$ for all $a,b,c \in S$.

Kazim, M. A. and Naseeruddin, MD. [1, Proposition 2.1] asserted that, in every LA-semigroup S, a medial law hold

$$(ab)(cd)=(ac)(bd) \quad \text{for all } a,b,c,d \in S.$$

Mushtaq, Q. and Khan, M. [2. p.322] introduced in every LA-semigroup *S* with left identity

$$(ab)(cd)=(db)(ca) \quad \text{for all } a,b,c,d \in S.$$

Further Khan, M., Faisal, and Amjid, V. [3] introduced if a LA-semigroup S with left identity, then the following law holds:

$$a(bc)=b(ac) \quad \text{for all } a,b,c,d \in S.$$

In this note we prefer to called left almost semigroup (LA-semigroup) as Abel-Grassmann's groupoid (abbreviated as an"AG-groupoid").

In [2]introduced the concepts of regular, weakly regular, left regular, right regular, completely regular and left quasi regular of an AG-groupoids as follows

Definition 1.1. [2. P1]. An element a of an AG-groupoid S is called a *regular* if there exists $x \in S$ such that $a=(ax)a$ and S is called *regular* if all elements of S are regular.

Definition 1.2. [2. P1]. An element a of an AG-groupoid S is called an *intra-regular* if there exist $x,y \in S$ such that $a=(x(aa))y$ and S is called *intra-regular* if all elements of S are intra-regular.

Definition 1.3. [2. P2]. An element a of an AG-groupoid S is called a *weakly regular* if there exist $x,y \in S$ such that $a=(ax)(ay)$ and S is called *weakly regular* if all elements of S are weakly regular.

Definition 1.4. [2. P2]. An element a of an AG-groupoid S is called a *left regular* if there exists $x \in S$ such that $a=x(aa)$ and S is called *left regular* if all elements of S are left regular.

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Definition 1.5. [2. P2]. An element a of an AG-groupoid S is called a *right regular* if there exists $x \in S$ such that $a = (aa)x$ and S is called *right regular* if all elements of S are right regular.

Definition 1.6. [2. P2]. An element a of an AG-groupoid S is called a *left quasi regular* if there exist $x, y \in S$ such that $a = (xa)(ya)$ and S is called *left quasi regular* if all elements of S are left qausi regular.

Definition 1.7. [2. P2]. An element a of an AG-groupoid S is called a *completely regular* if a is regular, left and right regular. S is called *completely regular* if it is regular, left and right regular.

2. DEFINITION OF Γ-AG-GROUPOIDS

Shah, T. and Rehman, I. [6, p.268] asserted that, in 1981, the notion of Γ-semigroups was introduced by Sen, M. K. Let S and Γ be any nonempty sets. If there exists a mapping $S \times \Gamma \times S \to S$ written $(a, \alpha, c) \mapsto a\alpha c$, S is called a Γ-semigroups if S satisfies the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. A Γ-AG-groupoids analogous to Γ-semigroups.

Definition 2.1. [6, p.268] Let S and Γ be any non-empty sets. We call S to be Γ-AG-groupoid if there exists a mapping $S \times \Gamma \times S \to S$, written $(a, \alpha, b) \mapsto a\alpha b$ such that S satisfies the identity $(a\alpha b)\beta c = (c\alpha b)\beta a$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Definition 2.2. [3, p.2]. Let S and Γ be any non-empty sets. We call S to be a Γ-medial if it satisfies $(a\alpha b)\beta(c\gamma d) = (a\alpha c)\beta(b\gamma d)$ and S is called a Γ-paramedial if it satisfies

$(a\alpha b)\beta(c\gamma d) = (d\alpha c)\beta(b\gamma a)$ for all $a, b, c, d \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

Definition 2.3. A Γ-AG-groupoids S with left identity, the following law hold

$a\alpha(b\beta c) = b\alpha(a\beta c)$, for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

In this paper, we introduce the concept of a Γ-regular, weakly Γ-regular, left Γ-regular, right Γ-regular, Γ-completely regular and left Γ-quasi regular of Γ-AG-groupoids which is defined analogous to [2] and investigate its properties.

3. MAIN RESULTS

Definition 2.4. [6. P274]. An element a of a Γ-AG-groupoid S is called a Γ*-regular* if there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$ and S is called Γ*-regular* if all elements of S are Γ-regular.

Definition 2.5. [2. P1]. An element a of a Γ-AG-groupoid S is called an *intra-*Γ*-regular* if there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha(a\beta a))\gamma y$ and S is called *intra-*Γ*-regular* if all elements of S are intra-Γ-regular.

Definition 2.6. An element a of a Γ-AG-groupoid S is called a *weakly* Γ*-regular* if there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (a\alpha x)\beta(a\gamma y)$ and S is called *weakly* Γ*-regular* if all elements of S are weakly Γ-regular.

Definition 2.7. An element a of a Γ-AG-groupoid S is called a *left* Γ*-regular* if there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha(a\beta a)$ and S is called *left* Γ*-regular* if all elements of S are left Γ-regular.

Definition 2.8. An element a of a Γ-AG-groupoid S is called a *right* Γ*-regular* if there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha a)\beta x$ and S is called *right* Γ*-regular* if all elements of S are right Γ-regular.

Definition 2.9. An element a of a Γ-AG-groupoid S is called a *left* Γ*-quasi regular* if there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha a)\beta(y\gamma a)$ and S is called *left* Γ*-quasi regular* if all elements of S are left Γ-qausi regular.

Definition 2.10. An element a of a AG-groupoid S is called a *completely* Γ*-regular* if a is Γ-regular and left (right) Γ-regular. S is called *completely* Γ*-regular* if it is Γ-regular, left and right Γ-regular.

Lemma 3.1. If S is Γ-regular (intra-Γ-regular, weakly Γ-regular, left Γ-regular, right Γ-regular, left Γ-quasi regular and completely Γ-regular) Γ-AG-groupoid, then $S = S\Gamma S$.

Proof. Let S be a Γ-regular and $a \in S$. Then there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$. Thus $a = (a\alpha x)\beta a \in S\Gamma S$ so $S \subseteq S\Gamma S$. Since S is a Γ-AG-groupoid we have $S\Gamma S \subseteq S$. Hence $S = S\Gamma S$. Similarly if S is an intra-Γ-regular, weakly Γ-regular, right Γ-regular, left Γ-regular, left Γ-quasi regular, completely Γ-regular, then can show that $S = S\Gamma S$. □

Theorem 3.2 If S is a Γ-AG-groupoid with left identity, then S is an intra-Γ-regular if and only if for all $a \in S$, $a = (x\alpha a)\gamma(a\omega z)$ for some $x, z \in S$ and $\alpha, \gamma, \omega \in \Gamma$.

Proof $(\Rightarrow)$ Let S be an intra-Γ-regular Γ-AG-groupoid with left identity, then for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha(a\beta a))\gamma y$. Now by using Lemma 3.1 let $y = u\omega v$ for some $u, v \in S$ and $\omega \in \Gamma$. Thus by using Definition 2.1, 2.2, 2.3, we have

$$
\begin{aligned}
a &= (x\alpha(a\beta a))\gamma y = (a\alpha(x\beta a))\gamma y = (y\alpha(x\beta a))\gamma a \\
&= (y\alpha(x\beta a)\gamma(x\lambda(a\eta a))\delta y) = ((u\omega v)\alpha(x\beta a)\gamma(x\lambda(a\eta a))\delta y) \\
&= ((a\omega x)\alpha(v\beta u)\gamma(x\lambda(a\eta a))\delta y)) = ((a\omega x)\alpha t)\gamma(x\lambda(a\eta a))\delta y)) \\
&= (((x\lambda(a\eta a))\delta y)\alpha t)\gamma(a\omega x) = (t\delta y)\alpha(x\lambda(a\eta a))\gamma(a\omega x) \\
&= ((a\eta a)\delta x)\alpha(y\lambda t)\gamma(a\omega x)) = (((a\eta a)\delta x)\alpha s)\gamma(a\omega x) \\
&= ((s\delta x)\alpha(a\eta a))\gamma(a\omega x) = ((a\eta a)\alpha(x\delta s))\gamma(a\omega x) \\
&= ((a\eta a)\alpha k)\gamma(a\omega x) = ((k\eta a)\alpha a)\gamma(a\omega x) \\
&= (z\alpha a)\gamma(a\omega x) = (x\alpha a)\gamma(a\omega z),
\end{aligned}
$$

where $v\beta u = t$, $y\lambda t = s$, $x\delta s = k$ and $k\eta a = z$ for some $t, s, k \in S$ and $\lambda, \eta, \delta \in \Gamma$.

$(\Leftarrow)$ Let $a \in S$, $a = (x\alpha a)\gamma(a\omega z)$ for some $x, z \in S$ and $\alpha, \omega \in \Gamma$. Thus by using Definition 2.1, 2.2, 2.3, we have

$$
\begin{aligned}
a &= (x\alpha a)\gamma(a\omega z) = a\gamma((x\alpha a)\omega z) = (x\lambda a)\beta(a\delta z)\gamma((x\alpha a)\omega z) \\
&= (a\beta((x\lambda a)\delta z))\gamma((x\alpha a)\omega z) = (((x\alpha a)\omega z)\beta((x\lambda a)\delta z))\gamma a \\
&= (((x\alpha a)\omega(x\lambda a))\beta(z\delta z))\gamma a = (((a\alpha x)\omega(a\lambda x))\beta(z\delta z))\gamma a \\
&= ((a\omega((a\alpha x)\lambda x))\beta(z\delta z)))\gamma a = (((z\delta z)\omega((a\alpha x)\lambda x))\beta a)\gamma a \\
&= (((a\alpha x)\omega((z\delta z)\lambda x))\beta a)\gamma a = (((((z\delta z)\lambda x)\alpha x)\omega a)\beta a)\gamma a \\
&= ((((x\lambda x)\alpha(z\delta z))\omega a)\beta a)\gamma a = ((a\omega a)\beta(x\lambda x)\alpha(z\delta z))\gamma a \\
&= (a\beta(x\lambda x)\alpha(z\delta z))\gamma(a\omega a) = (a\beta t)\gamma(a\omega a),
\end{aligned}
$$

where $(x\lambda x)\alpha(z\delta z) = t$ for some $t \in S$ and $\lambda, \delta \in \Gamma$. Now by using Definition 2.1, 2.2, we have

$$
\begin{aligned}
a &= (a\beta t)\gamma(a\omega a) = ((a\lambda t)\eta(a\delta a)\beta t)\gamma(a\omega a) = ((a\lambda a)\eta(t\delta a)\beta t)\gamma(a\omega a) \\
&= (t\eta(t\delta a)\beta(a\lambda a))\gamma(a\omega a) = (u\beta(a\lambda a))\gamma v,
\end{aligned}
$$

where $t\eta(t\delta a) = u$ and $(a\omega a) = v$ for some $u, v \in S$ and $\delta, \omega \in \Gamma$. Thus S is an intra-Γ-regular. □

Lemma 3.3 If S is a Γ-AG-groupoid, then the following are equivalent.

(1) S is weakly Γ -regular.

(2) S is intra- Γ -regular.

Proof (1) $\Rightarrow$ (2) Let S be a weakly Γ -regular Γ -AG-groupoid with left identity, then for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (a\alpha x)\beta(a\gamma y)$ and by Lemma 3.1 let $x = u\lambda v$ for some $u, v \in S$ and $\lambda \in \Gamma$. Now by using Definition 2.1, 2.2, 2.3, we have

$$a = (a\alpha x)\beta(a\gamma y) = (y\alpha a)\beta(x\gamma a) = (y\alpha a)\beta((u\lambda v)\gamma a)$$
$$= (y\alpha a)\beta((a\lambda v)\gamma u) = (a\alpha v)\beta((y\lambda a)\gamma u) = (a\alpha(y\lambda a))\beta(v\gamma u)$$
$$= (a\alpha(y\lambda a))\beta t = (y\alpha(a\lambda a))\beta t,$$

where $v\gamma u = t$ for some $t \in S$. Thus S is intra- Γ -regular.

(2) $\Rightarrow$ (1) Let S be an intra- Γ -regular, for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha(a\beta a))\gamma y$ and by Lemma 3.1 let $x = u\lambda v$ for some $u, v \in S$ and $\lambda \in \Gamma$. Now by using Definition 2.1, 2.2, 2.3, we have

$$a = (y\alpha(a\lambda a))\beta t = (a\alpha(y\lambda a))\beta t = (a\alpha(y\lambda a))\beta(v\gamma u)$$
$$= (a\alpha v)\beta((y\lambda a)\gamma u) = (y\lambda a)\beta((a\alpha v)\gamma u) = (y\lambda a)\beta((u\alpha v)\gamma a)$$
$$= (y\lambda a)\beta(x\gamma u) = (a\lambda x)\beta(a\gamma y),$$

where $x = u\alpha v$ for some $u, v \in S$ and $\alpha \in \Gamma$. Thus S is weakly Γ -regular. □

Lemma 3.4 If S is a Γ -AG-groupoid, then the following are equivalent.

(1) S is weakly Γ -regular.

(2) S is right Γ -regular.

Proof (1) $\Rightarrow$ (2) Let S be a weakly Γ -regular Γ -AG-groupoid with left identity, then for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (a\alpha x)\beta(a\gamma y)$ and let $x\gamma y = t$ for some $t \in S$. Now by Γ -medial, we have $a = (a\alpha x)\beta(a\gamma y) = (a\alpha a)\beta(x\gamma y) = (a\alpha a)\beta t$. Thus S is right Γ -regular.

(2) $\Rightarrow$ (1) Let S be a right Γ -regular, for any $a \in S$ there exists $t \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha a)\beta t$ and let $x\gamma y = t$ for some $x, y \in S$. Now by Γ -medial, we have.
$a = (a\alpha a)\beta t = (a\alpha a)\beta(x\gamma y) = (a\alpha a)\beta(x\gamma y)$ Thus S is weakly Γ -regular. □

Lemma 3.5 If S is a Γ -AG-groupoid, then the following are equivalent.

(1) S is weakly Γ -regular.

(2) S is left Γ -regular.

Proof (1) $\Rightarrow$ (2) Let S be a weakly Γ -regular Γ -AG-groupoid with left identity, then for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (a\alpha x)\beta(a\gamma y)$ and let $y\alpha x = t$ for some $t \in S$. Now by Definition 2.2, we have $a = (a\alpha x)\beta(a\gamma y) = (a\alpha a)\beta(x\gamma y) = (y\alpha x)\beta(a\gamma a) = t\beta(a\gamma a)$. Thus S is left Γ -regular.

(2) $\Rightarrow$ (1) Let S is left Γ -regular, for any $a \in S$ there exists $t \in S$ and $\beta, \gamma \in \Gamma$ such that $a = t\beta(a\gamma a)$ and let $y\alpha x = t$ for some $x, y \in S$. Now by Definition 2.2, we have
$a = t\beta(a\gamma a) = (y\alpha x)\beta(a\gamma a) = (y\alpha a)\beta(x\gamma a) = (a\alpha x)\beta(a\gamma y)$.

Thus S is weakly Γ -regular. □

Lemma 3.6. Every weakly Γ -regular Γ -AG-groupoid with left identity is Γ -regular.

Proof. Assume that S is a weakly Γ - regular Γ -AG-groupoid with left identity then for any $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (a\alpha x)\beta(a\gamma y)$. Let $x\gamma y = t$ for some $t \in S$ and $t\omega(((y\lambda x)\eta a)) = u \in S$ for some $\lambda, \omega, \eta \in \Gamma$. Now by Definition 2.1, we have

$$
\begin{aligned}
a &= (a\alpha x)\beta(a\gamma y) = ((a\gamma y)\alpha x)\beta a \\
&= ((x\gamma y)\alpha a)\beta a = (t\alpha a)\beta a; && \text{by Definition 2.1 and } x\gamma y = t \\
&= (t\alpha(a\lambda x)\omega(a\eta y))\beta a; && \text{where } a = (a\alpha x)\beta(a\gamma y) \\
&= (t\alpha(a\lambda a)\omega(x\eta y))\beta a; && \text{by } \Gamma\text{-medial law} \\
&= (t\alpha(y\lambda x)\omega(a\eta a))\beta a; && \text{by } \Gamma\text{-paramedial law} \\
&= (t\alpha(a\omega((y\lambda x)\eta a)))\beta a; && \text{by Definition 2.3} \\
&= (a\alpha(t\omega((y\lambda x)\eta a)))\beta a; && \text{by Definition 2.3} \\
&= (a\alpha u)\beta a; && \text{where } t\omega(((y\lambda x)\eta a))) = u.
\end{aligned}
$$

Thus S is a Γ -regular. □

Theorem 3.7. If S is a Γ -AG-groupoid, then the following are equivalent.

(1) S is weakly Γ -regular.

(2) S is completely Γ -regular.

Proof. (1)$\Longrightarrow$(2) Let S be a weakly Γ -regular. Then by Lemma 3.4, 3.5, 3.6, we have S is a completely Γ -regular.

(2)$\Longrightarrow$(1) Let S be a completely Γ -regular. Then by Lemma 3.5, we have S is a weakly Γ -regular.

Lemma 3.8 If S is a Γ -AG-groupoid, then the following are equivalent.

(1) S is weakly Γ -regular.

(2) S is left Γ -quasi regular.

Proof *(1)$\Longrightarrow$(2) Let* S *be a weakly* Γ *-regular* Γ *-AG-groupoid with left identity, then for any* $a \in S$ *there exists* $x, y \in S$ *and* $\alpha, \beta, \gamma \in \Gamma$ *such that* $a = (a\alpha x)\beta(a\gamma y)$ *. Then*

$$
\begin{aligned}
a &= (a\alpha x)\beta(a\gamma y) \\
&= (y\alpha a)\beta(x\gamma a) && \text{by } \Gamma\text{-paramedial law} \\
&= (x'\alpha a)\beta(y'\gamma a) && \text{where } y = x' \text{ and } x = y'
\end{aligned}
$$

Thus S is left Γ - quasi regular.

(2)$\Longrightarrow$(1) Let S *be a* left Γ -quasi regular Γ *-AG-groupoid with left identity, then for any* $a \in S$ *there exists* $x, y \in S$ *and* $\alpha, \beta, \gamma \in \Gamma$ *such that* $a = (x\alpha a)\beta(y\gamma a)$ *. Then*

$$
\begin{aligned}
a &= (x\alpha a)\beta(y\gamma a) \\
&= (a\alpha y)\beta(a\gamma x) && \text{by } \Gamma\text{-paramedial law} \\
&= (a\alpha x')\beta(a\gamma y') && \text{where } y = x' \text{ and } x = y'
\end{aligned}
$$

Thus S is weakly Γ -regular. □

The next Theorem will conclude of research.

Theorem 3.9. If S is a Γ-AG-groupoid, then the following are equivalent.

(1) S is weakly Γ-regular.

(2) S is intra-Γ-regular.

(3) S is right Γ-regular.

(4) S is left Γ-regular.

(5) S is left Γ-qausi regular.

(6) S is completely Γ-regular.

(7) for all $a \in S$ there exist $x, y \in S$ and $\alpha, \omega \in \Gamma$ such that $a = (x\alpha a)(a\omega y)$.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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