Gazi University Journal of Science
GU J Sci
29(4): 929-935 (2016)

Oscillation Theorems for Second-Order Nonlinear Differential Equations

Adil MISIR[1,♠], Süleyman ÖĞREKÇİ[2]

[1] *Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey,*

[2] *Department of Mathematics, Sciences & Arts Faculty, Amasya University, Amasya, Turkey,*

Received: 18/10/2016 *Accepted: 27/10/2016*

ABSTRACT

In this paper, we are concerned with the oscillations in forced second order nonlinear differential equations with nonlinear damping terms. By using clasical variational principle and averaging technique, new oscillation criteria are established, which improve and extend some recent results. Examples are also given to illustrate the results.

2000 Mathematics Subject Classification: 34C10, 34C15, 34K11
Keywords: *Differential Equations, Oscillation, Damping.*

1. INTRODUCTION

In this paper, we are concerned with the oscillatory behavior of the forced second-order differential equation

$$\left(r(t)k_1\left(x(t),x'(t)\right)\right)' + p(t)k_2\left(x(t),x'(t)\right)+q(t)f\left(x(t)\right)=e(t), t\geq t_0 \tag{1.1}$$

where $t_0 \geq 0$ is a fixed real number, $r \in C^1([t_0,\infty),(0,\infty))$, $p,q,e \in C([t_0,\infty),\mathbb{R})$, $f \in C(\mathbb{R},\mathbb{R})$, $k_1 \in C^1(\mathbb{R}^2,\mathbb{R})$, $k_2 \in C(\mathbb{R}^2,\mathbb{R})$, and $vk_1(u,v)>0$ for all $v \neq 0$.

A function $x:[t_0,t_1)\to(-\infty,\infty)$, $t_1>t_0$, is called a solution of Eq. (1.1) if $x(t)$ satisfies Eq. (1.1) for all $t\in[t_0,t_1)$. In this paper we restrict our attention to these solutions $x(t)$ of Eq. (1.1) which exists on $[t_0,\infty)$ and satisfy

♠Corresponding author, e-mail: adilm@gazi.edu.tr

$\sup\{|x(t)|: t > t_x\} \neq 0$ for all $t_x \geq t_0$. Such a solution of Eq. (1.1) is called oscillatory if the set of its zeros is unbounded from above, otherwise, it is said to be nonoscillatory. Eq. (1.1) is called oscillatory, if its all solutions are oscillatory.

There are a great number of papers devoted to particular cases of Eq. (1.1) such as

$$\left(r(t)x'(t)\right)' + q(t)x(t) = e(t), \tag{1.2}$$

$$\left(r(t)x'(t)\right)' + q(t)f(x(t)) = e(t), \tag{1.3}$$

$$\left(r(t)\psi(x(t))\phi(x'(t))\right)' + q(t)f(x(t)) = e(t), \tag{1.4}$$

$$\left(r(t)k_1(x(t),x'(t))\right)' + p(t)k_2(x(t),x'(t))x'(t) + q(t)f(x(t)) = e(t). \tag{1.5}$$

Numerous oscillation criteria have been obtained for Eqs. (1.2)-(1.4) (see, for example [1]-[6] and references cited therein). The oscillatory behavior of Eq. (1.5) with $e(t) \equiv 0$ has been first studied by Rogovchenko and Rogovchenko [7], later, several oscillation criteria have been established (see for example [8]-[12]). Shi [14], Meng and Huang [13] have established oscillation criteria for the forced equation Eq. (1.5).

The aim of this paper is to obtain some new oscillation criteria for Eq. (1.1) which extend the above mentioned results and improve the results given in [13].

2. MAIN RESULTS

Firstly we introduce the general mean and some well known properties that will be used in the proofs of our results.

Let

$$D(s_i,t_i) = \{u \in C^1[s_i,t_i] : u(t) \neq 0, u(s_i) = u(t_i) = 0\} \; for\; i = 1,2. \tag{2.1}$$

We take the integral operator $A(\cdot; s_i, t_i)$;

$$A(h; s_i, t_i) = \int_{s_i}^{t_i} H^2(t)h(t)dt, \;\; s_i \leq t \leq t_i, i = 1,2, \tag{2.2}$$

where $h \in C([t_0,\infty))$. It is easily seen that the linear operator $A(\cdot; s_i, t_i)$ satisfies

$$A(h'; s_i, t_i) = -A\left(2\frac{H'}{H}h; s_i, t_i\right) \geq -A\left(\left|2\frac{H'}{H}\right||h|; s_i, t_i\right), \text{for } i = 1,2. \tag{2.3}$$

In this section, we shall make use of the following conditions:

1. $f(x)$ is differentiable and $xf(x) > 0$ for all $x \neq 0$,
2. $f'(x)$ exists and $f'(x) \geq 0$ for all $x \neq 0$,
3. $k_2(u,v)f(u) \geq \beta_1 |k_1(u,v)|^{(\alpha+1)/\alpha} |f(u)|^{(\alpha-1)/\alpha}$ for some $\alpha > 0$, $\beta_1 > 0$ and for all $(u,v) \in \mathbb{R}^2$,
4. $f(x)/x \geq K|x|^{\gamma-1}$ for $x \neq 0$, where $K > 0$ and $\gamma \geq 1$ are suitable constants,
5. $k_2(u,v)u \geq \beta_2 |k_1(u,v)|^{(\alpha+1)/\alpha} |u|^{(\alpha-1)/\alpha}$ for some $\alpha > 0$, $\beta_2 > 0$ and for all $(u,v) \in \mathbb{R}^2$,
6. $uk_2(u,v) \geq 0$ for all $u \neq 0$,
7. $vk_1(u,v) \geq \beta_3 |k_1(u,v)|^{(\alpha+1)/\alpha} |u|^{(\alpha-1)/\alpha}$, for some $\alpha > 0$, $\beta_3 > 0$ and for all $(u,v) \in \mathbb{R}^2$.

Theorem 1 *Suppose the conditions* $(A_1)-(A_3)$ *hold And for any* $T \geq t_0$*, there exists* $T \leq s_1 < t_1 \leq s_2 \leq t_2$ *such that*

$$e(t) \leq 0 \ \ for\, t \in [s_1, t_1]\, and\, e(t) \geq 0 \ \ for\, t \in [s_2, t_2], \tag{2.4}$$

and $p(t) > 0$ *on* $[s_1, t_1] \cup [s_2, t_2]$. *Let* $D(s_i, t_i)$ *and* $A(\cdot; s_i, t_i)$ *are defined by (2.1) and (2.2) respectively. If there exists* $H \in D(s_i, t_i)$ *such that*

$$A(q; s_i, t_i) > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{1}{\beta_1^\alpha} A\left(\frac{r^{\alpha+1}}{p^\alpha} \left| 2\frac{H'}{H} \right|^{\alpha+1}; s_i, t_i \right), \tag{2.5}$$

for $i = 1,2,$ *then Eq. (1.1) is oscillatory.*

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. We may assume that $x(t) > 0$ on $[T_0, \infty)$ for some large $T_0 \geq t_0$. By the assumptions, there exists s_1, t_1, s_2 and t_2 such that $T_0 \leq s_1 < t_1 \leq s_2 \leq t_2$ and (2.4) holds and $p(t) \geq 0$ on $[s_1, t_1] \cup [s_2, t_2]$. Define

$$w(t) = \frac{r(t) k_1(x(t), x'(t))}{f(x(t))}. \tag{2.6}$$

Then differentiating (2.6) and using Eq. (1.1) we obtain

$$w'(t) = -q(t) - \frac{p(t) k_2(x(t), x'(t)) f(x(t))}{f^2(x(t))} - \frac{r(t) k_2(x(t), x'(t)) x'(t) f'(x(t))}{f^2(x(t))} + \frac{e(t)}{f(x(t))}.$$

By using assumptions $(A_1)-(A_3)$ we obtain for $t \in [s_1, t_1] \cup [s_2, t_2]$

$$w'(t) \leq -q(t) - \frac{\beta_1 p(t)}{r^{(\alpha+1)/\alpha}(t)} |w(t)|^{(\alpha+1)/\alpha} + \frac{e(t)}{f(x(t))} \tag{2.7}$$

On the interval $[s_1, t_1]$, inequality (2.7) implies that $w(t)$ satisfies

$$w'(t) \leq -q(t) - \frac{\beta_1 p(t)}{r^{(\alpha+1)/\alpha}(t)} |w(t)|^{(\alpha+1)/\alpha}. \tag{2.8}$$

Applying operator $A(\cdot; s_i, t_i)$ for $i = 1,$ to inequality (2.8) we obtain

$$A(q; s_1, t_1) \leq A\left(\left| 2\frac{H'}{H} \right| |w| - \frac{\beta_1 p}{r^{(\alpha+1)/\alpha}} |w|^{(\alpha+1)/\alpha}; s_1, t_1 \right), \tag{2.9}$$

where $D(s_i, t_i)$ is given by hypotheses. Setting

$$F(v) = \left| 2\frac{H'}{H} \right| v - \frac{\beta_1 p}{r^{(\alpha+1)/\alpha}} v^{(\alpha+1)/\alpha}, \quad v > 0,$$

we have $F'(v^*) = 0$ and $F''(v^*) < 0,$ where $v^* = r^{\alpha+1} \left(\frac{\alpha}{\alpha+1} \frac{1}{\beta_1 p} \left| 2\frac{H'}{H} \right| \right)^\alpha$, which implies that $F(v)$ obtains its maximum at v^*. So we have

$$F(v) \le F(v^*) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r^{\alpha+1}}{(\beta_1 p)^\alpha} \left| 2\frac{H'}{H} \right|^{\alpha+1}. \tag{2.10}$$

Then we get, by using (2.10) in (2.9) we get,

$$A(q;s_1,t_1) \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{1}{\beta_1^\alpha} A\left(\frac{r^{\alpha+1}}{p^\alpha} \left| 2\frac{H'}{H} \right|^{\alpha+1}; s_1, t_1 \right), \tag{2.11}$$

which contradicts to (2.5) for $i = 1$.

If $x(t) < 0$ on $[T_0, \infty)$ for some large $T_0 \ge t_0$, we get inequality (2.7) again, which implies that (2.8) holds on the interval $[s_2, t_2]$. Applying operator $A(\cdot; s_i, t_i)$ for $i = 2$, to (2.8) this time, we get the same contradiction to (2.5) for $i = 2$. Thus the proof is complete.

Lemma 1 *[15]If A and B are non-negative, then*

$$\frac{1}{m}A + \frac{1}{n}B \ge A^{1/m}B^{1/n}, \quad \frac{1}{m} + \frac{1}{n} = 1.$$

Theorem 2 *Suppose the conditions (A_4) and (A_5) hold. And for any $T \ge t_0$, there exists $T \le s_1 < t_1 \le s_2 \le t_2$ such that (2.4) holds, $p(t) > 0$ and $q(t) \ge 0$ on $[s_1, t_1] \cup s_2, t_2]$. Let $D(s_i, t_i)$ and $A(\cdot; s_i, t_i)$ are defined by (2.1) and (2.2) respectively. If there exists $H \in D(s_i, t_i)$ such that*

$$A(Q; s_i, t_i) > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{1}{\beta_2^\alpha} A\left(\frac{r^{\alpha+1}}{p^\alpha} \left| 2\frac{H'}{H} \right|^{\alpha+1}; s_i, t_i \right), \tag{2.12}$$

for $i = 1,2,$ where

$$Q(t) = \gamma(\gamma-1)^{(1-\gamma)/\gamma} [Kq(t)]^{1/\gamma} |e(t)|^{(\gamma-1)/\gamma}$$

with the convention $0^0 = 1$. Then Eq. (1.1) is oscillatory.

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. We may assume that $x(t) > 0$ on $[T_0, \infty)$ for some large $T_0 \ge t_0$. By the assumptions, there exists s_1, t_1, s_2 and t_2 such that $T_0 \le s_1 < t_1 \le s_2 \le t_2$ and (2.4) holds, $p(t) \ge 0$ and $q(t) \ge 0$ on $[s_1, t_1] \cup [s_2, t_2]$. Define

$$w(t) = \frac{r(t)k_1(x(t), x'(t))}{x(t)}. \tag{2.13}$$

Then differentiating (2.13) and using Eq. (1.1) we obtain

$$w'(t) = -\frac{q(t)f(x(t))}{x(t)} - \frac{p(t)k_2(x(t), x'(t))x(t)}{x^2(t)} - \frac{r(t)k_1(x(t), x'(t))x'(t)}{x^2(t)} + \frac{e(t)}{x(t)}.$$

By using (A_4) and (A_5) we obtain for $t \in [s_1, t_1] \cup [s_2, t_2]$,

$$w'(t) \le -q(t)K|x(t)|^{\gamma-1} - \frac{\beta_2 p(t)}{r^{(\alpha+1)/\alpha}(t)} |w(t)|^{(\alpha+1)/\alpha} + \frac{e(t)}{x(t)}. \tag{2.14}$$

On the interval $[s_1, t_1]$, inequality (2.14) implies that $w(t)$ satisfies

$$q(t)K|x(t)|^{\gamma-1}+\frac{|e(t)|}{x(t)}\le -w'(t)-\frac{\beta_2 p(t)}{r^{(\alpha+1)/a}(t)}|w(t)|^{(\alpha+1)/\alpha}. \tag{2.15}$$

For $\gamma>1$, by setting $m=\gamma,\ n=\gamma/(\gamma-1),\ A=\gamma Kq(t)|x(t)|^{\gamma-1},\ B=\gamma/(\gamma-1)\frac{|e(t)|}{x(t)}$ and using Lemma 1, we obtain

$$q(t)K|x(t)|^{\gamma-1}+\frac{|e(t)|}{x(t)}\ge Q(t). \tag{2.16}$$

Hence, on the interval $[s_1, t_1]$, $w(t)$ satisfies

$$w'(t)\le -Q(t)-\frac{\beta_2 p(t)}{r^{(\alpha+1)/a}(t)}|w(t)|^{(\alpha+1)/\alpha}. \tag{2.17}$$

Note that the inequality holds for $\gamma=1$ also.

Applying operator $A(\cdot;s_i,t_i)$ for $i=1,$ to (2.17) we obtain a contradiction to (2.12), this part of the proof is similar to Theorem 1 and hence omitted.

If $x(t)<0$ on $[T_0,\infty)$ for some large $T_0\ge t_0$, it is easy to see that (2.17) holds for $t\in[s_2,t_2]$. Then applying operator $A(\cdot;s_i,t_i)$ for $i=2$, we still obtain contradiction. Thus the proof is complete.

Theorem 3 *Suppose the conditions* (A_4), (A_6) *and* (A_7) *hold. And for any* $T\ge t_0$, *there exists* $T\le s_1<t_1\le s_2\le t_2$ *such that (2.4) holds,* $p(t)\ge 0$ *and* $q(t)\ge 0$ *on* $[s_1,t_1]\cup[s_2,t_2]$. *Let* $D(s_i,t_i)$ *and* $A(\cdot;s_i,t_i)$ *are defined by (2.1) and (2.2) respectively. If there exists* $H\in D(s_i,t_i)$ *such that*

$$A(Q;s_i,t_i)>\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{1}{\beta_3^\alpha}A\left(r\left|2\frac{H'}{H}+\frac{\rho'}{\rho}\right|^{\alpha+1};s_i,t_i\right), \tag{2.18}$$

for $i=1,2,$ *where*

$$Q(t)=\gamma(\gamma-1)^{(1-\gamma)\gamma}[Kq(t)]^{1/\gamma}|e(t)|^{(\gamma-1)\gamma}$$

with the convention $0^0=1$. *Then Eq. (1.1) is oscillatory.*

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. We may assume that $x(t)>0$ on $[T_0,\infty)$ for some large $T_0\ge t_0$. By the assumptions, there exists $s_1,\ t_1,\ s_2$ and t_2 such that $T_0\le s_1<t_1\le s_2\le t_2$ and (2.4) holds, $p(t)\ge 0$ and $q(t)\ge 0$ on $[s_1,t_1]\cup[s_2,t_2]$. Define $w(t)$ as in (2.13). Then differentiating (2.13) and using Eq. (1.1) we obtain

$$w'(t)=-\frac{q(t)f(x(t))}{x(t)}-\frac{p(t)k_2(x(t),x'(t))x(t)}{x^2(t)}-\frac{r(t)k_1(x(t),x'(t))x'(t)}{x^2(t)}+\frac{e(t)}{x(t)}.$$

By using (A_4), (A_6) and (A_7) we obtain for $t\in s_1,t_1]\cup s_2,t_2]$,

$$w'(t)\le -q(t)K|x(t)|^{\gamma-1}-\frac{\beta_3}{r^{1/a}(t)}|w(t)|^{(\alpha+1)/\alpha}+\frac{e(t)}{x(t)}.$$

Rest of the proof is similar with previous theorem, hence omitted.

Remark 1 *If the hypotheses on the function* $e(t)$ *is replaced by the following condition*

$$e(t) \le 0 \;\; for\, t \in [s_1, t_1]\, and\; e(t) \ge 0 \;\; for\, t \in [s_2, t_2],$$

we will find the condition of the above theorems are valid as well.

Remark 2 *If we choose* $k_2(u,v) = vk_3(u,v)$ *Eq. (1.1) coincides with Eq. (1.5), which has been studied in [13]. In this case, we find that our conditions* $(A_1 - A_5)$ *are weaker than the corresponding conditions* $(A_1 - A_7)$ *imposed in the theorems given in [13]. Furthermore, we do not give any condition on the function* p *in our last theorem. Thus, we do not only extend, but also improve the results given in [13].*

Example 1 *Consider the equation*

$$\left(\left(\cos^2 t + 1\right)x^2(t)x'(t)\right)' + t^{-\lambda}x^3(t)\left(x'(t)\right)^2 + Kt^{\lambda}x(t) = \sin t, \tag{2.19}$$

where $t \ge t_0 > 1$ *and* $\lambda > 0.$ *It is easy to verify that the conditions* $(A_1) - (A_3)$ *hold for the functions*

$$k_1(u,v) = u^2 v, k_2(u,v) = u^3 v^2, \; f(x) = x$$

$$for \;\; \alpha = \beta_1 = 1.$$

Moreover let $H(t) = \sqrt{2}t^{-\lambda}\sin t$ *and* . *For any* $T \ge 1$, *choose* k *sufficiently large so that* $2k\pi \ge T$ *and* $s_1 = 2k\pi, \; t_1 = (2k+1)\pi.$ *Then we have*

$$A(q; s_1, t_1) = \int_{2k\pi}^{(2k+1)\pi} H^2(t)q(t)dt = K\int_{2k\pi}^{(2k+1)\pi} \sin^2 t dt = K\pi.$$

On the other hand

$$\begin{aligned}
\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{1}{\beta_1^{\alpha}}A\left(\frac{r^{\alpha+1}}{p^{\alpha}}\left|2\frac{H'}{H}\right|^{\alpha+1}; s_1, t_1\right) &= \frac{1}{4}A\left(\frac{(\cos^2 t+1)^2}{t^{-\lambda}}\left|2\frac{\cos t}{\sin t} - \lambda t^{-1}\right|^2; s_1, t_1\right) \\
&= \frac{1}{2}\int_{2k\pi}^{(2k+1)\pi}\left(\cos^2 t + 1\right)^2 \sin^2 t\left|2\frac{\cos t}{\sin t} - \lambda t^{-1}\right|^2 dt \\
&\le 2\int_{2k\pi}^{(2k+1)\pi}\left((\lambda+2)^2 + 2\lambda\right)dt \\
&= 2\left((\lambda+2)^2 + 2\lambda\right)\pi.
\end{aligned}$$

So, the inequality (2.5) hold for $K > 2\left((\lambda+2)^2 + 2\lambda\right)^2.$ Similarly, for $s_2 = (2k+1)\pi$ and $t_2 = (2k+2)\pi$, we can show that the inequality (2.5) holds for $K > 2\left((\lambda+2)^2 + 2\lambda\right)^2.$ Thus Eq. (2.19) is oscillatory if $K > \left((\lambda+2)^2 + 2\lambda\right)$ by Theorem 1.

Example 2 *Consider the equation*

$$\left(\left(\cos^2 t + 1\right)\frac{x^2(t)x'(t)}{1+x^2(t)}\right)' + t^{-\lambda}\frac{x^3(t)\left(x'(t)\right)^2}{1+x^2(t)} + Mt^{\lambda}x(t)\left(\cos x(t) + 2\right) = \sin t, \tag{2.20}$$

where $t \geq t_0 > 1$ *and* $\lambda > 0$. *In this example, the monotonicity condition* (A_2) *does not hold for the function* $f(x) = x(\cos x + 2)$. *But the condition* (A_4) *holds for* $K = \gamma = 1$. *The condition* (A_5) *also holds for* $\alpha = \beta_2 = 1$. *Thus Eq. (2.20) is oscillatory if* $M > 2\left((\lambda+2)^2 + 2\lambda\right)$ *by Theorem 2.*

Example 3 *Consider the equation*

$$\left((\cos^2 t + 1)\frac{x'(t)}{1+x^2(t)}\right)' + Nt^{\lambda} x(t) = \sin t, \quad t \geq t_0 > 1, \lambda > 0. \tag{2.21}$$

where $t \geq t_0 > 1$ *and* $\lambda > 0$. *Note that Theorem 1 and Theorem 2 are inapplicable to (2.21). But the conditions of Theorem 3 are fulfilled. According to a similar calculation with the examples above, we obtain that Theorem 3 ensures oscillation of (2.21) when* $\frac{\left((\lambda+2)^2 + 2\lambda\right)}{\lambda - 1}\frac{(2k\pi)^{1-\lambda} - ((2k+1)\pi)^{1-\lambda}}{\pi} < N$ *for some* k *which satisfies* $2k\pi \geq T$.

Remark 3 *In fact, the equations (2.19), (2.20) and (2.21) are particular cases of (1.1) and in the form of (1.5) which has been studied in [13]. But the conditions imposed in their theorems do not hold for these equations, so their theorems are not applicable to our examples.*

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] M.S. Keener, Solutions of a certain linear nonhomogeneous second order differential equations, Appl. Anal. 1 (1971) 57–63.

[2] A.Skidmore, W. Leighton, On the equation $y'' + p(x)y = f(x)$ J. Math. Anal. Appl. 43 (1973) 46–55.

[3] A. Skidmore, J. J. Bowers, Oscillation behavior of $y'' + p(x)y = f(x)$, J. Math. Anal. Appl. 49 (1975) 317–323.

[4] S.M. Rainkin, Oscillation theorems for second order nonhomogeneous linear differential equations, J. Math. Anal. Appl. 53 (1976) 550–553.

[5] J.S.W. Wong, Second order nonlinear forced oscillations, SIAM J. Math. Anal. 19 (1988) 667–675.

[6] F. Jiang, F. Meng, New oscillation criteria for a class of second-order nonlinear forced differential equations, J. Math. Anal. Appl. 336 (2007) 1476–1485.

[7] S.P. Rogovchenko, Y.V. Rogovchenko, Oscillation theorems for differential equation with a nonlinear damping term, J. Math. Anal. Appl. 279 (2003) 121–134.

[8] A. Tiryaki, A. Zafer, Interval oscillation of a general class of second-order nonlinear differential equations with nonlinear damping, Nonlinear Anal. 60 (2005) 49–63.

[9] A. Tiryaki, A. Zafer, Oscillation of Second-Order Nonlinear Different ial Equations with Nonlinear Damping, Mathematical and Computer modelling 39 (2004) 197-208.

[10] X. Zhao, F. Meng, Oscillation of second-order nonlinear ODE with damping, Appl. Math.Comput. 182 (2006) 1861–1871.

[11] Y. Huang, F. Meng, Oscillation of second-order nonlinear ODE with damping, Appl. Math. Comput. 199 (2008) 644–652.

[12] A. Zhao, Y. Wang, J. Yan, Oscillation criteria for second-order nonlinear differential equations with nonlinear damping, Computers and Mathematics with Applications 56 (2008) 542–555.

[13] F. Meng, Y. Huang, Interval oscillation criteria for a forced second-order nonlinear differential equations with damping, Appl. Math.Comput. 218 (2011) 1857–1861.

[14] W. Shi, Interval oscillation criteria for a forced second-order differential equation with nonlinear damping, Mathematical and Computer modelling 43 (2006) 170–177.

[15] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, second ed., Cambridge University Press, Cambridge, 1988.