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Research Article

On the computation of some code sets of the added Sierpinski triangle

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Abstract

In recent years, the intrinsic metrics have been formulated on the classical fractals. In particular, Sierpinski-like triangles such as equilateral, isosceles, scalene, added and mod−3 Sierpinski triangle have been considered in many different studies. The intrinsic metrics can be defined in different ways. One of the methods applied to obtain the intrinsic metric formulas is to use the code representations of the points on these self-similar sets. To define the intrinsic metrics via the code representations of the points on fractals makes also possible to investigate different geometrical, topological properties and geodesics of these sets. In this paper, we investigate some circles and closed sets of the added Sierpinski triangle and express them as the code sets by using its intrinsic metric.

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1. Introduction

Fractals are interesting and fascinating shapes with the models such as Cantor set, Sierpinski triangle, Menger sponge, Mandelbrot set and Julia sets ([6,9,10]). Apart from many different features, these structures have a common feature like the self-similarity. Many properties of these sets have been investigated from every aspect for years. Especially the Sierpinski triangle, S, has been considered as a fundamental model in various studies. In recent years, studies in which the intrinsic metric is formulated on this fractal come to the fore. As seen in [7,8,11–14,19], there are different ways to formulate this metric. To define the intrinsic metrics by using the code representations of the points on self-similar sets satisfies some advantages while determining geodesics or investigating geometrical and topological features of these sets (for details see [14–17]). Moreover, the intrinsic metrics are used in many studies obtained the chaotic dynamical systems on fractals such as Sierpinski triangle, Box fractal, Sierpinski propeller and Sierpinski tetrahedron (see [1–4,18]).

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Throughout this paper, we are interested in the added Sierpinski triangle, $\tilde{S}$, which can be obtained as the attractor of the iterated function system $\{\mathbb{R}^2; f_0, f_1, f_2, f_3\}$ where

$$\begin{aligned} f_0(x,y) &= \Big(\frac{x}{4}+\frac{3}{8}, \frac{y}{4}+\frac{\sqrt{3}}{8}\Big), \\ f_1(x,y) &= \Big(\frac{x}{2}, \frac{y}{2}\Big), \\ f_2(x,y) &= \Big(\frac{x}{2}+\frac{1}{2}, \frac{y}{2}\Big), \\ f_3(x,y) &= \Big(\frac{x}{2}+\frac{1}{4}, \frac{y}{2}+\frac{\sqrt{3}}{4}\Big). \end{aligned}$$

In [20], the intrinsic metric is formulated on $\tilde{S}$ via the code representations of the points. Note that it is much more complicated to formulate the intrinsic metric on $\tilde{S}$ compared to S (for the details see [14] and [20]). So, the computations of many interesting code sets of $\tilde{S}$ can be difficult. Moreover, the intrinsic metric on the added Sierpinski triangle has a special importance since it is the first formula expressed using code representations of points on a fractal, which is the attractor of an iterated function system with different contraction coefficients. In addition, there are very few studies on this fractal in the literature (see [5, 20]). That is why it is worthwhile to explore different code sets.

This paper presents some closed sets and circles with the code sets. For this aim, we consider the intrinsic metric defined on $\tilde{S}$ and compute these code sets by using the metric formulas and the code representations of the points on $\tilde{S}$. We obtain some closed sets and circles with code sets and only give the proofs of Proposition 2.3 and Proposition 2.5. Different cases, such as Example 2.7 and 2.8 with Corollary 2.4, 2.6, can be similarly proven. In addition, we illustrate them in Figure 2, 3, 4, 5, 6, 7, 8, 9.

The intrinsic metric formula on the added Sierpinski triangle: We want to briefly recall the intrinsic metric defined on the added Sierpinski triangle owing to the fact that it is too long to construct the intrinsic metric formula on (for the details such as coding and geometric interpretation see [20]). By the associated iterated function system, the code representations of the points on the added Sierpinski triangle is obtained as follows:

Let $\sigma = a_1a_2\ldots a_{k-1}$ and $f_\sigma(\widetilde{S}) = \widetilde{S}_\sigma$ where $a_i \in \{0,1,2,3\}$ for $i = 0,1,2,\ldots,k-1$. The middle part, the left-bottom part, the right-bottom and the upper parts of $\widetilde{S}_\sigma$ are expressed by $f_0(\widetilde{S}_\sigma) = \widetilde{S}_{\sigma 0}$, $f_1(\widetilde{S}_\sigma) = \widetilde{S}_{\sigma 1}$, $f_2(\widetilde{S}_\sigma) = \widetilde{S}_{\sigma 2}$ and $f_3(\widetilde{S}_\sigma) = \widetilde{S}_{\sigma 3}$ respectively. Consequently, we have

$$\widetilde{S}_\sigma = \widetilde{S}_{\sigma 0} \cup \widetilde{S}_{\sigma 1} \cup \widetilde{S}_{\sigma 2} \cup \widetilde{S}_{\sigma 3}$$

(see Figure 1).

The sub-added triangles of $\widetilde{S}$ have the nested set sequence relation such that

$$\widetilde{S} \supseteq \widetilde{S}_{a_1} \supseteq \widetilde{S}_{a_1a_2} \supseteq \widetilde{S}_{a_1a_2a_3} \ldots \supseteq \widetilde{S}_{a_1a_2a_3\ldots a_k} \ldots .$$

As a result of Cantor intersection theorem, $\bigcap_{k=1}^{\infty} \widetilde{S}_{a_1a_2a_3\ldots a_k}$ gives an unique point A on $\widetilde{S}$ and $a_1a_2a_3\ldots a_k\ldots$ is called as the code representation of the point A.

Let the code representations of the points A and B on $\tilde{S}$ be $a_1a_2\ldots a_{k-1}a_ka_{k+1}\ldots$ and $b_1b_2\ldots b_{k-1}b_kb_{k+1}\ldots$ respectively, where $a_i,\ b_i \in \{0,1,2,3\}$. Assume that $k = \min\{i \mid a_i \neq b_i\}$ and $\sigma = a_1a_2\ldots a_{k-1}$ and the number of elements of the set $\{i \mid a_i = b_i = 0, i < k\}$ is t. Suppose also that

$$\begin{aligned} M &= \{i+1 \mid a_i = 0, i > k\} = \{m_1, m_2, m_3, \ldots\} \\ L &= \{i+1 \mid b_i = 0, i > k\} = \{l_1, l_2, l_3, \ldots\} \end{aligned}$$

where $m_1 < m_2 < m_3 < \ldots$ and $l_1 < l_2 < l_3 < \ldots$.

The intrinsic metric is formulated in [20] as follows:

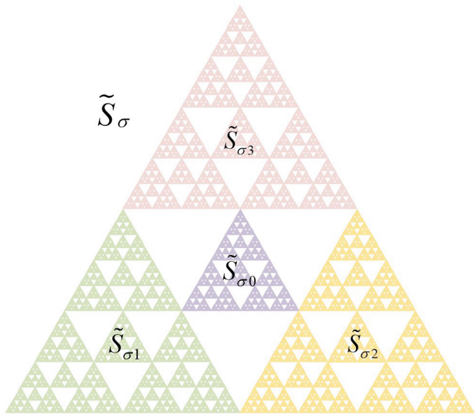

Figure 1. The sub-added Sierpinski triangle

Theorem 1.1. *Let $a_1a_2\dots a_{k-1}a_ka_{k+1}\dots$ and $b_1b_2\dots b_{k-1}b_kb_{k+1}\dots$ be two representations respectively of the points A and B on the added Sierpinski triangle such that $a_i = b_i$ for $i = 1, 2, \dots, k-1$ and $a_k \neq b_k$ and $a_i, b_i \in \{0, 1, 2, 3\}$. If $a_k \neq 0 \neq b_k$, then the intrinsic metric between the points A and B is formulated as*

$$d(A,B) = \min\left\{ \mathcal{A} + \mathcal{B},\ \frac{1}{2^{k+t}} + \mathcal{A}' + \mathcal{B}',\ \frac{1}{2^{k+t+1}} + \mathcal{A}'' + \mathcal{B}''\right\}, \tag{1.1}$$

and if $a_k \neq 0,\ b_k = 0$, then this formula is obtained as

$$d(A,B) = \min\left\{\mathcal{A}'' + \frac{1}{2}\mathcal{B},\ \frac{1}{2^{t+k+1}} + \mathcal{A}' + \frac{1}{2}\mathcal{B}',\ \frac{1}{2^{t+k+1}} + \mathcal{A} + \mathcal{C}\right\}. \tag{1.2}$$

In the following, we give $\mathcal{A}, \mathcal{A}', \mathcal{A}'', \mathcal{B}, \mathcal{B}', \mathcal{B}'', \mathcal{C}$ (for details see [20]):

- Let $a_k \neq 0 \neq b_k$.

$$\mathcal{A} = \sum_{i=k+1}^{m_1-1} \frac{\alpha_i}{2^{i+t}} + \frac{1}{2}\sum_{i=m_1}^{m_2-1} \frac{\alpha_i}{2^{i+t}} + \cdots + \frac{1}{2^r}\sum_{i=m_r}^{m_{r+1}-1} \frac{\alpha_i}{2^{i+t}} + \cdots \tag{1.3}$$

$$\mathcal{B} = \sum_{i=k+1}^{l_1-1} \frac{\beta_i}{2^{i+t}} + \frac{1}{2}\sum_{i=l_1}^{l_2-1} \frac{\beta_i}{2^{i+t}} + \cdots + \frac{1}{2^p}\sum_{i=l_p}^{l_{p+1}-1} \frac{\beta_i}{2^{i+t}} + \cdots \tag{1.4}$$

$$\mathcal{A}' = \sum_{i=k+1}^{m_1-1} \frac{\gamma_i}{2^{i+t}} + \frac{1}{2}\sum_{i=m_1}^{m_2-1} \frac{\gamma_i}{2^{i+t}} + \cdots + \frac{1}{2^r}\sum_{i=m_r}^{m_{r+1}-1} \frac{\gamma_i}{2^{i+t}} + \cdots \tag{1.5}$$

$$\mathcal{B}' = \sum_{i=k+1}^{l_1-1} \frac{\delta_i}{2^{i+t}} + \frac{1}{2}\sum_{i=l_1}^{l_2-1} \frac{\delta_i}{2^{i+t}} + \cdots + \frac{1}{2^p}\sum_{i=l_p}^{l_{p+1}-1} \frac{\delta_i}{2^{i+t}} + \cdots \tag{1.6}$$

where $a_k \neq c_k \neq b_k$ for $c_k \in \{1, 2, 3\}$ and

$$\gamma_i = \begin{cases} 0, & a_i = c_k \\ 1, & a_i \neq c_k \end{cases} \qquad \delta_i = \begin{cases} 0, & b_i = c_k \\ 1, & b_i \neq c_k \end{cases},$$

$$\alpha_i = \begin{cases} 0, & a_i = b_k \\ 1, & a_i \neq b_k \end{cases} \qquad \beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases}.$$

For the computation of $\mathcal{A}''$ ($\mathcal{B}''$ is computed similarly), there are three cases:

i) Let $a_{k+1} \neq a_k$ and $a_{k+1} \neq 0$. For $a_\mu \neq a_{k+1},\ a_\mu \neq a_k$ and $a_\mu \neq 0$,

$$\mathcal{A}'' = \sum_{i=k+2}^{m_1-1} \frac{\varphi_i}{2^{i+t}} + \frac{1}{2}\sum_{i=m_1}^{m_2-1} \frac{\varphi_i}{2^{i+t}} + \cdots + \frac{1}{2^r}\sum_{i=m_r}^{m_{r+1}-1} \frac{\varphi_i}{2^{i+t}} + \cdots \tag{1.7}$$

where

$$\varphi_i = \begin{cases} 0, & a_i = a_\mu \\ 1, & \text{otherwise.} \end{cases}$$

ii) Suppose that $a_{k+1} = 0$. For

$$r = \min\{i \mid a_i \neq 0,\ a_i \neq a_k,\ i \geq k+2\},$$

$$\mathcal{A}'' = \frac{1}{2^{k+t+2}} + \frac{1}{2}\sum_{i=k+2}^{m_2-1} \frac{\varphi_i}{2^{i+t}} + \frac{1}{2^2}\sum_{i=m_2}^{m_3-1} \frac{\varphi_i}{2^{i+t}} + \cdots + \frac{1}{2^r}\sum_{i=m_r}^{m_{r+1}-1} \frac{\varphi_i}{2^{i+t}} + \cdots \tag{1.8}$$

where

$$\varphi_i = \begin{cases} 0, & a_i = a_r \\ 1, & \text{otherwise.} \end{cases}$$

Note that, we obtain $\varphi_i = 1$ for $i = k+2, k+3, k+4, \ldots$ if

$$\{i \mid a_i \neq 0,\ a_i \neq a_k,\ i \geq k+2\} = \emptyset.$$

iii) a) Let $a_k = a_{k+1} = \cdots = a_{s-1} \neq a_s \neq 0$ $(s > k+1)$. We define

$$r = \min\{i \mid a_i \neq 0,\ a_i \neq a_k,\ i > s\}.$$

In this case, we obtain

$$\varphi_i = \begin{cases} 0, & a_i = a_r \\ 1, & \text{otherwise} \end{cases}$$

for $i \neq s$ and $\varphi_s = 0$ for $i = s$. If

$$\{i \mid a_i \neq 0,\quad a_i \neq a_k,\quad i > s\} = \emptyset,$$

then we also get $\varphi_i = 1$ for $i \neq s$ and $\varphi_s = 0$ for $i = s$.

b) Let $a_k = a_{k+1} = \cdots = a_{s-1}$ and $a_s = 0$ $(s > k+1)$. In this case, we get

$$\varphi_i = \begin{cases} 0, & a_i = a_r \\ 1, & \text{otherwise} \end{cases}$$

for $i \neq s$ and $\varphi_s = \frac{1}{2}$ for $i = s$ where $r = \min\{i \mid a_i \neq 0,\ a_i \neq a_k,\ i \geq k+2\}$. If

$$\{i \mid a_i \neq 0,\ a_i \neq a_k,\ i \geq k+2\} = \emptyset,$$

then we obtain $\varphi_i = 1$ for $i \neq s$ and $\varphi_s = \frac{1}{2}$ for $i = s$.

c) If $a_k = a_{k+1} = \cdots = a_i = \cdots$, then $\varphi_i = 1$ for $i = k+2, k+3, k+4, \ldots$

$$\mathcal{A}'' = \frac{1}{2^{k+t+1}} + \sum_{i=k+2}^{m_1-1} \frac{\varphi_i}{2^{i+t}} + \frac{1}{2}\sum_{i=m_1}^{m_2-1} \frac{\varphi_i}{2^{i+t}} + \cdots + \frac{1}{2^r}\sum_{i=m_r}^{m_{r+1}-1} \frac{\varphi_i}{2^{i+t}} + \cdots. \tag{1.9}$$

- Let $a_k \neq 0,\ b_k = 0$.

$$\mathcal{B} = \sum_{i=k+1}^{l_1-1} \frac{\beta_i}{2^{i+t}} + \frac{1}{2}\sum_{i=l_1}^{l_2-1} \frac{\beta_i}{2^{i+t}} + \cdots + \frac{1}{2^r}\sum_{i=l_p}^{l_{p+1}-1} \frac{\beta_i}{2^{i+t}} + \cdots \tag{1.10}$$

$$\mathcal{B}' = \sum_{i=k+1}^{l_1-1} \frac{\delta_i}{2^{i+t}} + \frac{1}{2}\sum_{i=l_1}^{l_2-1} \frac{\delta_i}{2^{i+t}} + \cdots + \frac{1}{2^r}\sum_{i=l_p}^{l_{p+1}-1} \frac{\delta_i}{2^{i+t}} + \cdots \tag{1.11}$$

$$\mathcal{C} = \frac{1}{2}\Big(\sum_{i=k+1}^{l_1-1} \frac{\beta'_i}{2^{i+t}} + \frac{1}{2} \sum_{i=l_1}^{l_2-1} \frac{\beta'_i}{2^{i+t}} + \cdots + \frac{1}{2^p} \sum_{i=l_p}^{l_{p+1}-1} \frac{\beta'_i}{2^{i+t}} + \cdots \Big) \tag{1.12}$$

where

$$\beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases},$$

$$\delta_i = \begin{cases} 0, & b_i = c_k \\ 1, & b_i \neq c_k \end{cases},$$

$$\beta'_i = \begin{cases} 0, & b_i = b'_k \\ 1, & b_i \neq b'_k \end{cases}$$

and $c_k \neq a_k$ for $c_k \in \{1, 2, 3\}$ and $b'_k \neq a_k$, $b'_k \neq c_k$ and $b'_k \in \{1, 2, 3\}$.

2. Some code sets of the added Sierpinski triangle

In this section, we give the code sets of some circles and closed discs of the added Sierpinski triangle and then we illustrate them. First of all, we give a few lemmas that we use them to prove some of propositions in the present paper.

Lemma 2.1. *Let A be a vertex point and B be any point of $\widetilde{S}_\sigma$ and suppose that the code representations of these points are $\sigma a_k a_k a_k \dots$ and $\sigma b_k b_{k+1} b_{k+2} \dots$ respectively, where $a_k \in \{1, 2, 3\}$ and $b_i \in \{0, 1, 2, 3\}$ for $i \in \{k, k+1, k+2, \dots\}$.*

a) *If $b_k \neq 0$, then $d(A, B) = \mathcal{A} + \mathcal{B}$.*

b) *If $b_k = 0$, then $d(A, B) = \mathcal{A}'' + \frac{1}{2}\mathcal{B}$.*

For the details of the proof see Lemma 3.5 in [20].

Lemma 2.2. *Let A be an arbitrary point of $\widetilde{S}_\sigma$ whose code representation is $\sigma a_k a_{k+1} a_{k+2} \dots$ for $a_i \in \{0, 1, 2, 3\}$ and $a_k \neq 0$. Suppose also that $\sigma 000 \dots$ is the code representation of $O_\sigma \in \widetilde{S}_\sigma$. In this case, we get*

$$d(A, O_\sigma) = \mathcal{A}'' + \frac{1}{2}\mathcal{B}.$$

Proof. If the formulas given in (1.10), (1.11) and (1.12) are used, then the following equalities are obviously obtained:

$$\begin{aligned} \frac{1}{2}\mathcal{B} = \frac{1}{2}\mathcal{B}' = \mathcal{C} &= \frac{1}{2}\left(\frac{1}{2^{k+t+1}} + \frac{1}{2}\frac{1}{2^{k+t+2}} + \frac{1}{2^2}\frac{1}{2^{k+t+3}} + \cdots \right) \\ &= \frac{1}{2^{k+t+2}}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots \right) = \frac{1}{3.2^{k+t}}. \end{aligned}$$

Furthermore, even if $\varphi_i = 1$ for $i = k+2, k+3, k+4, \dots$, for Case (i) and (ii), the maximum value of $\mathcal{A}''$ is calculated as

$$\mathcal{A}'' = \frac{1}{2^{k+t+2}} + \frac{1}{2^{k+t+3}} + \frac{1}{2^{k+t+4}} + \frac{1}{2^{k+t+5}} + \cdots = \frac{1}{2^{k+t+1}},$$

and we then get

$$\mathcal{A}'' + \frac{1}{2}\mathcal{B} \leq \frac{1}{2^{k+t+1}} + \mathcal{A}' + \frac{1}{2}\mathcal{B}' \quad \text{and} \quad \mathcal{A}'' + \frac{1}{2}\mathcal{B} \leq \frac{1}{2^{k+t+1}} + \mathcal{A} + \mathcal{C}.$$

A similar situation is also valid for Case (iii). Therefore, the proof is completed. □

Proposition 2.3. *Let* P, Q *and* R *be vertices of* $\widetilde{S}$ *with the code representations* $111\ldots$, $222\ldots$ *and* $333\ldots$ *respectively. In this case, circles with radii* $\frac{1}{2^{n-1}}$ $(n = 1, 2, 3, \ldots)$ *centered at* P, Q, R *are determined by the following code sets:*

$$S\Big(P, \frac{1}{2^{n-1}}\Big) = \{111\ldots 1x_n x_{n+1} x_{n+2}\ldots | x_{n+i} \in \{2, 3\}, i = 0, 1, 2, 3, \ldots\},$$
$$S\Big(Q, \frac{1}{2^{n-1}}\Big) = \{222\ldots 2x_n x_{n+1} x_{n+2}\ldots | x_{n+i} \in \{1, 3\}, i = 0, 1, 2, 3, \ldots\},$$
$$S\Big(R, \frac{1}{2^{n-1}}\Big) = \{333\ldots 3x_n x_{n+1} x_{n+2}\ldots | x_{n+i} \in \{1, 2\}, i = 0, 1, 2, 3, \ldots\}.$$

Proof. We know that the only code representation of P is $111\ldots$. Let us investigate the code representations of $X \in \widetilde{S}$ such as $x_1x_2\ldots x_{n-1}x_nx_{n+1}\ldots$ for $x_i \in \{0, 1, 2, 3\}$ satisfying $d(X, 111\ldots) = \frac{1}{2^{n-1}}$. The first $n-2$ terms of X and P must be exactly equal to each other. That is, it must be $x_i = 1$ for $i = 1, 2, 3, \ldots, n-2$. Note that if x_{n-2} (similarly, x_i for $i = 1, 2, \ldots, n-1$) were not 1, then

$$\mathcal{B} = \sum_{i=n-1}^{\infty} \frac{\beta_i}{2^i} = \sum_{i=n-1}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n-1}} + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots = \frac{1}{2^{n-2}} \tag{2.1}$$

and $d(X, P)$ would be greater than or equal to $\frac{1}{2^{n-2}}$. We thus obtain the code representation of X as the form

$$111\ldots 1x_{n-1}x_nx_{n+1}\ldots.$$

Note that, to compute $d(X, P)$ we use the formulas given in Lemma 2.1 owing to the fact that P is a vertex point.

- If $x_{n-1} \in \{2, 3\}$ and $x_i = 1$ $(n \geq 2)$ for $i = 1, 2, \ldots, n-2$, then it is computed that

$$\mathcal{B} = \sum_{i=n}^{\infty} \frac{\beta_i}{2^i} = \sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots = \frac{1}{2^{n-1}}. \tag{2.2}$$

  Therefore, we get

$$d(X, 111\ldots) = \sum_{i=n}^{\infty} \frac{\alpha_i + \beta_i}{2^i} = \sum_{i=n}^{\infty} \frac{\alpha_i}{2^i} + \frac{1}{2^{n-1}} = \frac{1}{2^{n-1}} \tag{2.3}$$

  and it must be

$$\sum_{i=n}^{\infty} \frac{\alpha_i}{2^i} = 0.$$

This shows that $x_i = 1$ for $i = n, n+1, n+2, \ldots$, that is the code representation of $X \in \widetilde{S}$ that satisfies Equation (2.3) must be one of the elements of the set

$$\{111\ldots 12111\ldots, 111\ldots 13111\ldots\}.$$

- If $x_n \in \{2, 3\}$ and $x_i = 1$ for $i = 1, 2, 3, \ldots, n-1$, then we have

$$\mathcal{B} = \sum_{i=n+1}^{\infty} \frac{\beta_i}{2^i} = \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \cdots = \frac{1}{2^n}. \tag{2.4}$$

  In this way, we obtain

$$d(X, 111\cdots) = \sum_{i=n+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i} = \sum_{i=n+1}^{\infty} \frac{\alpha_i}{2^i} + \frac{1}{2^n} = \frac{1}{2^{n-1}} \tag{2.5}$$

  and it follows that

$$\sum_{i=n+1}^{\infty} \frac{\alpha_i}{2^i} = \frac{1}{2^n}.$$

This is possible in the case of $x_i \neq 1$ for $i = n+1, n+2, n+3, \ldots$. It means that the code representations of X are the elements of the set

$$\{111\ldots 1x_nx_{n+1}x_{n+2}\ldots | x_i \in \{2,3\}, i = n, n+1, n+2, \ldots\}.$$

- If $x_{n-1} = 0$ and $x_i = 1$ $(n \geq 2)$ for $i = 1, 2, 3, \ldots, n-2$, then the formula

$$d(X, 111\ldots) = \mathcal{A}'' + \frac{1}{2}\mathcal{B}$$

must be used. Since the code representations of P and X are

$$111\ldots 111\ldots$$

and

$$111\ldots 10x_nx_{n+1}x_{n+2}\ldots$$

respectively, we compute

$$\mathcal{A}'' = \frac{1}{2^n} + \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots = \frac{1}{2^{n-1}} \tag{2.6}$$

(see Theorem 1.1 (iii-c)) and thus we get

$$d(P, X) = \mathcal{A}'' + \frac{1}{2}\mathcal{B} = \frac{1}{2^{n-1}} + \frac{1}{2}\mathcal{B} = \frac{1}{2^{n-1}}. \tag{2.7}$$

Therefore, it must be $\mathcal{B} = 0$. This happens in the case of $x_i = 1$ for $i = n, n+1, n+2, n+3, \ldots$. As a result, one of the code representations of the point X that provides $d(X, 111\ldots) = \dfrac{1}{2^{n-1}}$ is $111\ldots 10111\ldots$.

- If $x_n = 0$ and $x_i = 1$ $(n \geq 2)$ for $i = 1, 2, 3, \ldots, n-1$, then we compute

$$\mathcal{A}'' = \frac{1}{2^{n+1}} + \sum_{i=n+2}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \cdots = \frac{1}{2^n}. \tag{2.8}$$

This shows that

$$d(X, 111\ldots) = \mathcal{A}'' + \frac{1}{2}\mathcal{B} = \frac{1}{2^n} + \frac{1}{2}\mathcal{B} = \frac{1}{2^{n-1}}. \tag{2.9}$$

It means that $\mathcal{B} = \dfrac{1}{2^{n-1}}$. Even if $x_i \neq 1$ for $i = n, n+1, n+2, \ldots$, it is impossible to satisfy the following equation:

$$\sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n-1}}.$$

Note that the points whose code representations are $11\ldots 1011\ldots$, $11\ldots 1211\ldots$, $111\ldots 1311\ldots$ have different code representations such as $11\ldots 11233\ldots$, $11\ldots 11222\ldots$, $111\ldots 11333\ldots$ respectively.

It follows that the code representations $11\ldots 1011\ldots$, $11\ldots 1211\ldots$ and $111\ldots 1311\ldots$ are the elements of the set

$$\{111\ldots x_nx_{n+1}x_{n+2}\ldots | x_{n+i} \in \{2,3\}, i = 0, 1, 2, \ldots\}.$$

Consequently, we obtain

$$S\Big(111\ldots, \frac{1}{2^{n-1}}\Big) = \{111\ldots 1x_nx_{n+1}x_{n+2}\ldots | x_{n+i} \in \{2,3\}, i = 0, 1, 2, \ldots\}.$$

The other cases are done in a similar way and thus the proof is completed (see Figure 2). □

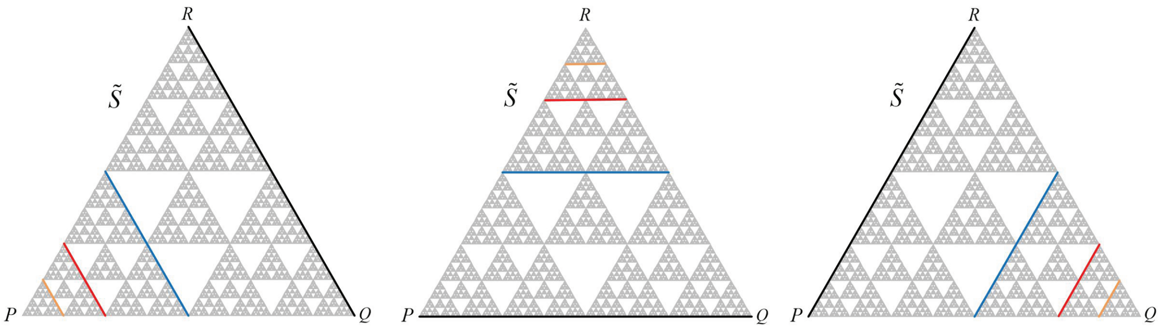

Figure 2. The circles with radii $\frac{1}{2^{n-1}}$ for $n = 1, 2, 3, 4$ centered at P, R, Q respectively.

Corollary 2.4. *Let P, Q and R be vertices of $\widetilde{S}$ with code representations $111\ldots$, $222\ldots$ and $333\ldots$ respectively. In this case, closed discs with radii $\frac{1}{2^{n-1}}$ $(n = 1, 2, 3, \ldots)$ centered at P, Q, R are determined by the following code sets:*

$$D\Big(P, \frac{1}{2^{n-1}}\Big) = \{111\ldots 1x_n x_{n+1} x_{n+2}\ldots | x_{n+i} \in \{0, 1, 2, 3\}, i = 0, 1, 2, 3, \ldots\},$$

$$D\Big(Q, \frac{1}{2^{n-1}}\Big) = \{222\ldots 2x_n x_{n+1} x_{n+2}\ldots | x_{n+i} \in \{0, 1, 2, 3\}, i = 0, 1, 2, 3, \ldots\},$$

$$D\Big(R, \frac{1}{2^{n-1}}\Big) = \{333\ldots 3x_n x_{n+1} x_{n+2}\ldots | x_{n+i} \in \{0, 1, 2, 3\}, i = 0, 1, 2, 3, \ldots\}$$

(see Figure 3).

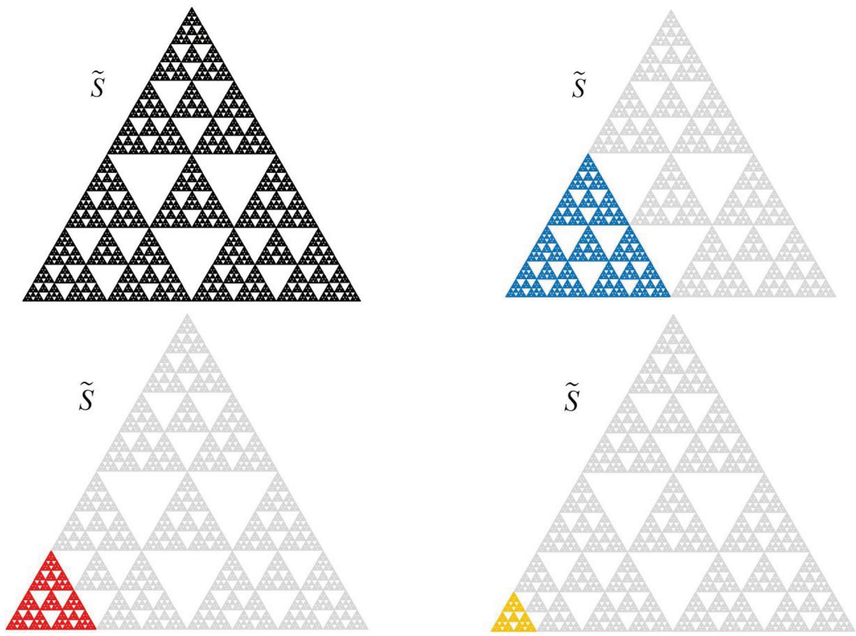

Figure 3. The closed discs with radii $\frac{1}{2^{n-1}}$ for $n = 1, 2, 3, 4$ centered at P respectively.

Proposition 2.5. *Let O_σ be a point of $\widetilde{S}$, whose code representation is $\sigma 000\ldots$. In this case, circles with radii $\frac{1}{2^{n+t+k}} + \frac{1}{3.2^{k+t}}$ centered at O_σ are obtained as follows:*

i) *For* $n=0$,

$$S\Big(O_\sigma, \frac{1}{2^{t+k}}+\frac{1}{3.2^{k+t}}\Big)=\{\sigma111\ldots,\sigma222\ldots,\sigma333\ldots\}.$$

ii) *For* $n=1$ *and* $x_k\neq x_{k+1}$ *and* $x_k\neq 0\neq x_{k+1}$,

$$S\Big(O_\sigma, \frac{1}{2^{t+k+1}}+\frac{1}{3.2^{k+t}}\Big)=\{\sigma x_k x_{k+1}x_{k+2}x_{k+3}\ldots|x_{k+i}\in\{x_k,x_{k+1}\}, i=2,3,\ldots\}$$
$$\cup\{\sigma x_k 0 x_k x_k\ldots|\ x_k\neq 0\}.$$

iii) *For* $n=2,3,4,\ldots$ *and* $x_k\neq x_{k+1}, x_k\neq 0\neq x_{k+1}$ *and* $x_\mu\neq x_k$, $x_\mu\neq x_{k+1}$, $x_\mu\neq 0$,

$$S\Big(O_\sigma,\ \frac{1}{2^{t+k+n}}+\frac{1}{3.2^{k+t}}\Big)=\{\sigma x_k x_{k+1}x_\mu\ldots x_\mu x_{k+n+1}x_{k+n+2}\ldots|\ x_{k+n+i}\in\{x_k,x_{k+1}\}, i=1,2,\ldots\}.$$

Proof. Suppose that

$$X=\sigma x_k x_{k+1}x_{k+2}\ldots,$$
$$O_\sigma=\sigma000\ldots000\ldots,$$

where $x_k\neq 0$.

Firstly, let us determine the code representations of the points on circle with radii $\frac{1}{2^{t+k}}+\frac{1}{3.2^{k+t}}$ centered at O_σ for $n=0$. The length of the shortest paths between points X and O_σ must be calculated from Lemma 2.2 with the formula

$$d(X,O_\sigma)=\mathcal{A}''+\frac{1}{2}\mathcal{B}.$$

Obviously, we compute

$$\begin{aligned}\frac{1}{2}\mathcal{B} &= \frac{1}{2}\left(\frac{1}{2^{k+t+1}}+\frac{1}{2}\frac{1}{2^{k+t+2}}+\frac{1}{2^2}\frac{1}{2^{k+t+3}}+\frac{1}{2^3}\frac{1}{2^{k+t+4}}+\ldots\right)\\ &= \frac{1}{2^{k+t+2}}\left(1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots\right)=\frac{1}{3.2^{k+t}}\end{aligned}$$

and

$$d(X,O_\sigma)=\mathcal{A}''+\frac{1}{2}\mathcal{B}=\mathcal{A}''+\frac{1}{3.2^{k+t}}=\frac{1}{2^{t+k}}+\frac{1}{3.2^{k+t}},$$

we thus get

$$\mathcal{A}''=\frac{1}{2^{t+k}}.$$

It is impossible to satisfy this equation by using the Cases (i) and (ii) given in Theorem 1.1. This equality is only provided with the Case (iii) given in Theorem 1.1. That is, it is possible with $\varphi_i=1$ for $i=k,k+1,k+2,\ldots$ and thus we obtain $x_k=x_{k+1}=\cdots=x_i=\cdots$ for $x_i\in\{1,2,3\}$ where $i=k,k+1,k+2,\ldots$. As a result, we compute

$$S\Big(O_\sigma,\ \frac{1}{2^{t+k}}+\frac{1}{3.2^{k+t}}\Big)=\{\sigma111\ldots,\sigma222\ldots,\sigma333\ldots\}=\{P_\sigma,Q_\sigma,R_\sigma\}.$$

Assume that $n=1$. We now determine the code set for the circles with radii $\frac{1}{2^{t+k+1}}+\frac{1}{3.2^{k+t}}$ centered at O_σ. With calculations similar to the above, we get

$$\mathcal{A}''=\frac{1}{2^{t+k+1}}. \tag{2.10}$$

To satisfy Equation (2.10), firstly we can use Case (i). It is possible with $M=\emptyset$ and $\varphi_i=1$ for $i=k+2,k+3,k+4,\ldots$. That is, it must be $x_k\neq x_{k+1}, x_{k+1}\neq 0$ where $x_\mu\neq 0$, $x_\mu\neq x_k$ and $x_\mu\neq x_{k+1}$. This requires to be $x_i\neq x_\mu$ for $i=k+2,k+3,\ldots$. It means that the code representations of X are the elements of the set

$$\{\sigma x_k x_{k+1}x_{k+2}x_{k+3}\ldots|x_k\neq 0\neq x_{k+1}, x_k\neq x_{k+1}, x_{k+i}\in\{x_k,x_{k+1}\}, i=2,3,\ldots\}.$$

To satisfy Equation (2.10), we can also use Case (ii). Assume that $x_{k+1} = 0$. It is possible with $M = \{k+2\}$ and $\varphi_i = 1$ for $i = k+2, k+3, k+4, \ldots$. This requires $x_i = x_k$ for $i = k+2, k+3, \ldots$. This shows that the code representations of X are the elements of the set

$$\{\sigma x_k 0 x_k x_k \ldots | x_k \in \{1,2,3\}\}.$$

We now use Case ($iii - a$) to satisfy Equation (2.10) (Note that it is impossible to satisfy Equation (2.10) by using Cases ($iii - b$) and ($iii - c$)). For this, it must be $\varphi_i = 0$ for $i = k+2, k+3, k+4, \ldots$. Therefore, we obtain either ($x_k = x_{k+1}$ and $x_i = x_{k+2}$ for $i = k+3, k+4, \ldots$ where $x_{k+2} \neq x_k$ and $x_{k+2} \neq 0$) or ($x_k = x_{k+1}$ and $x_i = x_{k+3}$ for $i = k+4, k+5, \ldots$ where $x_{k+3} \neq x_{k+2}, x_{k+3} \neq x_k$, $x_{k+3} \neq 0, x_{k+2} \neq x_k$ and $x_{k+2} \neq 0$). Consequently, the code representations of X are either the elements of the set

$$\{\sigma x_k x_k x_{k+2} x_{k+2} x_{k+2} \ldots | x_k, x_{k+2} \in \{1,2,3\}, x_k \neq x_{k+2}\}$$

or

$$\{\sigma x_k x_k x_{k+2} x_{k+3} x_{k+3} x_{k+3} \ldots | x_k, x_{k+2}, x_{k+3} \in \{1,2,3\} \text{ are different from each other}\}$$

respectively. Note that, the elements of these sets have different code representations as follows:

$$\{\sigma x_k x_{k+2} x_k x_k x_k \ldots | x_k, x_{k+2} \in \{1,2,3\}, x_k \neq x_{k+2}\}$$

and

$$\{\sigma x_k 0 x_k x_k x_{k+} x_k \ldots | x_k \neq 0\}$$

respectively.

As a consequence, we get

$$S\Big(O_\sigma, \frac{1}{2^{t+k+1}} + \frac{1}{3.2^{k+t}}\Big) = \{\sigma x_k x_{k+1} x_{k+2} x_{k+3} \ldots | x_k \neq x_{k+1}, x_{k+i} \in \{x_k, x_{k+1}\}, i = 2,3,\ldots\}$$
$$\cup \{\sigma x_k 0 x_k x_k \ldots |\ x_k \neq 0\}$$

where $x_k \neq 0 \neq x_{k+1}$ for $n = 1$.

Finally, we will show the code set of the circles with radii $\frac{1}{2^{n+t+k}} + \frac{1}{3.2^{k+t}}$ centered at O_σ for $n = 2,3,4,\ldots$. Since we compute $\dfrac{1}{2}\mathcal{B}$ as $\frac{1}{3.2^{k+t}}$ in the same way, we have

$$\mathcal{A}'' = \frac{1}{2^{t+k+n}}. \tag{2.11}$$

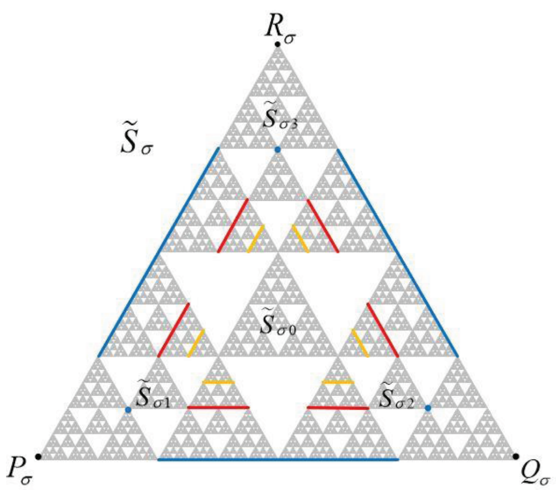

Figure 4. The circles with radii $\frac{1}{2^{n+t+k}} + \frac{1}{3.2^{k+t}}$ for $n = 0,1,2,3$ centered at O_σ.

To compute $\mathcal{A}''$ for $n \geq 2$, we can't use Case (iii) due to the fact that there is $\frac{1}{2^{t+k+1}}$ in Formula (1.9).

To satisfy Equation (2.11), we can use Case (i). It is possible with $\varphi_{k+i} = 0$ for $i = 2, 3, 4, \ldots, n$ and $\varphi_{k+i} = 1$ for $i = n+1, n+2, \ldots$. To be clear, it must be $x_k \neq x_{k+1}$, $x_{k+1} \neq 0$, $x_k \neq x_\mu \neq x_{k+1}$ and $x_\mu \neq 0$. That requires to be $x_i = x_\mu$ for $i = k+2, k+3, \ldots, k+n$ and $x_i \in \{x_k, x_{k+1}\}$ for $i = k+n+1, k+n+2, \ldots$. It means that the code representations of X are the elements of the set

$$G = \{\sigma x_k x_{k+1} x_\mu \ldots x_\mu x_{k+n+1} x_{k+n+2} \ldots \mid x_{k+i} \in \{x_k, x_{k+1}\}, i = n+1, n+2, \ldots\}.$$

For $n \geq 3$, Equation (2.11) is also satisfied while $\varphi_{k+i} = 0$ for $i = 2, 3, 4, \ldots, n-1, n+1, n+2, \ldots$ and $\varphi_{k+n} = 1$. In this case, it must be $x_{k+n} \neq x_\mu$ and $x_{k+i} = x_\mu$ for $i = 2, 3, 4, \ldots, n-1, n+1, n+2, \ldots$. Obviously, the code representations of these points are of the form

$$\sigma x_k x_{k+1} x_\mu \ldots x_\mu x_{k+n} x_\mu x_\mu x_\mu \ldots$$

which are the elements of the set G (that is, different code representations of the same points).

Note that, to satisfy Equation (2.11), we can also use Case (ii) for $n = 2$. If $x_{k+1} = 0$, then it is possible with $\varphi_i = 0$ for $i = k+2, k+3, k+4, \ldots$. This requires $0 \neq x_i \neq x_k$ for $i = k+2, k+3, \ldots$ and $x_{k+2} = x_{k+3} = x_{k+4} = \cdots$. It means that the code representation of X is the element of the set

$$\{\sigma x_k 0 x_{k+2} x_{k+2} x_{k+2} \ldots | x_{k+2} \neq x_k,\ x_{k+2} \in \{1, 2, 3\}\}.$$

Furthermore, the elements of this set are different code representations of the same points in G. As a result, we obtain

$$S\Big(O_\sigma,\ \frac{1}{2^{t+k+n}} + \frac{1}{3.2^{k+t}}\Big) = \{\sigma x_k x_{k+1} x_\mu \ldots x_\mu x_{k+n+1} x_{n+2} \ldots | x_{k+n+i} \in \{x_k, x_{k+1}\}, i = 1, 2, \ldots\}$$

where $x_k \neq x_{k+1}, x_k \neq 0 \neq x_{k+1}$ and $x_\mu \neq x_k$, $x_\mu \neq x_{k+1}$, $x_\mu \neq 0$, for $n = 2, 3, 4, \ldots$ (see Figure 4).

Therefore, the proof is completed. □

Corollary 2.6. *Let O_σ be a point of $\widetilde{S}$ with the code representation $\sigma 000\ldots$. In this case, the code sets of closed discs with radii $\frac{1}{2^{n+t+k}} + \frac{1}{3.2^{k+t}}$ centered at O_σ can be expressed as follows:*

i) *For $n = 0$,*

$$D\Big(O_\sigma, \frac{1}{2^{t+k}} + \frac{1}{3.2^{k+t}}\Big) = \{\sigma x_k x_{k+1} x_{k+2} x_{k+3} \ldots | x_{k+i} \in \{0, 1, 2, 3\}, i = 0, 1, 2, \ldots\} = \widetilde{S}_\sigma.$$

ii) *For $n = 1$ and $x_k \neq x_{k+1}$ if $x_k \in \{1, 2, 3\}$,*

$$D\Big(O_\sigma, \frac{1}{2^{t+k+1}} + \frac{1}{3.2^{k+t}}\Big) = \{\sigma x_k x_{k+1} x_{k+2} x_{k+3} \ldots | x_{k+i} \in \{0, 1, 2, 3\}, i = 0, 1, 2, \ldots\}.$$

iii) *For $n = 2, 3, 4, \ldots$ and $x_k \neq x_{k+1}$, $x_\mu \neq x_k$, $x_\mu \neq x_{k+1}$, $x_\mu \neq 0$,*

$$D\Big(O_\sigma,\ \frac{1}{2^{t+k+n}} + \frac{1}{3.2^{k+t}}\Big) =$$
$$\{\sigma x_k x_{k+1} x_\mu \ldots x_\mu x_{k+n+1} x_{k+n+2} \ldots | x_k \neq 0 \neq x_{k+1}, x_{k+n+i} \in \{0, 1, 2, 3\}, \quad i = 1, 2, \ldots\}$$
$$\cup \quad \{\sigma 0 x_{k+1} x_{k+2} x_{k+3} \ldots | x_{k+i} \in \{0, 1, 2, 3\}, i = 1, 2, 3, \ldots\}$$

(see Figure 5).

By making similar calculations, circles and closed disks with different centers and radii can be obtained. For the diversity, we will give some specific examples and show the obtained sets on the graphs without proofs.

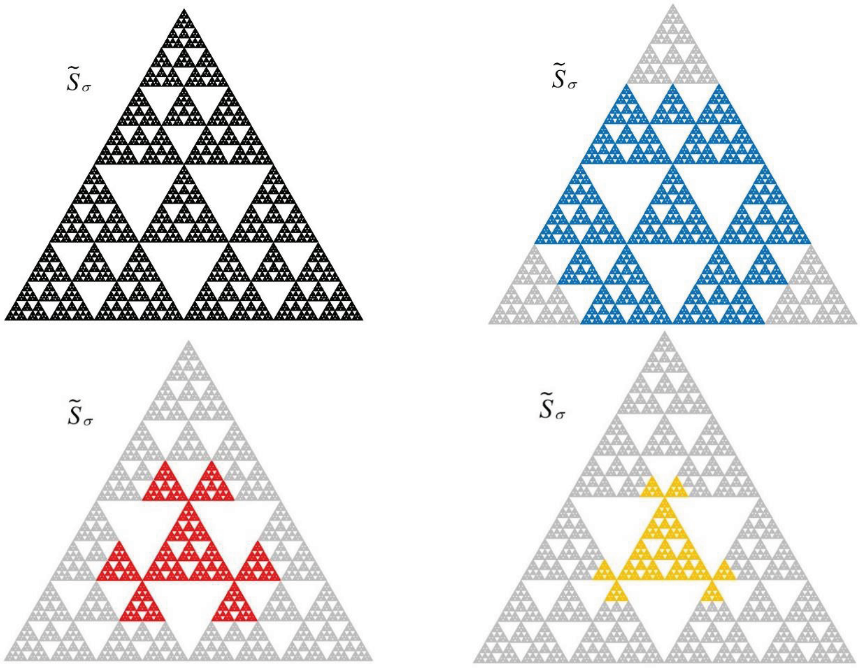

Figure 5. The closed discs with radii $\frac{1}{2^{n+t+k}} + \frac{1}{3.2^{k+t}}$ for $n = 0, 1, 2, 3$ centered at O_σ.

Example 2.7. Let A be a point of $\widetilde{S}$, whose code representation is $0111\ldots$. In this case, the circles with radii $\frac{1}{2^n}$ for $n = 1, 2, 3\ldots$ centered at A are determined by the following code sets:

i) For $n = 1$,

$$\{30333\ldots, 31333\ldots, 111\ldots, 21222\ldots, 20222\ldots\}$$
$$\cup\{x_1x_2x_3x_4x_5\ldots|x_i \in \{2,3\}, i = 1, 2, 3, \ldots \text{ and } x_1 \neq x_2\}.$$

ii) For $n = 2$,

$$\{10111\ldots\} \cup \{13x_3x_4x_5\ldots|x_i \in \{1,3\}, i = 3, 4, 5, \ldots\}$$
$$\cup\{12x_3x_4x_5\ldots|x_i \in \{1,2\}, i = 3, 4, 5, \ldots\} \cup \{0x_2\, x_3x_4x_5\ldots|x_i \in \{2,3\}, i = 2, 3, 4, \ldots\}.$$

iii) For $n = 3, 4, 5, \ldots$,

$$\{13x_3\ldots x_nx_{n+1}\ldots|x_i = 2, i = 3, 4, \ldots, n \text{ and } x_j \in \{1,3\}, j = n+1, n+2, \ldots\}$$
$$\cup\{12x_3\ldots x_nx_{n+1}\ldots|x_i = 3, i = 3, 4, 5, \ldots, n \text{ and } x_j \in \{1,2\}, j = n+1, n+2, n+3, \ldots\}$$
$$\cup\{01x_3\ldots x_nx_{n+1}\ldots|x_i = 1, i = 3, 4, 5, \ldots, n \text{ and } x_j \in \{2,3\}, j = n+1, n+2, n+3, \ldots\}$$

(see Figure 6).

Example 2.8. Let A be a point of $\widetilde{S}$, whose code representation is $0111\ldots$. In this case, the closed discs with radii $\frac{1}{2^n}$ for $n = 1, 2, 3\ldots$ centered at A are determined by the following code sets:

i) For $n = 1$,

$$\{x_1x_2x_3\ldots|x_i \in \{0,1,2,3\}, i = 1, 2, 3, \ldots \text{ and } x_1 \notin \{2,3\} \text{ for } x_1 = x_2\}.$$

ii) For $n = 2$,

$$\{0x_2x_3x_4\ldots|\, x_i \in \{0,1,2,3\}, i = 2, 3, 4, \ldots\} \cup \{10x_3x_4x_5\ldots|x_i \in \{0,1,2,3\}, i = 3, 4, 5, \ldots\} \cup$$
$$\{12x_3x_4x_5\ldots|x_i \in \{0,1,2,3\}, i = 3, 4, 5, \ldots\} \cup \{13x_3x_4x_5\ldots|x_i \in \{0,1,2,3\}, i = 3, 4, 5, \ldots\}.$$

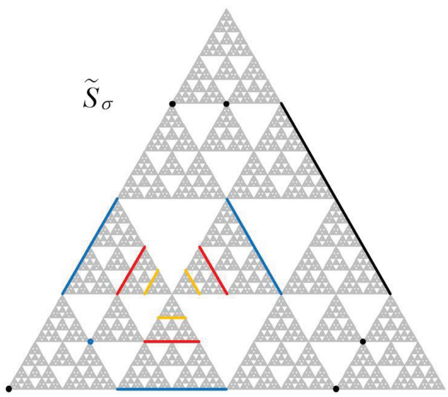

Figure 6. The circles with radii $\frac{1}{2^n}$ for $n = 1, 2, 3, 4$ centered at $0111\ldots$.

iii) For $n = 3, 4, 5, \ldots,$

$$\{13x_3 \ldots x_n x_{n+1} \ldots | x_i = 2, i = 3, 4, \ldots, n \text{ and } x_j \in \{0, 1, 2, 3\}, j = n + 1, n + 2, \ldots\}$$

$$\cup\{12x_3 \ldots x_n x_{n+1} \ldots | x_i = 3, i = 3, 4, 5, \ldots, n \text{ and } x_j \in \{0, 1, 2, 3\}, j = n+1, n+2, n+3, \ldots\}$$

$$\cup\{01x_3 \ldots x_n x_{n+1} \ldots | x_i = 1, i = 3, 4, 5, \ldots, n \text{ and } x_j \in \{0, 1, 2, 3\}, j = n+1, n+2, n+3, \ldots\}$$

(see Figure7).

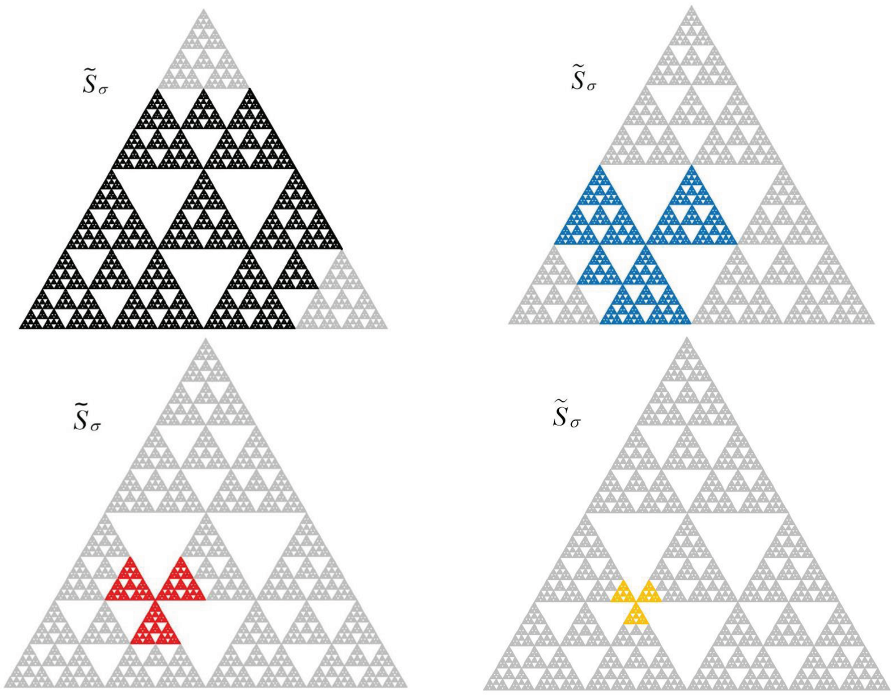

Figure 7. The closed discs with radii $\frac{1}{2^n}$ for $n = 1, 2, 3, 4$ centered at $0111\ldots$

In the following, to increase the number of different examples we only give two figures without defining code sets. Figure 8 and Figure 9 show the circles and closed discs with radii $\frac{1}{2^{n+t+k}}$ centered at the point with code representation $\sigma01222\ldots$ for $n = 0, 1, 2, 3,$ respectively.

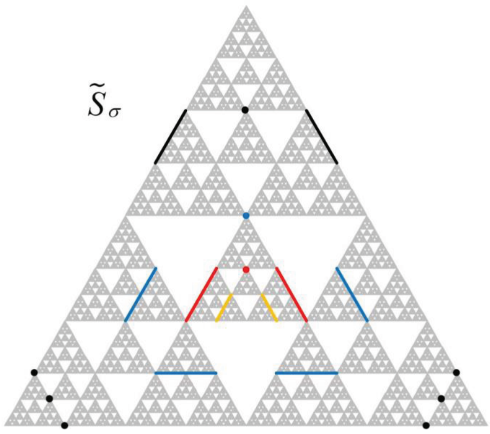

Figure 8. The circles with radii $\frac{1}{2^{n+t+k}}$ for $n = 0, 1, 2, 3$ centered at the point with code representation $\sigma 01222\ldots$.

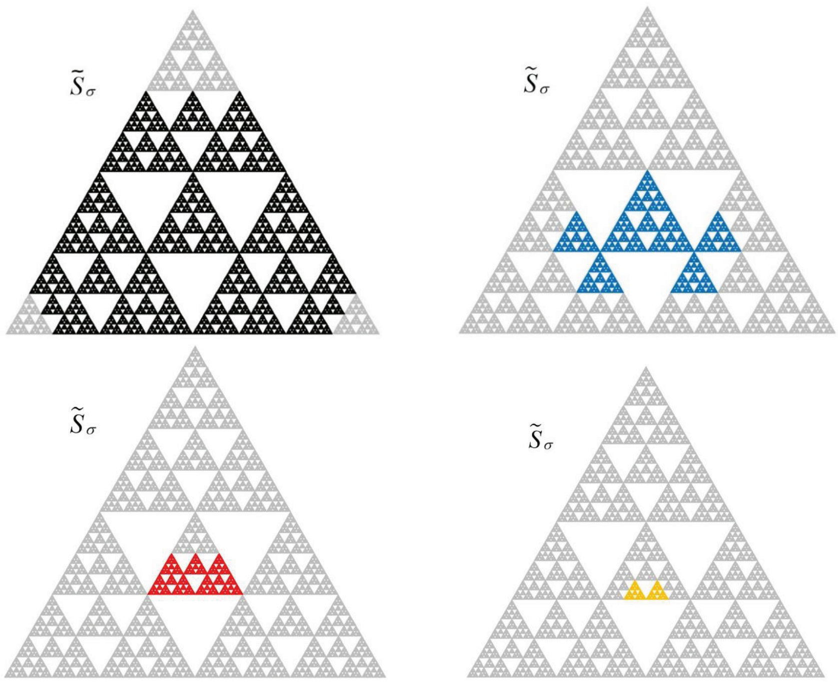

Figure 9. The closed discs with radii $\frac{1}{2^{n+t+k}}$ $n = 0, 1, 2, 3$ centered at the point with the code representation $\sigma 01222\ldots$.

3. Conclusions

In this paper, we compute some code sets of the added Sierpinski triangle by using the intrinsic metric and depict them. As seen in these figures and computations, some code sets are more understandable, while others can be more complex. Furthermore, a general formula cannot be obtained, especially for circles and closed sets of $\widetilde{S}$ as the code sets. As a result, one can obtain many different and interesting code sets of $\widetilde{S}$ in the light of the present paper.

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