Bitlis Eren Üniversitesi Fen Bilimleri Dergisi
BİTLİS EREN UNIVERSITY JOURNAL OF SCIENCE
ISSN: 2147-3129/e-ISSN: 2147-3188
VOLUME: 12 NO: 2 PAGE: 337-343 YEAR: 2023
DOI:10.17798/bitlisfen.1194877

On the Coefficient Bound Estimates and Fekete-Szegö Problem

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Keywords: Coefficient estimates, Fekete-Szegö problem, Analytic function.

Abstract
In this study, we introduce and examine a certain subclass of analytic functions in the open unit disk in the complex plane. Here, we give coefficient bound estimates and investigate the Fekete-Szegö problem for the introduced class. Some interesting special cases of the results obtained here are also discussed.

1. Introduction

In the study, by A we denote the class of all complex valued functions f which are analytic in the open unit disk $\mathcal{A}=\{t\in\mathbb{C}:|t|<1\}$ in the complex plane $\mathbb{C}$ and written in the form

$$f(t)=t+a_2t^2+\cdots+a_nt^n+\cdots$$
$$=t+\sum_{n=2}^{\infty}a_nt^n,\ t\in\mathbb{C}. \tag{1}$$

Then, the family of all univalent functions in A is denoted by S. Next, let $\alpha\in[0,1)$ then, $S^*(\alpha)$ denotes the starlike function classes of order α and $C(\alpha)$ denotes the convex function classes of order α in $\mathcal{A}$. By definition, we have

$$S^*(\alpha)=\left\{f\in S:\operatorname{Re}\left(\frac{tf'(t)}{f(t)}\right)>\alpha,\ t\in\mathcal{A}\right\}\text{ and}$$
$$C(\alpha)=\left\{f\in S:\operatorname{Re}\left(1+\frac{tf''(t)}{f'(t)}\right)>\alpha,\ t\in\mathcal{A}\right\}.$$

Moreover, if f and g are analytic functions in $\mathcal{A}$, then we say that f is subordinate to g and denote this condition by $f(t)\prec g(t)$ when an analytic function ω can be found such that it satisfies the conditions

$$\omega(0)=0,\ |\omega(t)|<1\text{ and }f(t)=g(\omega(t)).$$

The researchers agree that the coefficient problem is one of the crucial subjects of the geometric function theory. Many different and interesting subclasses of analytic functions have been defined and investigated by many researchers and some estimates on the first two coefficients for the functions of these classes have been found by them (see [2, 3, 10, 15, 19, 22]).
In the literature, the functional $H_2(1)=a_3-a_2^2$ is known as the Fekete-Szegö functional. Actually, the further generalized functional $H_2(1)=a_3-\mu a_2^2$ for real or complex number μ is known as the Fekete-Szegö functional in analytic functions theory (see [7]). In this theory, the Fekete-Szegö problem is to estimate the upper bound of $\left|a_3-\mu a_2^2\right|$. Many researchers have investigated this problem for different subclasses of analytic functions (see [12, 13, 21]). Very recently, the Fekete-Szegö problem for the subclass of bi-univalent functions in relation with a shell shaped region was studied by Mustafa and Mrugusundaramoorthy in [14] and associated with a nephroid domain in [20]. Also, the Fekete-Szegö problem is investigated for subclasses of bi-univalent functions with respect to the symmetric points defined by Bernoulli polynomials in [1], for bi-univalent

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Received: 26.10.2022, Accepted: 13.01.2023

functions related to the Legendre polynomials in [4], for m-fold symmetric bi-univalent functions in [16].

In [18] for the function $f \in A$, the following differential operator was introduced by Sălăgean and is known in the literature as the Sălăgean operator

$$S^0 f(t) = f(t),\ S^1 f(t) = tSf(t) = tf'(t),$$
$$S^2 f(t) = tS(Sf(t)) = tf''(t),...,$$
$$S^n f(t) = tS(S^{n-1} f(t)),$$
$$n = 1, 2, ...$$

From this definition, it can be clearly seen that

$$S^n f(t) = t + \sum_{k=2}^{\infty} k^n a_k t^k,\ t \in \mathcal{A},\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Sălăgean differential operator is used in recent research related to Janowski type $p-$harmonic functions [11], meromorphic harmonic functions [5] and starlike functions [9].

Now, we define some new subclasses of analytic and univalent functions as follows.

Definition 1.1. We will say a function $f \in S$ is in the class $S^*(n,\varphi)$ if it satisfies

$$\frac{z\left(S^n f(t)\right)'}{S^n f(t)} \prec \varphi(t),\ t \in \mathcal{A}$$

for $n = 0, 1, 2, ...$.

In the case $n = 0$, we have the subclass $S^*(\varphi) = S^*(0,\varphi)$.

In the Definition 1.1, $\varphi(t) = t + \sqrt{1+t^2}$ and the branch of the square root is chosen with the initial value $\varphi(0) = 1$. It can be clearly seen that by $\varphi(t) = t + \sqrt{1+t^2}$, the unit disc $\mathcal{A}$ is mapped onto a shell shaped region on the right half plane and φ is univalent and analytic in $\mathcal{A}$. Respect to real axis, the range $\varphi(\mathcal{A})$ is symmetric and φ has positive real part in $\mathcal{A}$ such that $\varphi(0) = \varphi'(0) = 1$. Furthermore, with respect to point $\varphi(0) = 1$, $\varphi(\mathcal{A})$ is a starlike domain.

Let, P be the set of the functions $\mathrm{k}(t)$ analytic in $\mathcal{A}$ and satisfying $\mathrm{Re}(\mathrm{k}(t)) > 0,\ t \in \mathcal{A}$ and $\mathrm{k}(0) = 1$ with power series

$$\mathrm{k}(t) = 1 + \mathrm{k}_1 t + \mathrm{k}_2 t^2 + \mathrm{k}_3 t^3 + \cdots + \mathrm{k}_n t^n + \cdots$$
$$= 1 + \sum_{n=1}^{\infty} \mathrm{k}_n t^n,\ t \in \mathcal{A}.$$

We will need the lemmas below (see [6, 8]) for the functions with positive real part so that we can show our main results.

Lemma 1.2. Let $\mathrm{k} \in \mathrm{P}$, then $|\mathrm{k}_n| \le 2$ for $n = 1, 2, ...$ and

$$\left|\mathrm{k}_2 - \frac{c}{2}\mathrm{k}_1^2\right| \le 2 \cdot \max\{1,\ |c-1|\}$$
$$= 2 \cdot \begin{cases} 1 & \text{if } c \in [0,2], \\ |c-1| & \text{elsewhere.} \end{cases}$$

Lemma 1.3. Let $\mathrm{k} \in \mathrm{P}$, then $|\mathrm{k}_n| \le 2$ for each $n = 1, 2, ...$ and

$$2\mathrm{k}_2 = \mathrm{k}_1^2 + \left(4 - \mathrm{k}_1^2\right)x,$$

$$4\mathrm{k}_3 = \mathrm{k}_1^3 + 2\left(4 - \mathrm{k}_1^2\right)\mathrm{k}_1 x - 2\left(4 - \mathrm{k}_1^2\right)\mathrm{k}_1 x^2$$
$$+ 2\left(4 - \mathrm{k}_1^2\right)\left(1 - |x|^2\right)z$$

for some x and z with $|x| < 1$ and $|z| < 1$.

Lemma 1.4. Let $\mathrm{k} \in \mathrm{P}$, $b \in [0,1]$ and $b(2b-1) \le d \le b$. Then,

$$\left|\mathrm{k}_3 - 2b\mathrm{k}_1\mathrm{k}_2 + d\mathrm{k}_1^3\right| \le 2.$$

Remark 1.5. As can be seen from the serial expansion the function φ given in Definition 1.1, belong to the class P.

In this paper, we give coefficient bound estimates and solve the Fekete-Szegö problem for the class $S^*(n,\varphi)$.

2. Main Results

In this section, firstly we present the below theorem on the coefficient bound estimates for the class $S^*(n,\varphi)$. In the study of bi-univalent functions, estimates on the first two Taylor–Maclaurin coefficients are usually given. We go further in the present paper and bounds of the first three coefficients as seen also in paper [17].

Theorem 2.1. Let the function f given by (1) be in the class $S^*(n,\varphi)$. Then,

$$|a_2| \le \frac{1}{2^n}, \ |a_3| \le \frac{1}{4\cdot 3^{n-1}} \text{ and } |a_4| \le \frac{5}{6\cdot 4^n}.$$

Proof. Let $f \in S^*(n,\varphi)$. Then, according to Definition 1.1 there is an analytic function $\omega : \mathcal{A} \to \mathcal{A}$ with $\omega(0)=0$ and $|\omega(t)|<1$ satisfying the following condition

$$\frac{t\left(S^n f(t)\right)'}{S^n f(t)} = \varphi(\omega(t)) = \omega(t)+\sqrt{1+\omega^2(t)}, \ t\in\mathcal{A} \tag{2}$$

Let, we define the function $t \in \mathrm{P}$ as follows

$$\begin{aligned}\mathrm{k}(t) &= \frac{1+\omega(t)}{1-\omega(t)} \\ &= 1+\mathrm{k}_1 t+\mathrm{k}_2 t^2+\mathrm{k}_3 t^3+\cdots+\mathrm{k}_n t^n+\cdots \\ &= 1+\sum_{n=1}^{\infty}\mathrm{k}_n t^n, \ t\in\mathcal{A}\end{aligned}$$

It follows from that

$$\begin{aligned}\omega(t) &= \frac{\mathrm{k}(t)-1}{\mathrm{k}(t)+1} \\ &= \frac{1}{2}\left[\mathrm{k}_1 t+\left(\mathrm{k}_2-\frac{\mathrm{k}_1^2}{2}\right)t^2+\left(\mathrm{k}_3-\mathrm{k}_1\mathrm{k}_2+\frac{\mathrm{k}_1^2}{4}\right)t^3+\cdots\right], \\ & t\in\mathcal{A}.\end{aligned} \tag{3}$$

Changing the formulation of the function ω in (2) with formulation in (3), we get

$$\begin{aligned}&\frac{t\left(S^n f(t)\right)'}{S^n f(t)} \\ &= 1+\frac{\mathrm{k}_1}{2}t+\left(\frac{\mathrm{k}_2}{2}-\frac{\mathrm{k}_1^2}{8}\right)t^2+\left(\frac{\mathrm{k}_3}{2}-\frac{\mathrm{k}_1\mathrm{k}_2}{4}\right)t^3+\ldots, \\ &t\in\mathcal{A}\end{aligned} \tag{4}$$

If necessary derivative operations and simplifications are made to the left-hand side of (4), we get

$$\begin{aligned}&1+2^n a_2 t+\left(2\cdot 3^n a_3-4^n a_2^2\right)t^2 \\ &+\left(3\cdot 4^n a_4-3\cdot 6^n a_2 a_3+8^n a_2^3\right)t^3+\cdots \\ &= 1+\frac{\mathrm{k}_1}{2}t+\left(\frac{\mathrm{k}_2}{2}-\frac{\mathrm{k}_1^2}{8}\right)t^2+\left(\frac{\mathrm{k}_3}{2}-\frac{\mathrm{k}_1\mathrm{k}_2}{4}\right)t^3+\ldots, \\ &t\in\mathcal{A}\end{aligned} \tag{5}$$

Then, by equalizing the coefficients of the terms of the same degree, are obtained the following equalities for a_2, a_3 and a_4

$$2^n a_2 = \frac{\mathrm{k}_1}{2}, \ 2\cdot 3^n a_3 - 4^n a_2^2 = \frac{\mathrm{k}_2}{2}-\frac{\mathrm{k}_1^2}{8},$$

$$3\cdot 4^n a_4 - 3\cdot 6^n a_2 a_3 + 8^n a_2^3 = \frac{\mathrm{k}_3}{2}-\frac{\mathrm{k}_1\mathrm{k}_2}{4};$$

that is,

$$a_2 = \frac{\mathrm{k}_1}{2^{n+1}}, \tag{6}$$

$$a_3 = \frac{1}{2}\left(\frac{4}{3}\right)^n a_2^2 + \frac{1}{4\cdot 3^n}\left(\mathrm{k}_2-\frac{\mathrm{k}_1^2}{4}\right), \tag{7}$$

$$a_4=\left(\frac{3}{2}\right)^n a_2a_3-\frac{2^n}{3}a_2^3+\frac{1}{6\cdot 4^n}\left(k_3-\frac{k_1k_2}{2}\right). \quad (8)$$

By applying the Lemma 1.2 to the equality (6), immediately obtained first result of theorem.
Now, considering equality (6) and applying the Lemma 1.3, then applying triangle inequality and Lemma 1.2 to the equality (7), we get the following inequality as

$$|a_3|\le\frac{3}{16\cdot 3^n}z^2+\frac{4-z^2}{8\cdot 3^n}\xi,\ \xi\in(0,1), \quad (9)$$

with $z=|k_1|$, $\xi=|x|<1$. If we maximize the right-hand side of the inequality (9) respect to the parameter ξ, we obtain

$$|a_3|\le\frac{3}{16\cdot 3^n}z^2+\frac{4-z^2}{8\cdot 3^n},\ z\in[0,2];$$

that is,

$$|a_3|\le\frac{1}{3^n}\left(\frac{z^2}{16}+\frac{1}{2}\right),\ z\in[0,2].$$

From the last inequality obtained the second result of theorem.
Finally, let's find an upper bound estimate for the coefficient a_4. From the equalities (6)-(8), we write

$$a_4=\frac{k_1}{2^{2n+3}}\left(k_2-\frac{k_1^2}{4}\right)+\frac{1}{6\cdot 4^n}\left(k_3-\frac{k_1k_2}{2}+\frac{k_1^3}{8}\right)$$

that is,

$$a_4=\frac{k_1}{2^{2n+3}}\left(k_2-\frac{c}{2}k_1^2\right)+\frac{1}{6\cdot 4^n}\left(k_3-2bk_1k_2+dk_1^3\right),$$

with $c=\frac{1}{2}$, $b=\frac{1}{4}$ and $d=\frac{1}{8}$. Using triangle inequality to the last equality, we get

$$|a_4|\le\frac{|k_1|}{2^{2n+3}}\left|k_2-\frac{c}{2}k_1^2\right|+\frac{1}{6\cdot 4^n}\left|k_3-2bk_1k_2+dk_1^3\right|.$$

Also, since

$$|k_1|\le 2,\ \left|k_2-\frac{c}{2}k_1^2\right|\le 2 \text{ and}$$
$$\left|k_3-2bk_1k_2+dk_1^3\right|\le 2,$$

according to Lemma 1.2 and Lemma 1.4, respectively, we obtain desired estimate for $|a_4|$.
Therefore, the proof of Theorem 2.1 is done.

In the case $n=0$, from the Theorem 2.1 obtained the following result.

Corollary 2.2. Assume that f given by (1) is in the class $S^*(\varphi)$. Then,

$$|a_2|\le 1,\ |a_3|\le\frac{3}{4} \text{ and } |a_4|\le\frac{5}{6}.$$

Now, we give the following theorem on the Fekete-Szegö problem for the class $S^*(n,\varphi)$.

Theorem 2.2. Assume that f given by (1) is in the class $S^*(n,\varphi)$ and $\mu\in\mathbb{C}$. Then,

$$|a_3-\mu a_2^2| \le\frac{1}{4\cdot 3^n}\cdot\begin{cases}2 & \text{if } 2\left|\left(\frac{4}{3}\right)^n-2\mu\right|\le\left(\frac{4}{3}\right)^n,\\ 2\left|\left(\frac{4}{3}\right)^n-2\mu\right|\left(\frac{3}{4}\right)^n+1 & \text{if } 2\left|\left(\frac{4}{3}\right)^n-2\mu\right|>\left(\frac{4}{3}\right)^n.\end{cases}$$

Proof. Let $f\in S^*(n,\varphi)$ and $\mu\in\mathbb{C}$. Then, from the expressions for the coefficients a_2 and a_3 in the equalities (6) and (7), we can write

$$a_3-\mu a_2^2=\frac{1}{2}\left[\left(\frac{4}{3}\right)^n-2\mu\right]a_2^2+\frac{1}{4\cdot 3^n}\left(k_2-\frac{k_1^2}{4}\right).$$

Considering equality (6) and applying Lemma 1.3, we write the following equality

$$a_3-\mu a_2^2=\frac{1}{8}\left\{\left[\left(\frac{4}{3}\right)^n-2\mu\right]\frac{k_1^2}{4^n}+\frac{1}{3^n}\left(\frac{k_1^2}{2}+\left(4-k_1^2\right)x\right)\right\}$$

for some x with $|x|<1$. From this, using triangle inequality we obtain

$$\left|a_3-\mu a_2^2\right|\le$$
$$\frac{1}{8}\left\{\left[\left|\left(\frac{4}{3}\right)^n-2\mu\right|\frac{1}{4^n}+\frac{1}{2\cdot 3^n}\right]z^2+\frac{4-z^2}{3^n}\xi\right\},$$
$$\xi\in(0,1)$$

with $z=|k_1|$, $\xi=|x|$. If we maximize the right-hand side of this inequality with respect to the parameter ξ, we get

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{8\cdot 3^n}\left\{\left[\left|\left(\frac{4}{3}\right)^n-2\mu\right|\left(\frac{3}{4}\right)^n-\frac{1}{2}\right]z^2+4\right\},$$
$$z\in[0,2]. \tag{10}$$

Now, we define the function $\sigma:[0,2]\to\mathbb{R}$ as follows

$$\sigma(z)=\left[\left|\left(\frac{4}{3}\right)^n-2\mu\right|\left(\frac{3}{4}\right)^n-\frac{1}{2}\right]z^2+4,\ z\in[0,2].$$

It is clear that the function σ is a decreasing function if

$$2\left|\left(\frac{4}{3}\right)^n-2\mu\right|\le\left(\frac{4}{3}\right)^n$$

and

$$\max\{\sigma(z):z\in[0,2]\}=\sigma(0)=4. \tag{11}$$

Additionally, the function σ is an increasing function if

$$2\left|\left(\frac{4}{3}\right)^n-2\mu\right|>\left(\frac{4}{3}\right)^n$$

and

$$\max\{\sigma(z):z\in[0,2]\}=\sigma(2)$$
$$=4\left|\left(\frac{4}{3}\right)^n-2\mu\right|\left(\frac{3}{4}\right)^n+2. \tag{12}$$

Considering (11) and (12) in the inequality (10), we arrive at the result.
That is, the proof of Theorem 2.2 is done.

In the cases $n=0$ and $\mu=0$, respectively from the Theorem 2.2, the following results are obtained.

Corollary 2.3. Let $f\in S^*(\varphi)$ and $\mu\in\mathbb{R}$, then

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{4}\cdot\begin{cases}2 & if\ \ 2|1-2\mu|\le 1,\\ 2|1-2\mu|+1 & if\ \ 2|1-2\mu|>1.\end{cases}$$

Corollary 2.4. Let $f\in S^*(n,\varphi)$, then

$$|a_3|\le\frac{1}{4\cdot 3^{n-1}}.$$

Corollary 2.5. Let $f\in S^*(n,\varphi)$, then

$$\left|a_3-a_2^2\right|\le$$
$$\frac{1}{4\cdot 3^n}\cdot\begin{cases}2 & if\ \ 2\left|\left(\frac{4}{3}\right)^n-2\right|\le\left(\frac{4}{3}\right)^n,\\ 2\left|\left(\frac{4}{3}\right)^n-2\right|\left(\frac{3}{4}\right)^n+1 & if\ \ 2\left|\left(\frac{4}{3}\right)^n-2\right|>\left(\frac{4}{3}\right)^n.\end{cases}$$

Remark 2.6. Result obtained in the Corollary 2.4 confirm the second inequality obtained in Theorem 2.1.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

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