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THE TYPE I HEAVY-TAILED ODD POWER GENERALIZED WEIBULL-G FAMILY OF DISTRIBUTIONS WITH APPLICATIONS

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Abstract. In this study, we propose a new heavy-tailed distribution, namely, the type I heavy-tailed odd power generalized Weibull-G family of distributions. Several statistical properties including hazard rate function, quantile function, moments, distribution of the order statistics and Rényi entropy are presented. Actuarial measures such as value at risk, tail value at risk, tail variance and tail variance premium are also derived. To obtain the estimates of the parameters of the new family of distributions, we adopt the maximum likelihood estimation method and assess the consistency property via a Monte Carlo simulation. Finally, we illustrate the usefulness of the new family of distributions by analyzing four real life data sets from different fields such as insurance, engineering, bio-medical and environmental sciences.

1. Introduction

Several researchers have developed probability models by adding one or more parameter(s) to well known classical distributions in order to improve their fitting power (flexibility). However, these extended distributions cannot model all real life data sets. Thus, serious efforts are still needed to propose and develop new flexible distributions. Several new generated distributions available in the literature include: Topp-Leone odd Burr III-G family of distributions by [23], the Marshall-Olkin exponentiated odd exponential half logistic-G family of distributions by [27], exponentiated odd Lomax exponential distribution by [16], the Marshall-Olkin odd

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exponential half logistic-G family of distributions by [26], the shifted Gompertz-G family of distributions by [17], type II half-logistic odd Fréchet class of distributions by [7], generalized modified exponential-G family of distributions by [19], truncated Cauchy power Weibull-G class of distributions by [6], modified alpha power family of distributions by [20] and type II general exponential class of distributions by [18] to mention a few.

Data sets in different fields, such as actuarial sciences, reliability, engineering, bio-medical sciences, economic, risk management are usually positive, right-skewed, unimodal with heavier tails. These data sets need to be modelled by heavy-tailed distributions. Thus, there is need for development of heavy-tailed distributions. Some heavy-tailed distributions available in the literature include among others; the type-I heavy-tailed Weibull distribution by [30], heavy-tailed beta-power transformed Weibull distribution by [31], heavy-tailed log-logistic distribution by [29], heavy-tailed exponential distribution by [1], the logit slash distribution by [21], and the LogPH class of distributions by [2].

The type-I heavy-tailed (TI-HT) family of distributions introduced by [30] have the cumulative distribution function (cdf) and probability density function (pdf) given by

$$G(x;\theta,\xi) = 1 - \left(\frac{1-F(x;\xi)}{1-(1-\theta)F(x;\xi)}\right)^{\theta}, \tag{1}$$

and

$$g(x;\theta,\xi) = \frac{\theta^2 f(x;\xi)\left(1-F(x;\xi)\right)^{\theta-1}}{\left(1-(1-\theta)F(x;\xi)\right)^{\theta+1}}, \tag{2}$$

for $\theta > 0, x \in \mathbb{R}$. [24] introduced the odd power generalized Weibull-G family of distributions with the cdf and pdf given as

$$F(x;\alpha,\beta,\xi) = 1 - \exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right), \tag{3}$$

and

$$\begin{aligned} f(x;\alpha,\beta,\xi) &= \alpha\beta\left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1} \\ &\times \exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\frac{g(x;\xi)}{\left(1-G(x;\xi)\right)^2}, \end{aligned} \tag{4}$$

respectively, for $\alpha, \beta > 0$ and parameter vector ξ.

The aim of this paper is to develop a new family of heavy-tailed distributions namely, type I heavy-tailed odd power generalized Weibull-G (TI-HT-OPGW-G) family of distributions by combining equations (1), (2), (3) and (4).

The general objectives of constructing this new family of distributions include the following:

- to generate distributions which are skewed, symmetric, J-shaped or reversed-J shaped;
- to define new family of distributions that possesses various types of hazard rate functions including monotonic as well as non-monotonic shapes;
- to construct new statistical distributions with better fits and properties than other competitive distributions;
- to construct heavy-tailed distributions for modeling various real data sets.

The rest of the work is organized in the following manner. Section 2 present the new TI-HT-OPGW-G family of distributions, reliability and hazard rate functions, sub-families, linear representation and quantile function. In Section 3, moments, moment generating function, the distribution of order statistics, and Rényi entropy are presented. In Section 4, some special models from the TI-HT-OPGW-G family of distributions are presented. Section 5 contains the estimation of the unknown parameters of the TI-HT-OPGW-G family of distributions via the method of maximum likelihood and a Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimators is given in Section 6. The results verified that the estimates are consistent as the mean tend to the true parameters, the root mean square error and average bias decreases when the sample size (n) increases. Section 7 contains actuarial measures. Four real data applications are given in Section 8, followed by some concluding remarks in Section 9.

2. The New Family of Distributions

This section present the type I heavy-tailed odd power generalized Weibull-G (TI-HT-OPGW-G) family of distributions. The cdf and pdf of the TI-HT-OPGW-G family of distributions are

$$F(x;\theta,\alpha,\beta,\xi) = 1 - \left(\frac{\exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)}{1 - \overline{\theta}\left[1 - \exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]} \right)^{\theta}, \quad (5)$$

and

$$\begin{aligned} f(x;\theta,\alpha,\beta,\xi) &= \theta^2\alpha\beta\left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1} \\ &\times \exp\left(\theta\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right)\frac{g(x;\xi)}{(1-G(x;\xi))^2} \\ &\times \left(1 - \overline{\theta}\left[1 - \exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-(\theta+1)}, \end{aligned} \quad (6)$$

respectively, for $\alpha, \beta, \theta > 0$ and parameter vector ξ. Note that $\overline{\theta} = 1 - \theta$.

2.1. **Reliability and Failure Rate Functions.** The survival function, and hazard rate function (hrf) of the TI-HT-OPGW-G family of distributions are given respectively by

$$S(x;\theta,\alpha,\beta,\xi)=\left(\frac{\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)}{1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]}\right)^{\theta}, \quad (7)$$

and

$$\begin{aligned} h(x;\theta,\alpha,\beta,\xi) &= \theta^2\alpha\beta\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1} \\ &\times \exp\left(\theta\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right)\frac{g(x;\xi)}{(1-G(x;\xi))^2} \\ &\times \left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-(\theta+1)} \\ &\times \left[\left(\frac{\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)}{1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]}\right)^{\theta}\right]^{-1}, \quad (8) \end{aligned}$$

for $\alpha,\beta,\theta>0$, $\overline{\theta}=1-\theta$, and parameter vector ξ.

2.2. **Sub-Families of TI-HT-OPGW-G Family of Distributions.**

- When $\theta=1$, we obtain the odd power generalized Weibull-G (OPGW-G) family of distributions (see [24]) with the cdf

$$F(x;\alpha,\beta,\xi)=1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right),$$

  for $\alpha,\beta>0$ and parameter vector ξ.
- When $\beta=1$, we obtain the new type I heavy-tailed Weibull-G (TI-HT-W-G) family of distributions with the cdf

$$F(x;\theta,\alpha,\xi)=1-\left(\frac{\exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)}{1-\overline{\theta}\left[1-\exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)\right]}\right)^{\theta},$$

  for $\theta,\alpha>0$ and parameter vector ξ. This is a new family of distributions.

- When $\alpha = 1$, we obtain the new type I heavy-tailed odd Nadarajah Haghighi-G (TI-HT-ONH-G) family of distributions with the cdf

$$F(x;\theta,\beta,\xi) = 1 - \left(\frac{\exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right]^{\beta}\right)}{1 - \overline{\theta}\left[1 - \exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right]^{\beta}\right)\right]} \right)^{\theta},$$

  for $\theta, \beta > 0$, $\overline{\theta} = (1-\theta)$, and parameter vector ξ. This is a new family of distributions.
- When $\theta = \beta = 1$, we obtain the Weibull-G (W-G) family of distributions (see [10]) with the cdf

$$F(x;\alpha,\xi) = 1 - \exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right),$$

  for $\alpha > 0$, and parameter vector ξ.
- If $\theta = \alpha = 1$, we obtain the odd Nadarajah Haghighi-G (ONH-G) (see [25]) family of distributions with the cdf

$$F(x;\beta,\xi) = 1 - \exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right]^{\beta}\right),$$

  for $\beta > 0$, and parameter vector ξ.
- If $\theta = \alpha = \beta = 1$, we obtain the odd exponential-G (OE-G) family of distributions with the cdf

$$F(x;\xi) = 1 - \exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right),$$

  for parameter vector ξ. This is a new family of distributions.
- If $\theta = \beta = 1, \alpha = 2$ we obtain the odd Rayleigh-G (OR-G) family of distributions with the cdf

$$F(x;\xi) = 1 - \exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{2}\right),$$

  for parameter vector ξ. This is a new family of distributions.

2.3. **Linear Representation.** Here, in this sub-section, we express the pdf of TI-HT-OPGW-G family of distributions as an infinite linear combination of exponentiated-G (Exp-G) family of distributions. Using the following generalized binomial and Taylor series expansions

$$(1+z)^{-a} = \sum_{k=0}^{\infty}(-1)^k \binom{a+k-1}{k} z^k \quad \text{for} \quad |z| < 1, \quad \text{and} \quad e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!},$$

respectively, we can write

$$f(x;\theta,\alpha,\beta,\xi) = \sum_{p=0}^{\infty} \varphi_{p+1} g_{p+1}(x;\xi), \tag{9}$$

where $g_{p+1}(x;\xi) = (p+1)[G(x;\xi)]^p g(x;\xi)$ is the Exp-G pdf with the power parameter $(p+1)$ and parameter vector ξ, and

$$\begin{aligned} \varphi_{p+1} &= \theta^2\alpha\beta \sum_{h,i,j,k,l,m=0}^{\infty} \overline{\theta}^h \binom{\theta+h}{h}\binom{h}{i}\binom{j}{k}\binom{\beta(k+1)-1}{l}\binom{\alpha(l+1)-1}{m} \\ &\times \binom{\alpha(l+1)-m+p}{p} \frac{(\theta+i)^j(-1)^{i+k+m}}{j!\,(p+1)}. \end{aligned} \tag{10}$$

Consequently, the mathematical and statistical properties of the TI-HT-OPGW-G family of distributions follow directly from those of the Exp-G family of distributions. (**See the Appendix for derivations**).

2.4. **Quantile Function.** The quantile function is used in Monte Carlo simulations to generate random numbers for a specified probability distribution. It is obtained by inverting the cdf of a distribution. If the random variable X is from the TI-HT-OPGW-G family of distributions, then the quantile function of X can be obtained as follows:

$$F(x;\theta,\alpha,\beta,\xi) = 1 - \left(\frac{\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)}{1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]} \right)^{\theta} = u, \tag{11}$$

for $0 \le u \le 1$, that is,

$$G(x;\xi) = \left(\left[\left(1-\log\left(\theta\left[(1-u)^{\frac{-1}{\theta}}-\overline{\theta}\right]^{-1}\right)\right)^{\frac{1}{\beta}}-1\right]^{\frac{-1}{\alpha}}+1\right)^{-1}. \tag{12}$$

Therefore, the quantile function of the TI-HT-OPGW-G family of distributions is given by

$$Q_X(u) = G^{-1}\left[\left(\left[\left(1-\log\left(\theta\left[(1-u)^{\frac{-1}{\theta}}-\overline{\theta}\right]^{-1}\right)\right)^{\frac{1}{\beta}}-1\right]^{\frac{-1}{\alpha}}+1\right)^{-1}\right]. \tag{13}$$

Consequently, variates of the TI-HT-OPGW-G family of distributions can be obtained using equation (13) for specified baseline cdf G.

3. Statistical Properties

In this section, we derived some statistical features of the TI-HT-OPGW-G family of distributions, specifically the moments, moment generating function, distribution of order statistics and Rényi entropy. Let the pdf of the TI-HT-OPGW-G family of distributions be denoted by $f(x)$.

3.1. **Moments and Generating Function.** Let $Y_{p+1} \sim Exponentiated - G(p+1,\xi)$, then the n^{th} raw moment, μ'_n of the TI-HT-OPGW-G family of distributions is given by

$$\mu'_n = E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx = \sum_{p=0}^{\infty} \varphi_{_{p+1}} E(Y_{p+1}^n),$$

where $E(Y_{p+1}^n)$ is the n^{th} moment of Y_{p+1} and $\varphi_{_{p+1}}$ is given by equation (10). The moment generating function (MGF), for $|t| < 1$, is given by:

$$M_X(t) = \sum_{p=0}^{\infty} \varphi_{_{p+1}} M_{p+1}(t),$$

where $M_{p+1}(t)$ is the mgf of Y_{p+1} and $\varphi_{_{p+1}}$ is given by equation (10).

3.2. **Order Statistics.** Order statistics have many applications in survival, reliability, failure analysis, and it is a natural way to perform a reliability analysis of a system. Suppose $X_1, X_2, ..., X_n$ are independent and identically distributed random variables from the TI-HT-OPGW-G family of distributions. The pdf of the r^{th} order statistic from the TI-HT-OPGW-G pdf $f(x)$ can be written as

$$f_{r:n}(x) = \frac{n!f(x)}{(r-1)!(n-r)!} \sum_{z=0}^{n-r} (-1)^z \binom{n-r}{z} [F(x)]^{z+r-1}. \tag{14}$$

Using equations (5) and (6), we have

$$\begin{aligned}
f(x)[F(x)]^{z+r-1} &= \theta^2\alpha\beta \left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1} \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1} \\
&\times \exp\left(\theta\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right) \frac{g(x;\xi)}{(1-G(x;\xi))^2} \\
&\times \left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-(\theta+1)} \\
&\times \left[1-\left(\frac{\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)}{1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]}\right)^{\theta}\right]^{z+r-1}
\end{aligned}$$

$$
\begin{aligned}
&= \theta^2\alpha\beta\sum_{q=0}^{\infty}\binom{z+r-1}{q}(-1)^q\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1}\\
&\times \exp\left(\theta(q+1)\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right)\frac{g(x;\xi)}{(1-G(x;\xi))^2}\\
&\times \left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-(\theta(q+1)+1)}.
\end{aligned}
$$

Now following the same steps leading to equation (9), we obtain

$$
f(x)[F(x)]^{z+r-1}=\sum_{p=0}^{\infty}\rho_{p+1}g_{p+1}(x;\xi), \tag{15}
$$

where $g_{p+1}(x;\xi)=(p+1)[G(x;\xi)]^p g(x;\xi)$ is the Exp-G pdf with the power parameter $(p+1)$ and parameter vector ξ, and

$$
\begin{aligned}
\rho_{p+1} &= \theta^2\alpha\beta\sum_{h,i,j,k,l,m,q=0}^{\infty}\overline{\theta}^h\binom{z+r-1}{q}\binom{\theta(q+1)+h}{h}\binom{h}{i}\binom{j}{k}\binom{\beta(k+1)-1}{l}\\
&\times \binom{\alpha(l+1)-1}{m}\binom{\alpha(l+1)-m+p}{p}\frac{(\theta(q+1)+i)^j(-1)^{q+i+k+m}}{j!\,(p+1)}.
\end{aligned}
$$

Thus, by substituting (15) into (14), the pdf of the r^{th} order statistic from the TI-HT-OPGW-G family of distributions can be written as

$$
f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\sum_{p=0}^{\infty}\sum_{z=0}^{n-r}(-1)^z\binom{n-r}{z}\rho_{p+1}g_{p+1}(x;\xi).
$$

3.3. **Rényi Entropy.** Rényi entropy is a very important tool in information theory, as a measure of randomness or uncertainty in the system. Rényi entropy is defined to be

$$
I_R(v)=\frac{1}{1-v}\log\left(\int_0^{\infty}[f(x;\theta,\alpha,\beta,\xi)]^v dx\right), v\neq 1, v>0. \tag{16}
$$

Rényi entropy for the TI-HT-OPGW-G family of distributions is given by

$$
\begin{aligned}
I_R(v) &= \frac{1}{1-v}\log\Bigg[\sum_{h,i,j,k,l,m,p=0}^{\infty}\overline{\theta}^h\binom{v(\theta+1)+h-1}{h}\binom{h}{i}\frac{(v\theta+i)^j(-1)^{i+k+m}}{j!}\binom{j}{k}\\
&\times \binom{\beta(k+v)-v}{l}\binom{\alpha(l+v)-v}{m}\binom{\alpha(l+v)+v-m+p-1}{p}\frac{(\theta^2\alpha\beta)^v}{\left[1+\frac{p}{v}\right]^v}\\
&\times \int_0^{\infty}\left(\left[1+\frac{p}{v}\right](G(x;\xi))^{\frac{p}{v}}(g(x;\xi))\right)^v dx\Bigg]
\end{aligned}
$$

$$= \frac{1}{1-v}\log\left[\sum_{p=0}^{\infty}\tau_{_p}\exp\left((1-v)I_{_{REG}}\right)\right], \qquad (17)$$

for $v>0$, $v\neq 1$, where $I_{_{REG}} = \frac{1}{1-v}\log\left[\int_0^{\infty}\left(\left[1+\frac{p}{v}\right](G(x;\xi))^{\frac{p}{v}}(g(x;\xi))\right)^{v}dx\right]$ is the Rényi entropy of Exp-G distribution with power parameter $(\frac{p}{v}+1)$ and

$$\begin{aligned}\tau_{_p} &= \sum_{h,i,j,k,l,m=0}^{\infty}\overline{\theta}^{h}\binom{v(\theta+1)+h-1}{h}\binom{h}{i}\frac{(v\theta+i)^{j}(-1)^{i+k+m}}{j!}\binom{j}{k}\\ &\times \binom{\beta(k+v)-v}{l}\binom{\alpha(l+v)-v}{m}\binom{\alpha(l+v)+v-m+p-1}{p}\frac{\left(\theta^2\alpha\beta\right)^{v}}{\left[1+\frac{p}{v}\right]^{v}}.\end{aligned}$$

Therefore, Rényi entropy of the TI-HT-OPGW-G family of distributions can be obtained from those of the Exp-G family of distributions. (**See the Appendix for derivations**).

3.4. **Moment of Residual and Reversed Residual Life.** Moments of the residual life distribution are used to obtain the mean, variance and coefficient of variation of residual life which are extensively used in reliability analysis.

The s^{th} moment of the residual life, say $\kappa_s(t)$ of a random variable X is

$$\kappa_s(t) = E\left[(X-t)^s \mid X>t\right] = \frac{1}{\overline{F}(t)}\int_t^{\infty}(x-t)^s f(x)dx.$$

Consequently, $\kappa_s(t)$ for the TI-HT-OPGW-G family of distributions is given as follows:

$$\kappa_s(t) = \frac{1}{\overline{F}(t)}\sum_{p,u=0}^{\infty}\binom{s}{u}(-t)^{s-u}\varphi_{_{p+1}}\int_t^{\infty}x^u g_{_{p+1}}(x;\xi)dx, \qquad (18)$$

where $\varphi_{_{p+1}}$ is as defined in equation (10) and $g_{_{p+1}}(x;\xi)$ denotes the Exp-G distribution with power parameter $(p+1)$. The mean excess function of the TI-HT-OPGW-G family of distributions is obtained from the above formula with $s=1$. The s^{th} moment of the reversed residual life, say $\vartheta_s(t)$ of a random variable X is

$$\vartheta_s(t) = E\left[(t-X)^s \mid X\leq t\right] = \frac{1}{F(t)}\int_0^{t}(t-x)^s f(x)dx.$$

Subsequently, $\vartheta_s(t)$ for the TI-HT-OPGW-G family of distributions is given as follows:

$$\vartheta_s(t) = \frac{1}{F(t)}\sum_{p,u=0}^{\infty}\binom{s}{u}(-t)^{s-u}\varphi_{_{p+1}}\int_0^{t}x^u g_{_{p+1}}(x;\xi)dx,$$

where φ_{p+1} is as defined in equation (10) and $g_{p+1}(x;\xi)$ denotes the Exp-G distribution with power parameter $(p+1)$. The mean inactivity time of the TI-HT-OPGW-G family of distributions is obtained from the above formula with $s=1$.

4. Some Special Cases

In this section, we present some special cases of the TI-HT-OPGW-G family of distributions. We considered cases when the baseline distributions are log-logistic, Weibull, Rayleigh and standard half logistic distributions.

4.1. TI-HT-OPGW-Log-Logistic (TI-HT-OPGW-LLoG) Distribution.

Given the cdf and pdf of the log-logistic distribution as $G(x;c)=1-(1+x^c)^{-1}$ and $g(x;c)=cx^{c-1}(1+x^c)^{-2}$ for $c>0$ and $x>0$, we define the cdf and pdf of the TI-HT-OPGW-LLoG distribution as follows

$$F(x;\theta,\alpha,\beta,c)=1-\left(\frac{\exp\left(1-\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta}\right)}{1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta}\right)\right]}\right)^{\theta},$$

and

$$\begin{aligned} f(x;\theta,\alpha,\beta,c) &= \theta^2\alpha\beta\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta-1}\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha-1} \\ &\times \exp\left(\theta\left(1-\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta}\right)\right)\frac{cx^{c-1}(1+x^c)^{-2}}{((1+x^c)^{-1})^2} \\ &\times \left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-(\theta+1)}, \end{aligned}$$

respectively, for $\theta,\alpha,\beta,c>0$, and $\overline{\theta}=1-\theta$. The hrf for the TI-HT-OPGW-LLoG distribution is given by

$$\begin{aligned} h(x;\theta,\alpha,\beta,c) &= \theta^2\alpha\beta\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta-1}\left(1-(1+x^c)^{-1}\right)^{\alpha-1} \\ &\times \left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-1} \\ &\times \frac{cx^{c-1}(1+x^c)^{-2}}{((1+x^c)^{-1})^{\alpha+1}}, \end{aligned}$$

for $\theta,\alpha,\beta,c>0$, and $\overline{\theta}=1-\theta$. Figure 1 shows the 3D plots of skewness and kurtosis of the TI-HT-OPGW-LLoG distribution. We observe that

- When we fix the parameters θ and β, the skewness and kurtosis of the TI-HT-OPGW-LLoG distribution changes from decreasing to increasing as α and c increases.
- When we fix the parameters α and β, the skewness and kurtosis of the TI-HT-OPGW-LLoG distribution increases as θ and c increases.

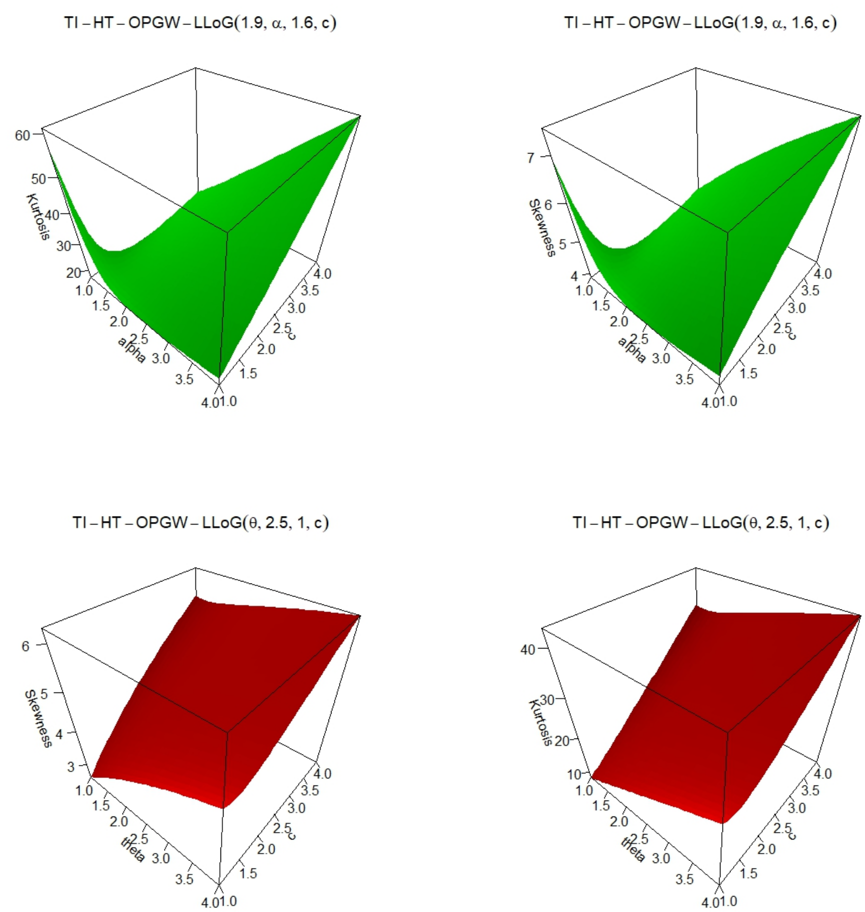

Figure 1. Plots of skewness and kurtosis for the TI-HT-OPGW-LLoG distribution

Figure 2 shows the plots of pdf and hazard functions of TI-HT-OPGW-LLoG distribution, respectively. The pdf can take several shapes including right-skewed, left-skewed, unimodal, J and reverse-J shapes. The TI-HT-OPGW-LLoG hrf displays increasing, decreasing, bathtub, upside-down bathtub and bathtub followed by upside-down bathtub shapes.

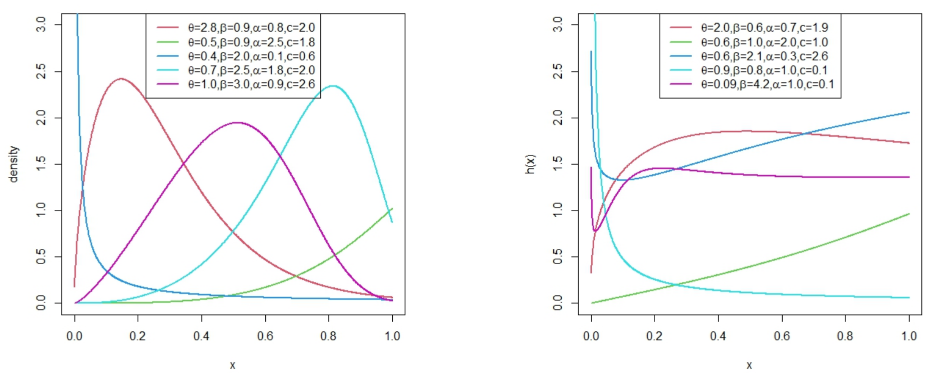

FIGURE 2. Plots of density and hazard rate function for TI-HT-OPGW-LLoG distribution

4.2. **TI-HT-OPGW-Weibull (TI-HT-OPGW-W) Distribution.** Suppose the cdf and pdf of the baseline distribution are given by $G(x;\lambda) = 1 - e^{-x^{\lambda}}$, and $g(x;\lambda) = \lambda x^{\lambda-1} e^{-x^{\lambda}}$, $x > 0, \lambda > 0$, then, the cdf and pdf of the TI-HT-OPGW-W distribution are given by

$$F(x;\theta,\alpha,\beta,\lambda) = 1 - \left(\frac{\exp\left(1 - \left[1 + \left(\frac{1-e^{-x^{\lambda}}}{e^{-x^{\lambda}}}\right)^{\alpha}\right]^{\beta}\right)}{1 - \overline{\theta}\left[1 - \exp\left(1 - \left[1 + \left(\frac{1-e^{-x^{\lambda}}}{e^{-x^{\lambda}}}\right)^{\alpha}\right]^{\beta}\right)\right]} \right)^{\theta},$$

and

$$\begin{aligned} f(x;\theta,\alpha,\beta,\lambda) &= \theta^2 \alpha\beta \left[1 + \left(\frac{1-e^{-x^{\lambda}}}{e^{-x^{\lambda}}}\right)^{\alpha}\right]^{\beta-1} \left(\frac{1-e^{-x^{\lambda}}}{e^{-x^{\lambda}}}\right)^{\alpha-1} \\ &\times \exp\left(\theta\left(1 - \left[1 + \left(\frac{1-e^{-x^{\lambda}}}{e^{-x^{\lambda}}}\right)^{\alpha}\right]^{\beta}\right)\right) \frac{\lambda x^{\lambda-1} e^{-x^{\lambda}}}{\left(e^{-x^{\lambda}}\right)^2} \end{aligned}$$

$$\times \left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{1-e^{-x^{\lambda}}}{e^{-x^{\lambda}}}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-(\theta+1)},$$

respectively, for $\theta,\alpha,\beta,\lambda>0$, and $\overline{\theta}=1-\theta$. The hrf for TI-HT-OPGW-W distribution is given by

$$\begin{aligned} h(x;\theta,\alpha,\beta,\lambda) &= \theta^{2}\alpha\beta\left[1+\left(\frac{1-e^{-x^{\lambda}}}{e^{-x^{\lambda}}}\right)^{\alpha}\right]^{\beta-1}\left(1-e^{-x^{\lambda}}\right)^{\alpha-1} \\ &\times \left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{1-e^{-x^{\lambda}}}{e^{-x^{\lambda}}}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-1} \\ &\times \frac{\lambda x^{\lambda-1}e^{-x^{\lambda}}}{\left(e^{-x^{\lambda}}\right)^{\alpha+1}}, \end{aligned}$$

for $\theta,\alpha,\beta,\lambda>0$, and $\overline{\theta}=1-\theta$.

Figure 3 shows the 3D plots of skewness andkurtosis of the TI-HT-OPGW-W distribution. We observe that

- When we fix the parameters θ and λ, the skewness and kurtosis of the TI-HT-OPGW-W distribution decrease as α and β increases.

- When we fix the parameters β and λ, the skewness and kurtosis of the TI-HT-OPGW-W distribution increases as θ and α increases.

Figure 4 shows the plots of pdf and hrf of TI-HT-OPGW-W distribution, respectively. The pdf can take several shapes including right-skewed, left-skewed, unimodal, J and reverse-J shapes. The TI-HT-OPGW-W hrf displays increasing, decreasing, bathtub, upside-down bathtub and bathtub followed by upside-down bathtub shapes.

4.3. **TI-HT-OPGW-Rayleigh (TI-HT-OPGW-R) Distribution.** If the baseline cdf and pdf are given by $G(x;\lambda)=1-\exp\left(-\frac{x^2}{2\lambda^2}\right)$ and $g(x;\lambda)=\frac{x}{\lambda^2}\exp\left(-\frac{x^2}{2\lambda^2}\right)$ for $x>0$, and $\lambda>0$, then, the cdf and pdf of the TI-HT-OPGW-R distribution are given by

$$F(x;\theta,\alpha,\beta,\lambda)=1-\left(\frac{\exp\left(1-\left[1+\left(\frac{1-\exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^{\alpha}\right]^{\beta}\right)}{1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{1-\exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^{\alpha}\right]^{\beta}\right)\right]}\right)^{\theta},$$

and

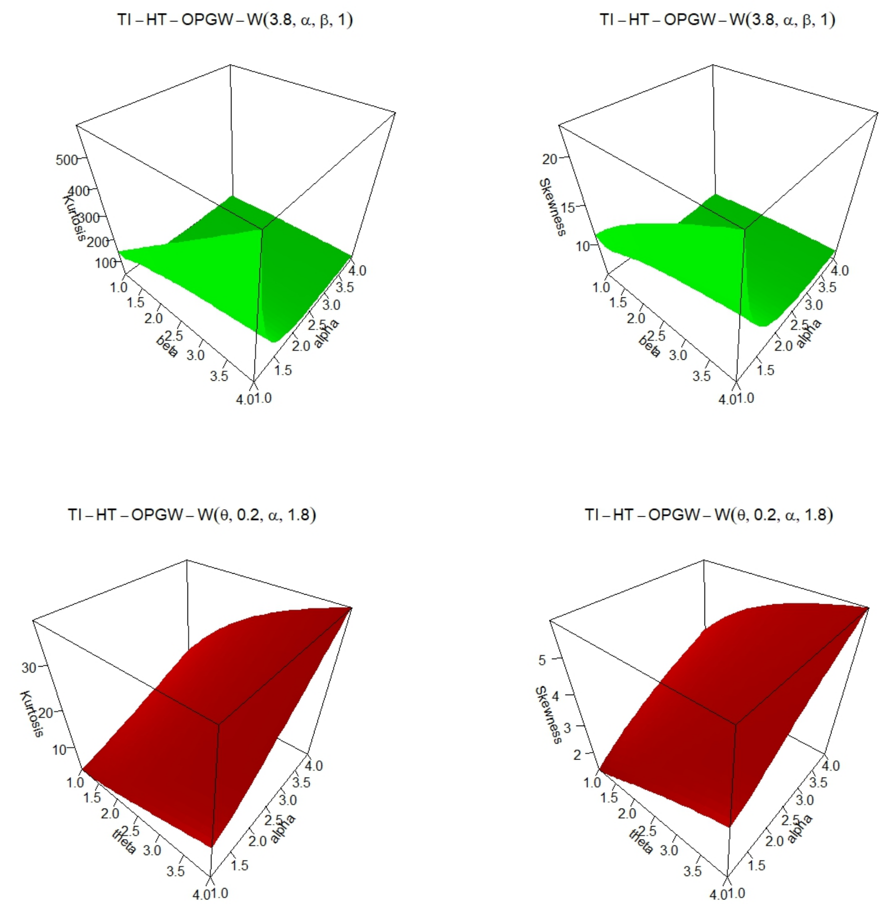

FIGURE 3. Plots of skewness and kurtosis for the TI-HT-OPGW-W distribution

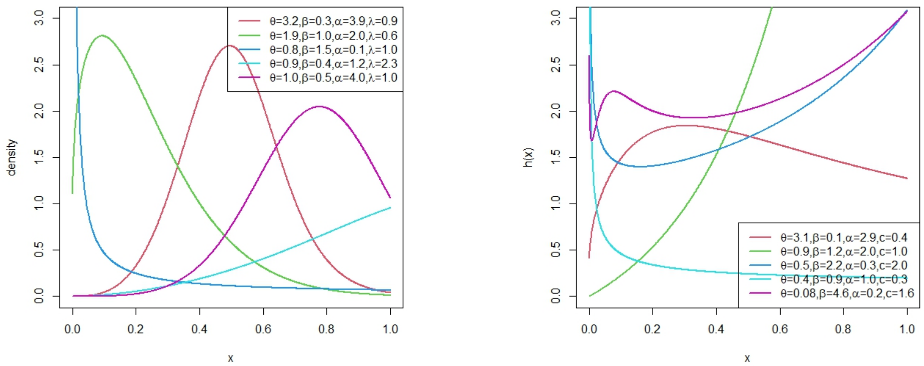

FIGURE 4. Plots of Density and Hazard Rate Function for TI-HT-OPGW-W Distribution

$$
\begin{aligned}
f(x;\theta,\alpha,\beta,\lambda) &= \theta^2\alpha\beta\left[1+\left(\frac{1-\exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^{\alpha}\right]^{\beta-1}\left(\frac{1-\exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^{\alpha-1} \\
&\times \exp\left(\theta\left(1-\left[1+\left(\frac{1-\exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^{\alpha}\right]^{\beta}\right)\right)\frac{\frac{x}{\lambda^2}\exp\left(-\frac{x^2}{2\lambda^2}\right)}{\left(\exp\left(-\frac{x^2}{2\lambda^2}\right)\right)^2} \\
&\times \left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{1-\exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-(\theta+1)},
\end{aligned}
$$

respectively, for $\theta,\alpha,\beta,\lambda>0$, and $\overline{\theta}=1-\theta$. The hrf for TI-HT-OPGW-R distribution is given by

$$
\begin{aligned}
h(x;\theta,\alpha,\beta,\lambda) &= \theta^2\alpha\beta\left[1+\left(\frac{1-\exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^{\alpha}\right]^{\beta-1}\left(1-\exp\left(-\frac{x^2}{2\lambda^2}\right)\right)^{\alpha-1} \\
&\times \left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{1-\exp\left(-\frac{x^2}{2\lambda^2}\right)}{\exp\left(-\frac{x^2}{2\lambda^2}\right)}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-1}
\end{aligned}
$$

$$\times \quad \frac{\frac{x}{\lambda^2}\exp\left(-\frac{x^2}{2\lambda^2}\right)}{\left(\exp\left(-\frac{x^2}{2\lambda^2}\right)\right)^{\alpha+1}},$$

for $\theta, \alpha, \beta, \lambda > 0$, and $\overline{\theta} = 1 - \theta$. Figure 5 shows the 3D plots of skewness and kurtosis of the TI-HT-OPGW-R distribution.

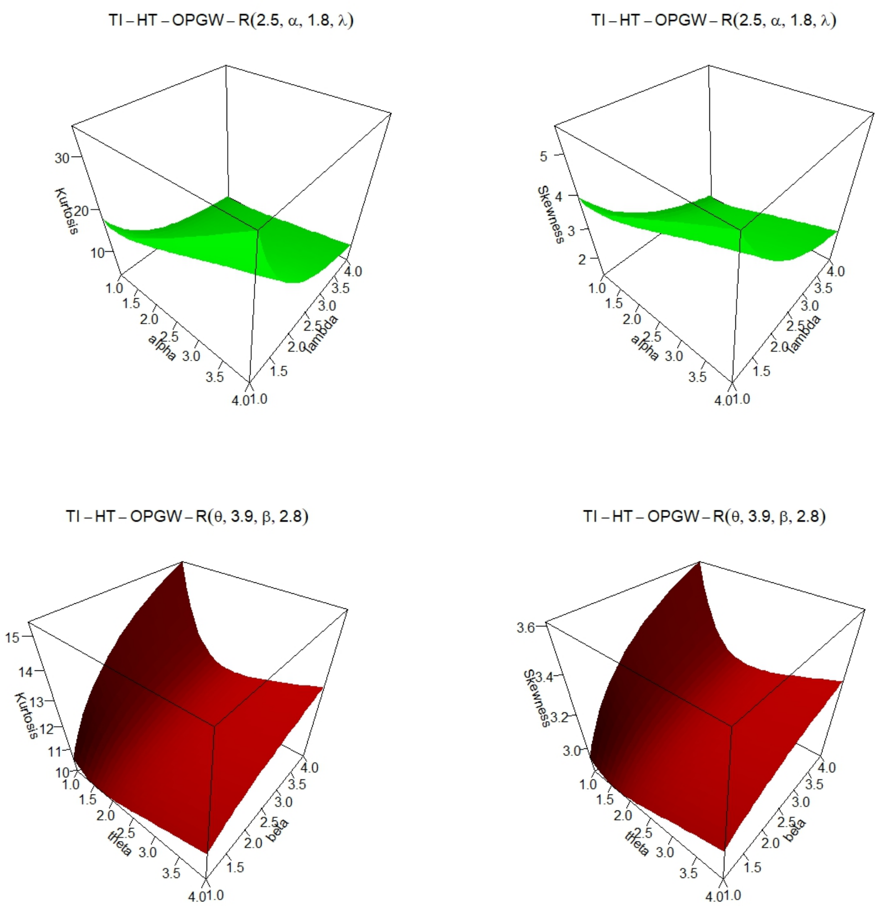

Figure 5. Plots of skewness and kurtosis for the TI-HT-OPGW-R distribution

We observe that

- When we fix the parameters θ and β, the skewness and kurtosis of the TI-HT-OPGW-R distribution decreases as α and λ increases.
- When we fix the parameters α and λ, the skewness and kurtosis of the TI-HT-OPGW-R distribution decreases and increases as θ and β increases.

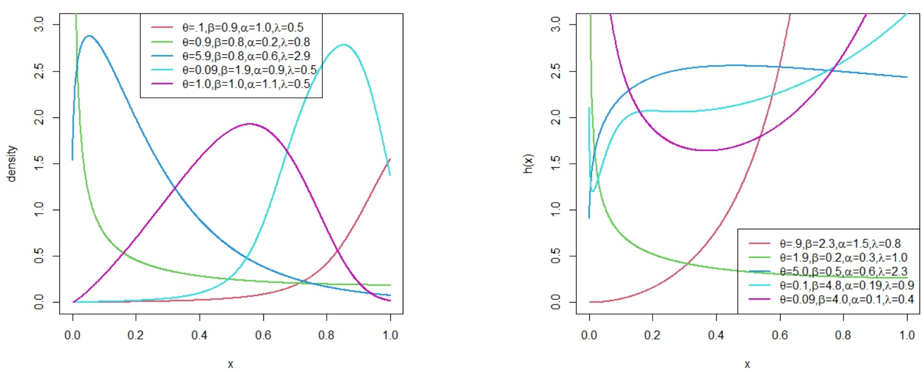

FIGURE 6. Plots of Density and Hazard Rate Function for TI-HT-OPGW-R Distribution

Figure 6 shows the plots of pdf and hrf of TI-HT-OPGW-R distribution, respectively. The pdf can take several shapes including right-skewed, left-skewed, unimodal, J and reverse-J shapes. The TI-HT-OPGW-R hrf displays increasing, decreasing, bathtub and upside-down bathtub shapes.

4.4. **TI-HT-OPGW-Standard Half Logistic (TI-HT-OPGW-SHL) Distribution.** Suppose the cdf and pdf of the baseline distribution are given by $G(x) = \frac{1-e^{-x}}{1+e^{-x}}$, and $g(x) = \frac{2e^{-x}}{(1+e^{-x})^2}$, $x > 0$, then, the cdf and pdf of the TI-HT-OPGW-SHL distribution are given by

$$F(x;\theta,\alpha,\beta) = 1 - \left(\frac{\exp\left(1 - \left[1 + \left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1-\frac{1-e^{-x}}{1+e^{-x}}}\right)^{\alpha}\right]^{\beta}\right)}{1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1-\frac{1-e^{-x}}{1+e^{-x}}}\right)^{\alpha}\right]^{\beta}\right)\right]} \right)^{\theta},$$

and

$$f(x;\theta,\alpha,\beta) \quad = \quad \theta^2\alpha\beta\left[1+\left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1-\frac{1-e^{-x}}{1+e^{-x}}}\right)^{\alpha}\right]^{\beta-1}\left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1-\frac{1-e^{-x}}{1+e^{-x}}}\right)^{\alpha-1}$$

$$\times \quad \left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1-\frac{1-e^{-x}}{1+e^{-x}}}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-(\theta+1)}$$

$$\times \quad \exp\left(\theta\left(1-\left[1+\left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1-\frac{1-e^{-x}}{1+e^{-x}}}\right)^{\alpha}\right]^{\beta}\right)\right)\frac{\frac{2e^{-x}}{(1+e^{-x})^2}}{\left(1-\frac{1-e^{-x}}{1+e^{-x}}\right)^2},$$

respectively, for $\alpha, \beta, \theta > 0$, and $\overline{\theta} = 1 - \theta$. The hrf for TI-HT-OPGW-SHL distribution is given by

$$\begin{aligned} h(x;\theta,\alpha,\beta) &= \theta^2\alpha\beta\left[1+\left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1-\frac{1-e^{-x}}{1+e^{-x}}}\right)^{\alpha}\right]^{\beta-1}\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\alpha-1} \\ &\times \left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{\frac{1-e^{-x}}{1+e^{-x}}}{1-\frac{1-e^{-x}}{1+e^{-x}}}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-1} \\ &\times \frac{\frac{2e^{-x}}{(1+e^{-x})^2}}{\left(1-\frac{1-e^{-x}}{1+e^{-x}}\right)^{\alpha+1}}, \end{aligned}$$

for $\theta, \alpha, \beta > 0$, and $\overline{\theta} = 1 - \theta$.

Figure 7 shows the 3D plots of skewness and kurtosis of the TI-HT-OPGW-SHL distribution. We observe that

- When we fix the parameters β, the skewness and kurtosis of the TI-HT-OPGW-SHL distribution decreases as θ and α increases.
- When we fix the parameters α, the skewness and kurtosis of the TI-HT-OPGW-SHL distribution increases as θ and β increases.

Figure 8 shows the plots of pdf and hrf of TI-HT-OPGW-SHL distribution, respectively. The pdf can take several shapes including right-skewed, left-skewed, unimodal, J and reverse-J shapes. The TI-HT-OPGW-SHL hrf displays increasing, decreasing, bathtub, and upside-down bathtub followed by bathtub shapes.

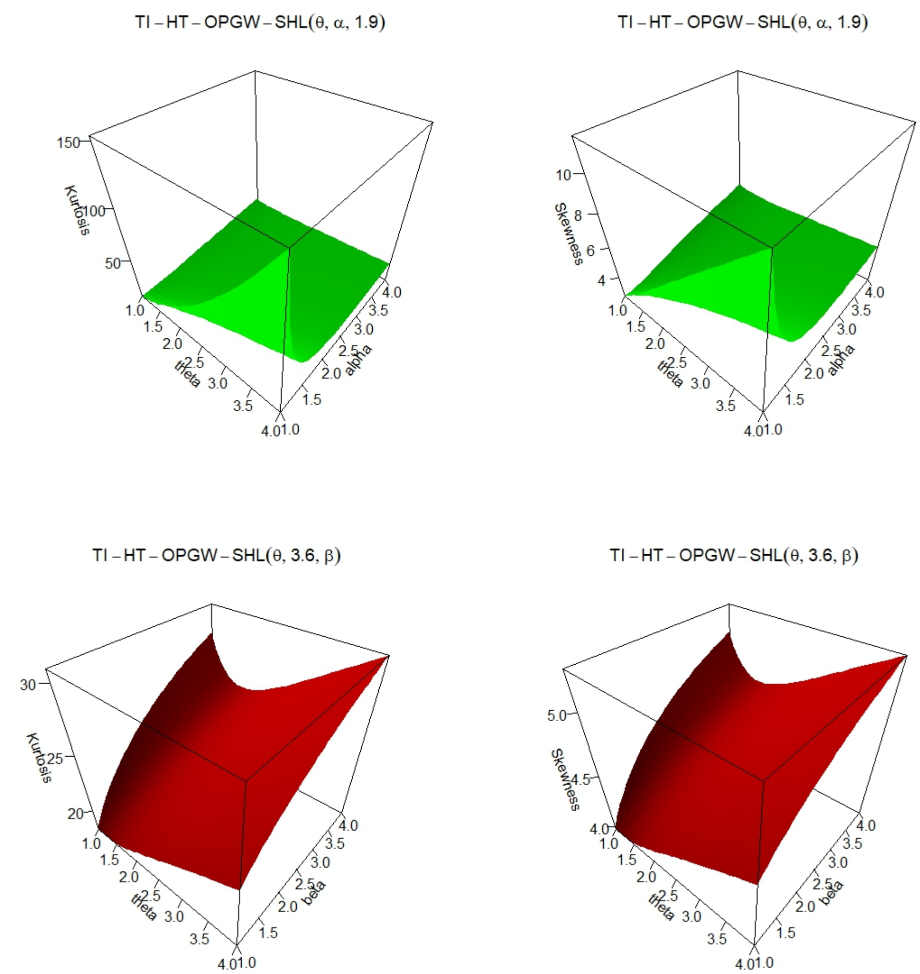

FIGURE 7. Plots for the skewness and kurtosis for the TI-HT-OPGW-SHL distribution

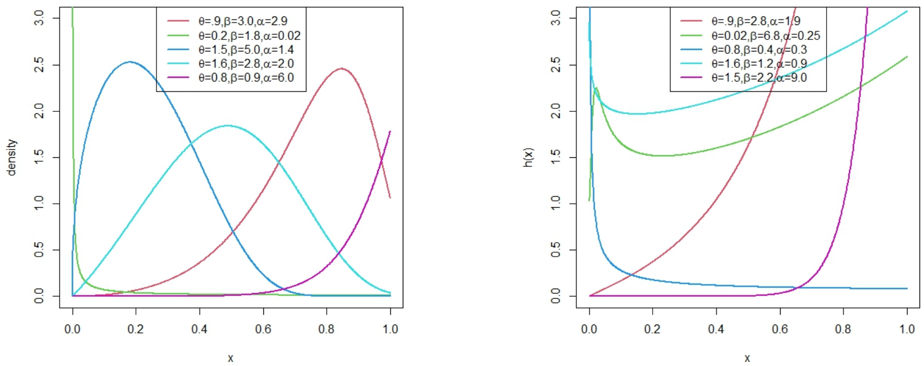

FIGURE 8. Plots of Density and Hazard Rate Functions for TI-HT-OPGW-SHL Distribution

5. PARAMETER ESTIMATION

In this section, we adopt the maximum likelihood estimation technique for estimating the parameters of the TI-HT-OPGW-G family of distributions. Let $x_1, x_2.......x_n$ be the realization from the TI-HT-OPGW-G family of distributions with the vector of model parameters $\mathbf{\Delta} = (\theta, \alpha, \beta, \xi)^T$. Then, the corresponding log-likelihood function $\ell_n = \ell_n(\mathbf{\Delta})$ of this sample has the form

$$\begin{aligned} \ell_n(\mathbf{\Delta}) &= n\ln(\theta^2\alpha\beta) + (\beta-1)\sum_{i=1}^{n}\log\left[1+\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)^{\alpha}\right] \\ &+ (\alpha-1)\sum_{i=1}^{n}\log\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right) + \sum_{i=1}^{n}\log\left(g(x_i;\xi)\right) \\ &+ \sum_{i=1}^{n}\left(\theta\left(1-\left[1+\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right) - 2\sum_{i=1}^{n}\log\left(1-G(x_i;\xi)\right) \\ &- (\theta+1)\sum_{i=1}^{n}\log\left(1-\overline{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]\right). \end{aligned} \quad (19)$$

To obtain the maximum likelihood estimates of the unknown parameters, denoted by $\hat{\mathbf{\Delta}}$, we set the nonlinear system of equations $(\frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \xi_k})^T = \mathbf{0}$, and solve them simultaneously. However, since these equations are not in closed form, the MLEs can be found by maximizing $\ell_n(\mathbf{\Delta})$ numerically with respect to the parameters, using a numerical method such as Newton-Raphson procedure. We maximized the likelihood function using the function nlm in R. The partial derivatives of the log-likelihood function with respect to each component of the parameter vector are given in the appendix. The observed Fisher information matrix is given by

$$J(\hat{\mathbf{\Delta}}) = \begin{pmatrix} J_{\theta,\theta}(\hat{\mathbf{\Delta}}) & J_{\theta,\alpha}(\hat{\mathbf{\Delta}}) & J_{\theta,\beta}(\hat{\mathbf{\Delta}}) & J_{\theta,\xi}(\hat{\mathbf{\Delta}}) \\ J_{\alpha,\theta}(\hat{\mathbf{\Delta}}) & J_{\alpha,\alpha}(\hat{\mathbf{\Delta}}) & J_{\alpha,\beta}(\hat{\mathbf{\Delta}}) & J_{\alpha,\xi}(\hat{\mathbf{\Delta}}) \\ J_{\beta,\theta}(\hat{\mathbf{\Delta}}) & J_{\beta,\alpha}(\hat{\mathbf{\Delta}}) & J_{\beta,\beta}(\hat{\mathbf{\Delta}}) & J_{\beta,\xi}(\hat{\mathbf{\Delta}}) \\ J_{\xi,\theta}(\hat{\mathbf{\Delta}}) & J_{\xi,\alpha}(\hat{\mathbf{\Delta}}) & J_{\xi,\beta}(\hat{\mathbf{\Delta}}) & J_{\xi,\xi}(\hat{\mathbf{\Delta}}) \end{pmatrix}, \tag{20}$$

where $J_{i,j} = -\frac{\partial^2 \ell_n(\hat{\mathbf{\Delta}})}{\partial_i \partial_j}$ for $i, j = \theta, \alpha, \beta, \xi$.

5.1. **Standard error computation.** The standard error of the estimate is a way to measure the accuracy of the predictions made by a model. It is also useful when calculating the confidence interval. It is given by

$$SE\left(\hat{\Delta}\right) = \sqrt{\left(J(\hat{\mathbf{\Delta}})\right)^{-1}}, \tag{21}$$

where $\left(J(\hat{\mathbf{\Delta}})\right)^{-1}$ is the inverse of the observed Fisher information matrix.

6. Monte Carlo Simulation Study

In this section, we evaluate the efficiency and consistency property of the MLEs of TI-HT-OPGW-W distribution via a Monte Carlo simulation based on the following: N=3000 samples of size n=50, 100, 200, 400, 800, 1600 generated from the TI-HT-OPGW-W distribution for different parameter values. The process is carried out as follows:

- 3000 random samples of size n=50, 100, 200, 400, 800, 1600 was generated from the TI-HT-OPGW-W distribution.
- Different combinations for the true parameters are selected such as for Table 1: $\theta = 0.3, \alpha = 1.8, \beta = 0.3, \lambda = 1.5$; $\theta = 0.3, \alpha = 1.8, \beta = 0.3, \lambda = 0.6$, and for $\theta = 1.5, \alpha = 0.6, \beta = 0.3, \lambda = 0.3$. For Table 2: $\theta = 1.8, \alpha = 0.8, \beta = 1.0, \lambda = 1.0$; $\theta = 0.5, \alpha = 0.8, \beta = 1.0, \lambda = 1.0$, and $\theta = 1.0, \alpha = 0.8, \beta = 1.0, \lambda = 2.5$. These values are unknown, hence chosen arbitrary and then estimated using the TI-HT-OPGW-W distribution.
- Two statistical quantities (ABIAS and RMSE) are calculated to evaluate the consistency of the MLEs.

The simulation results from the TI-HT-OPGW-W distribution are presented in Tables 1 and 2. These tables report the average estimates (Mean), average bias (ABIAS) and root mean squared errors (RMSEs). The ABIAS and RMSE for the

TABLE 1. Monte Carlo Simulation Results 1

		(0.3, 1.8, 0.3, 1.5)			(0.3, 1.8, 0.3, 0.6)			(1.5, 0.6, 0.3, 0.3)		
parameter	Sample Size	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
θ	50	0.4131	0.3320	0.1131	0.4025	0.3280	0.1025	1.9393	1.2849	0.4393
	100	0.3709	0.2615	0.0709	0.3627	0.2634	0.0627	1.8984	1.2093	0.3984
	200	0.3329	0.1659	0.0329	0.3323	0.1811	0.0323	1.8523	1.1403	0.3523
	400	0.3138	0.1020	0.0138	0.3130	0.1066	0.0130	1.7255	0.8857	0.2255
	800	0.3054	0.0655	0.0054	0.3038	0.0654	0.0038	1.7153	0.6723	0.2153
	1600	0.3033	0.0413	0.0033	0.3010	0.0421	0.0010	1.5500	0.4883	0.0500
α	50	2.8345	4.2902	1.0345	2.7212	3.9147	0.9212	0.8534	0.5010	0.2534
	100	2.1531	1.6306	0.3531	2.1017	1.6751	0.3017	0.7606	0.3709	0.1606
	200	1.9319	0.8064	0.1319	1.9023	0.8054	0.1023	0.6981	0.2622	0.0981
	400	1.8671	0.3683	0.0671	1.8506	0.3632	0.0506	0.6492	0.1579	0.0492
	800	1.8460	0.2244	0.0460	1.8195	0.2293	0.0195	0.6227	0.0930	0.0227
	1600	1.8287	0.1493	0.0287	1.8035	0.1513	0.0035	0.6227	0.0930	0.0227
β	50	0.2388	0.2563	-0.0611	0.3370	0.3062	0.0370	0.2218	0.1673	-0.0781
	100	0.2858	0.2929	-0.0141	0.3244	0.2383	0.0244	0.2387	0.1480	-0.0612
	200	0.3140	0.2744	0.0140	0.3152	0.1753	0.0152	0.2433	0.1306	-0.0566
	400	0.3065	0.2098	0.0065	0.3064	0.1087	0.0064	0.2606	0.1106	-0.0393
	800	0.3043	0.1543	0.0043	0.3034	0.0677	0.0034	0.2637	0.1009	-0.0362
	1600	0.2963	0.0612	-0.0036	0.3006	0.0408	0.0006	0.2858	0.0782	-0.0141
λ	50	2.2032	1.2778	0.7032	0.7148	0.3051	0.1148	0.4404	0.3475	0.1404
	100	1.9400	0.9330	0.4400	0.6682	0.2302	0.0682	0.3811	0.2420	0.0811
	200	1.7680	0.6684	0.2680	0.6349	0.1569	0.0349	0.3441	0.1859	0.0441
	400	1.6370	0.3791	0.1370	0.6156	0.0933	0.0156	0.3261	0.1673	0.0261
	800	1.5583	0.2246	0.0583	0.6034	0.0581	0.0034	0.3166	0.1470	0.0166
	1600	1.5143	0.0984	0.0143	0.6004	0.0386	0.0004	0.3035	0.1027	0.0035

estimated parameter, say, $\hat{\theta}$, are given by:

$$ABIAS(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

From Tables 1 and 2, it can be verified that the mean tend to the true parameters, the RMSEs and average bias decreases when n increases, thus showing that the estimates are consistent.

7. Actuarial Measures

In actuarial sciences, actuaries need to evaluate the exposure of market risk in a portfolio of instruments. This section introduces some important risk measures for the TI-HT-OPGW-G family of distributions such as value-at-risk (VaR), tail-value-at-risk (TVaR), tail variance (TV) and tail variance premium (TVP).

7.1. **Value-at-Risk (VaR).**

Value-at-risk (VaR) is an important and well known risk measure. It is also known as the quantile risk measure. The VaR of a random variable X is the q^{th} quantile of its cdf. If X is a random variable from TI-HT-OPGW-G family of distributions, then

TABLE 2. Monte Carlo Simulation Results 2

		(1.0,0.8, 1.0, 1.0)			(0.5, 0.8, 1.0, 1.0)			(1.0, 0.8,1.0, 2.5)		
parameter	Sample Size	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
θ	50	1.4765	1.2119	0.4765	0.9008	0.8341	0.4008	1.4641	1.1880	0.4641
	100	1.3266	1.0090	0.3266	0.7723	0.6818	0.2723	1.2951	0.9814	0.2951
	200	1.1939	0.6730	0.1939	0.6529	0.4772	0.1529	1.1603	0.7228	0.1603
	400	1.0873	0.3151	0.0873	0.5586	0.2767	0.0586	1.0603	0.3698	0.0603
	800	1.0480	0.2065	0.0480	0.5255	0.1497	0.0255	1.0305	0.1995	0.0305
	1600	1.0343	0.1418	0.0343	0.5160	0.1074	0.0160	1.0176	0.1385	0.0176
α	50	0.6694	0.4028	-0.1305	0.7109	0.2963	-0.0890	0.6270	0.4801	-0.1729
	100	0.7054	0.2882	-0.0945	0.7314	0.2234	-0.0685	0.6734	0.3882	-0.1265
	200	0.7451	0.2180	-0.0548	0.7490	0.1776	-0.0509	0.7296	0.3072	-0.0703
	400	0.7697	0.1535	-0.0302	0.7791	0.1314	-0.0208	0.7621	0.2162	-0.0378
	800	0.7810	0.1067	-0.0189	0.7929	0.0878	-0.0070	0.7853	0.1568	-0.0146
	1600	0.7890	0.0757	-0.0109	0.7955	0.0575	-0.0044	0.7915	0.1107	-0.0084
β	50	1.3852	0.9164	0.3852	0.7714	0.6878	-0.2285	1.3469	0.8878	0.3469
	100	1.2787	0.7790	0.2787	0.8657	0.6477	-0.1342	1.2998	0.8349	0.2998
	200	1.2316	0.5893	0.2316	0.9453	0.5986	-0.0546	1.2538	0.6952	0.2538
	400	1.1744	0.4825	0.1744	1.0284	0.4209	0.0284	1.1921	0.5701	0.1921
	800	1.1190	0.3480	0.1190	1.0134	0.3097	0.0134	1.1502	0.4588	0.1502
	1600	1.0720	0.2444	0.0720	0.9976	0.1883	-0.0023	1.0938	0.3273	0.0938
λ	50	1.5395	0.9209	0.5395	1.4667	0.7605	0.4667	3.7769	2.2398	1.2769
	100	1.3303	0.6322	0.3303	1.3161	0.5817	0.3161	3.2118	1.5261	0.7118
	200	1.1754	0.4109	0.1754	1.1890	0.4010	0.1890	2.8540	0.9718	0.3540
	400	1.0872	0.2718	0.0872	1.0970	0.2440	0.0970	2.6753	0.6612	0.1753
	800	1.0468	0.1834	0.0468	1.0400	0.1368	0.0400	2.5916	0.4537	0.0916
	1600	1.0246	0.1229	0.0246	1.0157	0.0936	0.0157	2.5420	0.3094	0.0420

$$VaR_q = x_q = G^{-1}\left[\left(\left(\left[\left(1-\log\left(\theta\left[(1-q)^{\frac{-1}{\theta}}-\overline{\theta}\right]^{-1}\right)\right)^{\frac{1}{\beta}}-1\right]^{\frac{-1}{\alpha}}+1\right)^{-1}\right], \quad (22)$$

where $q \in (0,1)$ is a specified level of significance.

7.2. **Tail-Value-at-Risk (TVaR).** TVaR is used to measure the expected loss given that an event outside a given probability level has occurred. Let X follows from TI-HT-OPGW-G family of distributions, then TVaR of X is defined as

$$\begin{aligned} TVaR_q &= \frac{1}{1-q}\int_{VaR_q}^{\infty} xf(x)dx \\ &= \frac{1}{1-q}\sum_{p=0}^{\infty}\int_{VaR_q}^{\infty} x\varphi_{p+1}g_{p+1}(x;\xi)dx, \end{aligned} \quad (23)$$

where $g_{p+1}(x;\xi) = (p+1)[G(x;\xi)]^p g(x;\xi)$ is the Exp-G pdf with the power parameter $(p+1)$ and parameter vector ξ, and φ_{p+1} is given by equation (10). Thus, TVaR of TI-HT-OPGW-G family of distributions can be obtained from those of Exp-G distribution.

7.3. **Tail Variance (TV).** The tail variance is one of the most important actuarial measures which looks at the variance beyond the VaR. The TV of the TI-HT-OPGW-G family of distributions can be defined as

$$\begin{aligned} TV_q &= E(X^2 \mid X > x_q) - (TVaR_q)^2 \\ &= \frac{1}{1-q}\int_{VaR_q}^{\infty} x^2 f(x)dx - (TVaR_q)^2 \\ &= \frac{1}{1-q}\sum_{p=0}^{\infty}\int_{VaR_q}^{\infty} x^2 \varphi_{p+1} g_{p+1}(x;\xi)dx - (TVaR_q)^2, \end{aligned} \tag{24}$$

where $g_{p+1}(x;\xi) = (p+1)[G(x;\xi)]^p g(x;\xi)$ is the Exp-G pdf with the power parameter $(p+1)$ and parameter vector ξ, and φ_{p+1} is given by equation (10). Thus, TV of TI-HT-OPGW-G family of distributions can be obtained from those of Exp-G distribution.

7.4. **Tail Variance Premium (TVP).** The TVP is an important actuarial measure that plays an essential role in insurance sciences. The TVP of the TI-HT-OPGW-G family of distributions takes the form

$$TVP_q = TVaR_q + \delta TV_q, \tag{25}$$

where $0 < \delta < 1$. The TVP of the TI-HT-OPGW-G family of distributions can be obtained by substituting the equations (23) and (24) into equation (25).

7.5. **Numerical Study for the Risk Measures.** This sub-section deals with the the numerical study of VaR, TVaR, TV and TVP for the TI-HT-OPGW-W distribution for different sets of parameters. The VaR, TVaR, TV and TVP of the TI-HT-OPGW-W distribution are compared with the type-I heavy-tailed Weibull (TI-HT-W) distribution, the half logistic generalized Weibull (HLGW) and Weibull distribution.
The process of obtaining the results is described as follows:

1. Random samples of size $n = 100$ are generated from each one of used distributions and parameters have been estimated via maximum likelihood method.
2. 1000 repetitions are made to calculate the VaR, TVaR, TV and TVP for these distributions.

Tables 3 and 4 present the simulated results of VaR, TVaR, TV and TVP of the compared distributions. A model with higher values of VaR, TVaR, TV and TVP is said to have a heavier tail. The simulated results provided in Tables 3 and 4 shows that the proposed TI-HT-OPGW-W distribution has higher values of the risk measures than the TI-HT-W, HLGW, and Weibull distributions.

TABLE 3. Simulation results 1 of VaR, TVaR, TV and TVP

Significance level		0.7	0.75	0.8	0.85	0.9	0.95	0.99
TI-HT-OPGW-W($\theta = 0.9, \alpha = 1.0, \beta = 1.0, \lambda = 0.5$)	VaR	1.9926	2.6118	3.4848	4.8008	7.0294	11.8902	
	TVaR	7.6930	8.7744	10.2127	12.2525	15.4771	21.8847	
	TV	73.5835	80.9009	90.1785	102.5321	120.3521	150.9723	
	TVP	59.2015	69.4501	82.3555	99.4047	123.7940	165.3084	
TI-HT-W($\theta = 0.9, \alpha = 1.0, \beta = 1.0$)	VaR	1.3277	1.6403	2.0526	2.6306	3.5303	5.2882	
	TVaR	3.5823	4.0032	4.5450	5.2855	6.4073	8.5284	
	TV	8.1989	8.7637	9.4695	10.4026	11.7593	14.1829	
	TVP	9.3215	10.5760	12.1206	14.1278	16.9907	22.0022	
HLGW($w = 0.9, \lambda = 1.0, \gamma = 1.0$)	VaR	1.8490	2.0965	2.3944	2.7727	3.2999	4.1968	
	TVaR	3.1698	3.4099	3.7023	4.0781	4.6077	5.5185	
	TV	1.8390	1.8495	1.8675	1.8971	1.9480	2.0518	
	TVP	4.4571	4.7971	5.1964	5.6907	6.3609	7.4678	
W($\lambda = 0.5$)	VaR	1.4484	1.9202	2.5892	3.6011	5.3150	9.0315	
	TVaR	5.9073	6.7544	7.8846	9.4938	12.0546	17.2314	
	TV	51.1712	56.9057	64.4260	74.9502	91.3972	124.0315	
	TVP	41.7272	49.4338	59.4254	73.2015	94.3121	135.0614	

TABLE 4. Simulation results 2 of VaR, TVaR, TV and TVP

Significance level		0.7	0.75	0.8	0.85	0.9	0.95	0.99
TI-HT-OPGW-W($\theta = 1.1, \alpha = 1.0, \beta = 0.7, \lambda = 0.85$)	VaR	1.9364	2.3451	2.8822	3.6337	4.8049	7.1085	
	TVaR	4.8986	5.4519	6.1648	7.1416	8.6279	11.4624	
	TV	16.0559	17.3282	18.9573	21.1721	24.5035	30.7003	
	TVP	16.1378	18.4481	21.3306	25.1379	30.6811	40.6277	
TI-HT-W($\theta = 1.1, \alpha = 1.0, \beta = 0.7$)	VaR	1.3891	1.7162	2.1476	2.7523	3.6937	5.5329	
	TVaR	3.7480	4.1884	4.7553	5.5301	6.7037	8.9230	
	TV	8.9912	9.6105	10.3845	11.4077	12.8957	15.5534	
	TVP	10.0419	11.3964	13.0630	15.2267	18.3099	23.6987	
HLGW($w = 1.0, \lambda = 1.0, \gamma = 1.7$)	VaR	1.0243	1.1469	1.2926	1.4750	1.7249	2.1408	
	TVaR	1.6523	1.7659	1.9030	2.0774	2.3201	2.7303	
	TV	0.3841	0.3812	0.3790	0.3781	0.3792	0.3856	
	TVP	1.9212	2.0518	2.2062	2.3988	2.6614	3.0967	
W($\lambda = 0.85$)	VaR	1.2421	1.4647	1.7443	2.1149	2.6543	3.6157	
	TVaR	2.5691	2.8131	3.1167	3.5156	4.0910	5.1058	
	TV	2.0797	2.1306	2.1905	2.2641	2.3622	2.5175	
	TVP	4.0249	4.4111	4.8691	5.4401	6.2170	7.4974	

8. Applications

In this section, we illustrate the fitting power of the new TI-HT-OPGW-W distribution by analyzing four real life data sets from different fields. The choice of this special case is motivated by the applicability of the Weibull distribution in different fields as compared to other baseline distributions considered in defining other presented special cases. The goodness-of-fit of the TI-HT-OPGW-W distribution was compared to those of other well-known heavy-tailed distributions and some generalizations of the Weibull distribution. These distributions are: alpha power Topp-Leone Weibull (APTLW) distribution by [9], the type-I heavy-tailed Weibull (TI-HT-W) distribution by [30], the heavy-tailed beta-power transformed Weibull (HTBPT-W) distribution by [31], the half logistic generalized Weibull (HLGW) distribution by [8], odd generalized half-logistic Weibull-Weibull (OGHLW-W) by [14], the Kumaraswamy-Weibull (KW) distribution by [15], and type II exponentiated half logistic Weibull (TIIEHLW) distribution by [4]. The pdf's of these competing models are given in the appendix.

For comparison purposes, we used well-known goodness-of-fit statistics such as -2log-likelihood statistic ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$) by [3], Consistent Akaike Information Criterion ($CAIC = AIC + 2\frac{p(p+1)}{n-p-1}$) by [11], Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$) by [28], (n is the number of observations, and p is the number of estimated parameters), Cramér-von Mises (W^*) statistic, and Anderson-Darling statistic (A^*) described by [13], Kolmogorov-Smirnov (K-S) statistic by [12], and its $p - value$. It is known that the smaller the values of all the goodness-of-fit statistics, except for the $p-$value of K-S statistic, the better the model for fitting the data set.

For the probability plot, we plotted $F(x_{(j)}) = F(x_{(j)}; \hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{\lambda})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[F(x_{(j)}) - \left(\frac{j-0.375}{n+0.25} \right) \right]^2 .$$

8.1. **Biomedical Sciences Data.** The data set is on remission times (months) of 128 bladder cancer patients by [22]. (**See the data in the Appendix**).

Table 5 gives the MLEs of the fitted distributions together with the standard errors (in parenthesis). Table 6, gives the values of all considered goodness-of-fit statistics. It is evident that the TI-HT-OPGW-W distribution provides the best fit among the competitors since it has the lowest value of $-2\ln(L)$, AIC, $CAIC$, BIC, W^*, A^*, K-S statistic, and larger $p - value$. From the plots in Figure 9, we can see that the TI-HT-OPGW-W distribution follows the fitted histogram closely and has the smallest sum of squares (SS) value from the probability plots. This supports the conclusion made from Table 6.

The total test time (TTT) scaled plots, observed and the fitted Kaplan-Meier survival curves, theoretical and empirical cumulative distribution function (ECDF) and hazard rate function (HRF) plots of the TI-HT-OPGW-W distribution are shown in Figure 10. From the Kaplan-Meier and ECDF plots, it is clear that the TI-HT-OPGW-W distribution is a good candidate for modeling the biomedical data. The TTT scaled plot demonstrates that the data follow an upside-down bathtub hazard rate shape. Furthermore, the hazard rate function exhibit a non-monotonic shape for the remission times data.

8.2. **Insurance Data.** This data set from the insurance field represents monthly metrics on unemployment insurance from July 2008 to April 2013 from the department of labor, licensing and regulation. It consists of 58 observations and 21 variables, we studied the variable number 6.

Table 5. Estimates of models for remission times data

	Estimates			
Model	θ	α	β	λ
TI-HT-OPGW-W	0.1575	4.6733	0.3432	0.2149
	(0.0633)	(2.6563)	(0.2056)	(0.0368)
	θ	α	β	λ
APTLW	0.3632	2.6272×10^{02}	0.8610	0.1120
	(0.1239)	(4.8767×10^{-05})	(0.1241)	(0.0540)
	α	θ	β	
TI-HT-W	1.1030	1.2551	0.0564	
	(0.6049)	(3.5043)	(0.3501)	
	α	γ	β	
HTBPT-W	1.0478	0.0938	0.9999	
	(0.0675)	(0.0190)	(0.7280)	
	w	λ	γ	
HLGW	6.2967	0.5000	0.0516	
	(9.9402)	(0.0776)	(0.0804)	
	a	b	c	λ
KW	8.8278	206.7178	0.1835	0.0301
	(4.6441)	(0.0352)	(0.0580)	(0.0679)
	a	λ	δ	γ
TIIEHLW	1.0297×10^{03}	1.0759×10^{02}	2.4004	0.0500
	(1.8761×10^{-06})	(1.9066×10^{-04})	(2.1528×10^{-02})	(3.2111×10^{-03})
	α	β	λ	γ
OGHLW-W	2.1269×10^{-05}	0.6471	14.4550	0.0774
	(3.4793×10^{-06})	(4.2738×10^{-04})	(1.9132×10^{-05})	(4.3136×10^{-03})

Table 6. Goodness-of-fit statistics for remission times data

	Statistics							
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
TI-HT-OPGW-W	822.0841	830.0841	830.4093	841.4922	0.0507	0.3255	0.0488	0.9196
APTLW	827.4507	835.4507	835.7759	846.8588	0.1275	0.7620	0.0727	0.5065
TI-HT-W	826.9924	832.9924	833.1859	841.5485	0.1150	0.6940	0.0684	0.5863
HTBPT-W	828.1738	834.1738	834.3673	842.7298	0.1313	0.7864	0.0700	0.5570
HLGW	851.9690	857.9690	858.1626	866.5251	0.2699	1.5921	0.1647	0.0019
KW	823.5203	831.5204	831.8456	842.9285	0.0776	0.4738	0.0567	0.8036
TIIEHLW	823.0934	831.0934	831.4186	842.5015	0.0710	0.4392	0.0549	0.8351
OGHLW-W	838.0349	846.0347	846.3599	857.4428	0.2477	1.4603	0.0952	0.1962

It is available at: https://catalog.data.gov/dataset/ unemployment-insurance-data-july-2008-to-april-2013. (**See the data in the Appendix**).

For these data, the MLEs, standard errors (in parenthesis) are given in Table 7. From the goodness-of-fit statistics given in Table 8, we observe that the proposed

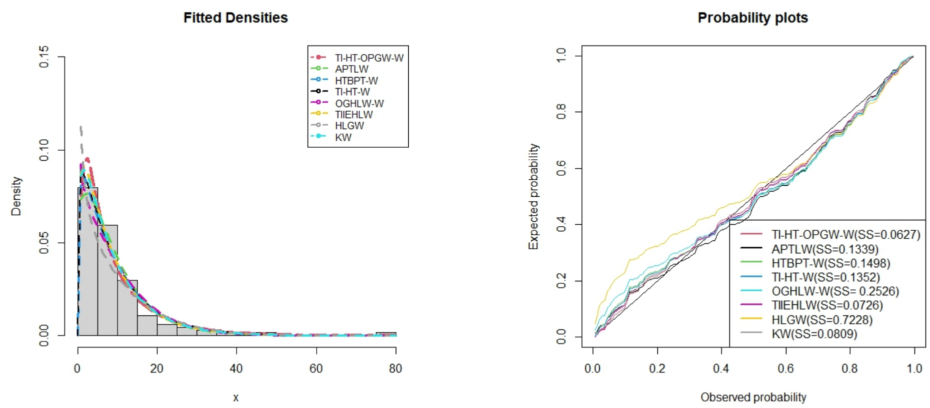

FIGURE 9. Fitted densities and probability plots for remission times data

TABLE 7. Estimates of models for insurance data

	Estimates			
Model	θ	α	β	λ
TI-HT-OPGW-W	0.3099	2.0529	0.0078	1.1668
	(0.1775)	(1.8481×10^{-05})	(0.0109)	(0.2846)
	θ	α	β	λ
APTLW	1.3112	2.3022	2.3222	7.6494×10^{-05}
	(2.3870×10^{-10})	(5.1581×10^{-11})	(2.4955×10^{-09})	(8.0428×10^{-06})
	α	θ	β	
TI-HT-W	1.1963	0.1221	0.0916	
	(0.1374)	(0.0537)	(0.0567)	
	α	γ	β	
HTBPT-W	1.2034	0.0262	7.1487×10^{-05}	
	(0.1125)	(0.01207)	(6.8034×10^{-05})	
	w	λ	γ	
HLGW	545.6400	1.0121	3.2890×10^{-05}	
	(1.2046×10^{-13})	(2.7123×10^{-10})	(2.0129×10^{-06})	
	a	b	c	λ
KW	1.2341×10^{02}	9.4224×10^{03}	0.1119	1.1220×10^{02}
	(2.0464×10^{-07})	(3.0326×10^{-10})	(6.1342×10^{-04})	(6.9732×10^{-08})
	a	λ	δ	γ
TIIEHLW	1.4145×10^{04}	2.3843×10^{03}	4.4752	0.0500
	(9.5420×10^{-08})	(3.7577×10^{-05})	(9.1240×10^{-02})	(5.0324×10^{-03})
	α	β	λ	γ
OGHLW-W	0.0008	1.2387	1.7631	0.3265
	(0.0004)	(0.2632)	(0.1435)	(0.0490)

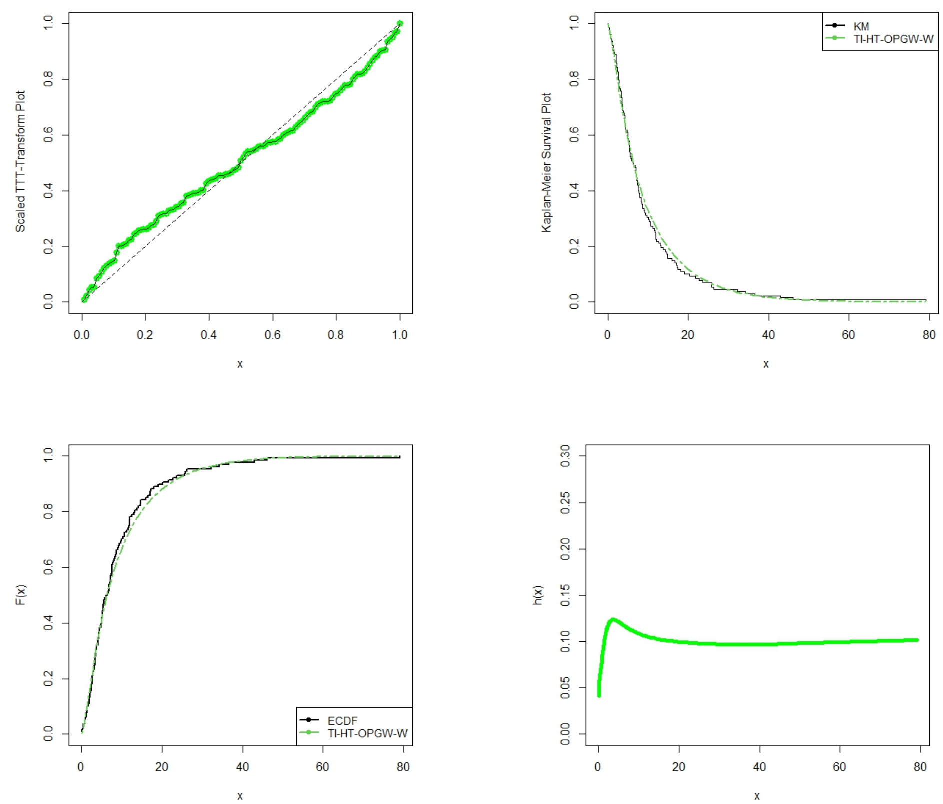

FIGURE 10. Fitted TTT, Kaplan-Meier Survival, ECDF and HRF plots for remission times data

TABLE 8. Goodness-of-fit statistics for insurance data

	Statistics							
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
TI-HT-OPGW-W	495.8290	503.8290	504.5838	512.0708	0.0578	0.3621	0.0719	0.9249
APTLW	496.5398	504.5398	505.2945	512.7816	0.1102	0.5583	0.1054	0.5394
TI-HT-W	525.6656	531.6656	532.1100	537.8469	0.3263	1.7175	0.2036	0.0162
HTBPT-W	505.3288	511.3291	511.7735	517.5104	0.2356	1.2313	0.1188	0.3853
HLGW	512.5666	518.5643	519.0087	524.7456	0.0841	0.4385	0.2242	0.0058
KW	498.6628	506.6629	507.4176	514.9047	0.1545	0.7836	0.1241	0.3328
TIIEHLW	498.9921	506.9921	507.7468	515.2338	0.1601	0.8126	0.1261	0.3145
OGHLW-W	496.0044	504.0044	504.7591	512.2462	0.0599	0.3634	0.0741	0.9069

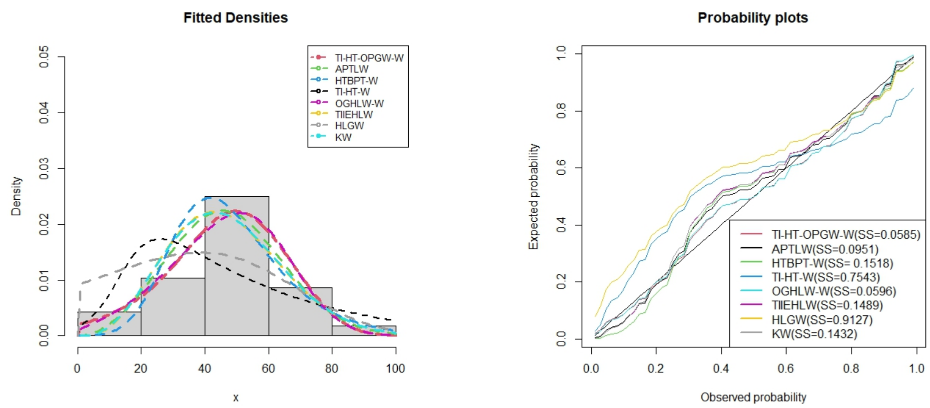

FIGURE 11. Fitted densities and probability plots for insurance data

TI-HT-OPGW-W distribution is the best choice to implement for fitting the insurance data since it has the smallest values of goodness-of-fit statistics and larger p-value compared to other fitted distributions. To support the results in Table 8, a visual illustrations is provided in Figure 11.

Figure 12 shows the TTT scaled plots, observed and the fitted Kaplan-Meier survival curves, ECDF and HRF plots. We can see that the TI-HT-OPGW-W distribution follows the empirical cdf, and Kaplan-Meier survival curves very closely. The TTT scaled plot shows an increasing hrf, allowing us to fit the heavy-tailed insurance data using TI-HT-OPGW-W distribution. Furthermore, the estimated hazard rate function for insurance data is an increasing shape.

8.3. **Agriculture Data.** This dataset represents the total factor productivity (TFP) growth agricultural production for thirty-seven African countries from 2001-2010, see `https://dataverse.harvard.edu/dataset.xhtml?persistent\Id=doi:10.7910/DVN/9IOAKR`, accessed on 30 June 2022. (**See the data in the Appendix**).

The parameter estimates, standard error (in parentheses) are given in Table 9. The goodness-of-fit statistics: AIC, BIC, CAIC, W^*, A^*, K-S statistic, and its p-value are given in Table 10. The values in Table 10 shows that the TI-HT-OPGW-W distribution gives the smallest values for the goodness-of-fit statistics and the largest p-value of K-S statistic. Thus, the TI-HT-OPGW-W distribution provides better fit than the rest of the distributions for the TFP growth data. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 13.

Figure 14 presents the TTT scaled plots, empirical and theoretical Kaplan-Meier survival plots, cumulative frequency curve of the observed data with the fitted cdf

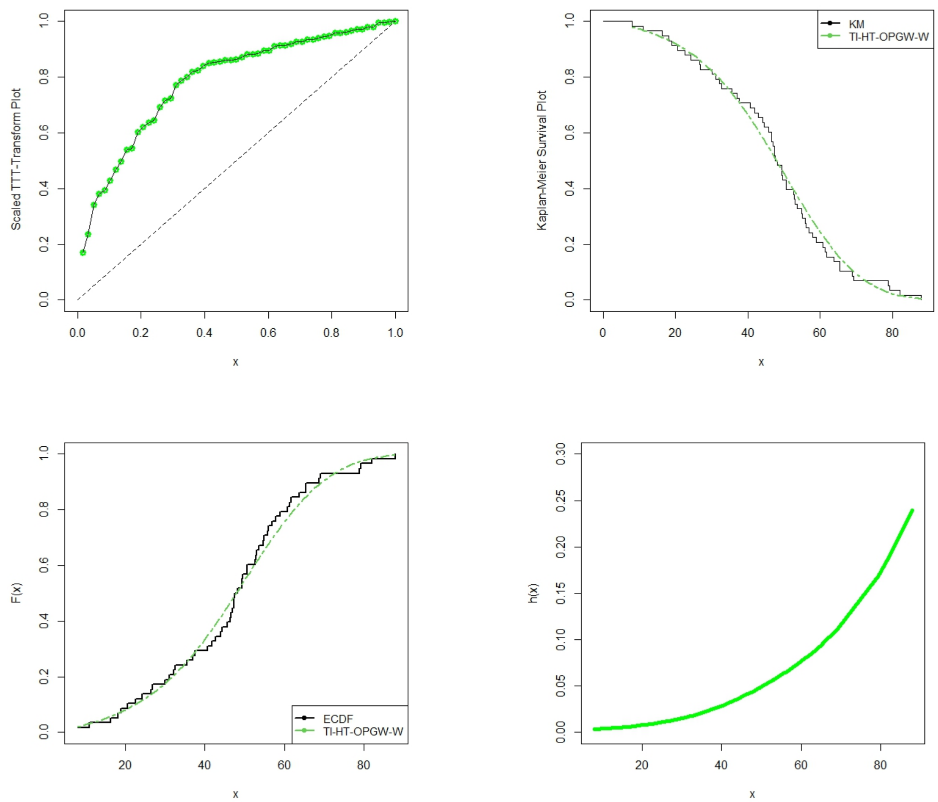

FIGURE 12. Fitted TTT, Kaplan-Meier survival, ECDF and HRF plots for insurance data

of the TI-HT-OPGW-W distribution and also, the HRF plot. It is visible that, the fitted empirical and theoretical plots are close to each other, hence we conclude that our model provide the better fit for the data. The TTT scaled plot clearly demonstrates that the data fit the increasing hazard rate structure. Furthermore, the estimated hazard rate function for TFP growth data is an increasing shape.

8.4. **Failure Times Data.** These data represent the failure time of a machine from Babel tyres factory (Iraq) in hours. The data was obtained from [5]. (**See the data in the Appendix**).

Table 11 and 12 presents numerical values of the MLEs with standard errors (in parenthesis), $-2\ln(L)$, AIC, $CAIC$, BIC, W^* , A^*, K-S statistic, and its $p-value$ for the failure times data. It can be seen that the TI-HT-OPGW-W distribution has

Table 9. Estimates of models for TFP growth data

	Estimates			
Model	θ	α	β	λ
TI-HT-OPGW-W	73.8740	0.9815	7.6304×10^{-05}	1.4386
	(2.6403×10^{-11})	(8.2976×10^{-10})	(1.2706×10^{-05})	(8.8095×10^{-10})
	θ	α	β	λ
APTLW	3.1266	12.1322	0.6406	1.0002
	(5.8242)	(29.4512)	(0.5086)	(1.0759)
	α	θ	β	
TI-HT-W	0.8309	0.2991	2.7494	
	(0.1212)	(0.1187)	(1.1363)	
	α	γ	β	
HTBPT-W	0.6588	1.1740	0.0286	
	(0.1140)	(0.2605)	(0.0330)	
	w	λ	γ	
HLGW	797.4800	0.5000	9.0808×10^{-04}	
	(1.1333×10^{-07})	(7.3374×10^{-02})	(9.6790×10^{-05})	
	a	b	c	λ
KW	0.1293	1.8134	7.4198	0.2015
	(0.0446)	(1.0046)	(2.5729)	(0.0385)
	a	λ	δ	γ
TIIEHLW	1.6908	3.9314	0.9271	0.6168
	(1.5011)	(4.8099)	(0.8245)	(0.4828)
	α	β	λ	γ
OGHLW-W	1.0774×10^{-04}	0.6420	13.6300	0.1275
	(1.2797×10^{-04})	(8.7916×10^{-02})	(3.5390×10^{-03})	(2.3757×10^{-02})

Table 10. Goodness-of-fit statistics for TFP growth data

	Statistics							
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
TI-HT-OPGW-W	107.0129	115.0129	116.2629	121.4566	0.0293	0.1812	0.0798	0.9722
APTLW	107.9218	115.9218	117.1718	122.3655	0.0368	0.2264	0.1049	0.8098
TI-HT-W	111.3917	117.3918	118.1191	122.2246	0.0368	0.2240	0.1197	0.6635
HTBPT-W	115.7192	121.7192	122.4464	126.5519	0.0311	0.1928	0.1807	0.1781
HLGW	117.7719	123.7719	124.4991	128.6046	0.0325	0.2050	0.2142	0.0670
KW	109.6151	117.6158	118.8658	124.0595	0.0745	0.4569	0.1043	0.8151
TIIEHLW	109.7821	117.7821	119.0321	124.2258	0.0643	0.4020	0.1255	0.6043
OGHLW-W	108.3969	116.3967	117.6467	122.8403	0.0428	0.2752	0.0889	0.9316

the lowest value for all goodness-of-fit statistics and a larger $p-value$ between all fitted distributions which gives it the superiority for fitting the failure times data. To support the best fitting power of the TI-HT-OPGW-W distribution, plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 15.

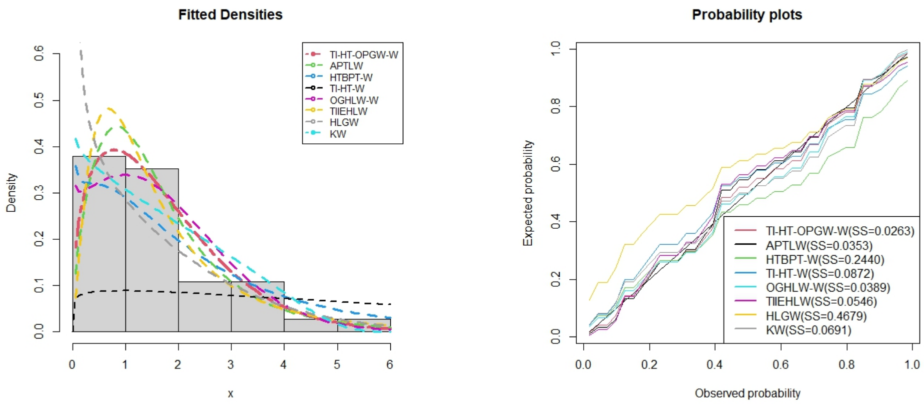

FIGURE 13. Fitted densities and probability plots for TFP growth data

TABLE 11. Estimates of models for failure times data

	Estimates			
Model	θ	α	β	λ
TI-HT-OPGW-W	0.1548	24.2130	0.0461	0.1683
	(0.1132)	(4.5574×10^{-05})	(0.0239)	(0.0168)
	θ	α	β	λ
APTLW	0.5500	0.0173	1.3216	0.0001
	(0.0748)	(0.0675)	(0.0378)	(0.0001)
	α	θ	β	
TI-HT-W	0.6980	0.1196	0.4071	
	(0.0756)	(0.0486)	(0.1391)	
	α	γ	β	
HTBPT-W	0.8987	0.0125	1.0002	
	(0.0783)	(0.0052)	(0.4193)	
	w	λ	γ	
HLGW	0.2370	0.9780	0.5827	
	(0.0500)	(0.1847)	(0.4439)	
	a	b	c	λ
KW	15.4546	62.0537	0.0990	0.4869
	(4.7782)	(0.1384)	(0.0165)	(1.1236)
	a	λ	δ	γ
TIIEHLW	27.4140	109.6900	2.6706	0.0500
	(3.2565×10^{-04})	(7.7848×10^{-04})	(8.9301×10^{-02})	(6.7886×10^{-03})
	α	β	λ	γ
OGHLW-W	2.1083×10^{-05}	0.4143	21.2990	0.0510
	(8.5735×10^{-06})	(6.8140×10^{-04})	(1.3243×10^{-05})	(5.8370×10^{-03})

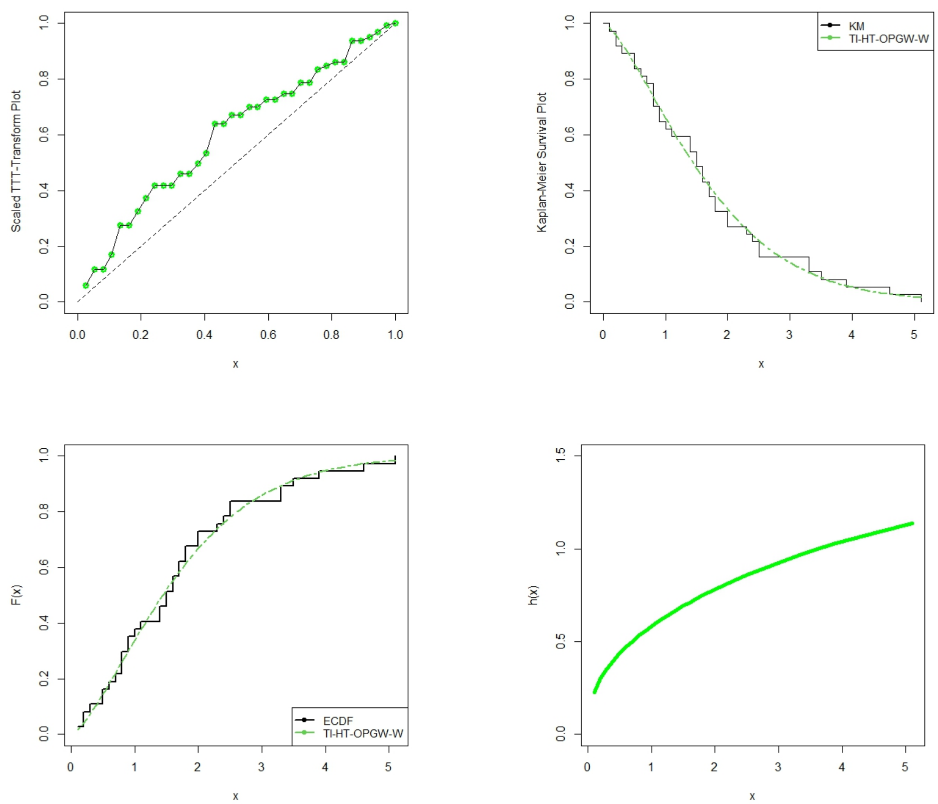

FIGURE 14. Fitted TTT, Kaplan-Meier survival, ECDF and HRF plots for TFP growth data

TABLE 12. Goodness-of-fit statistics for failure times data

	Statistics							
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
TI-HT-OPGW-W	327.3327	335.3327	336.9994	340.8019	0.0660	0.3691	0.1344	0.6705
APTLW	328.3628	336.3628	338.0295	341.8320	0.0845	0.4480	0.1495	0.5353
TI-HT-W	332.8800	338.8801	339.8401	342.9819	0.0687	0.3736	0.2413	0.0682
HTBPT-W	333.4933	339.4933	340.4533	343.5952	0.1034	0.5659	0.2861	0.0173
HLGW	333.0898	339.0899	340.0499	343.1918	0.0598	0.3901	0.1786	0.3132
KW	327.7001	335.7001	337.3668	341.1693	0.0696	0.3815	0.1365	0.6518
TIIEHLW	327.5437	335.8219	337.4885	341.2911	0.0731	0.3973	0.1377	0.6409
OGHLW-W	331.0865	339.0866	340.7532	344.5557	0.1181	0.6422	0.1616	0.4348

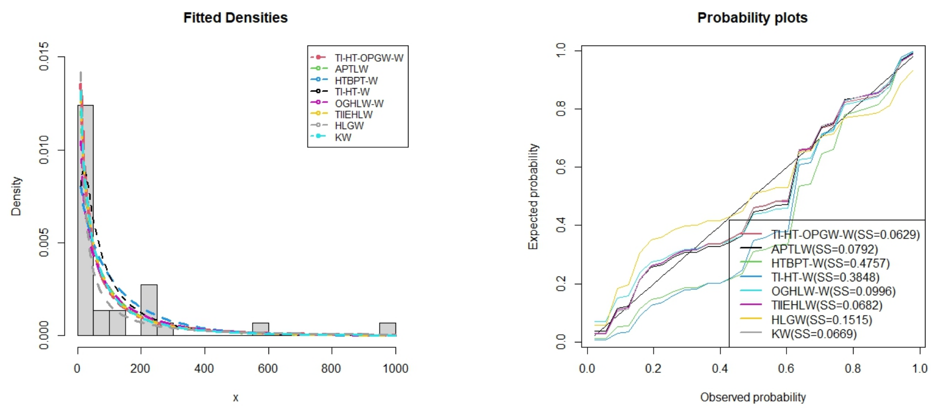

FIGURE 15. Fitted densities and probability plots for failure times data

Figure 16 depicts the TTT scaled plots, observed and the fitted Kaplan-Meier survival curves, theoretical and ECDF and HRF plots. The green line in the Kaplan-Meier and ECDF plots illustrates the good fit to the data. The TTT scaled plot demonstrates that the data follow a decreasing hazard rate shape. Furthermore, the hazard rate function exhibit a decreasing shape for the failure times data.

9. Concluding Remarks

We propose and study a new heavy-tailed family of distributions called type I heavy-tailed odd power generalized Weibull-G (TI-HT-OPGW-G) distribution. Many of its statistical properties such as quantile function, linear representation, moments, moment generating function, distribution of order statistics and Rényi entropy were derived. The maximum likelihood estimation method was derived and evaluated via a simulation study. Actuarial measures for the proposed distribution were also derived. Numerical comparisons of the actuarial measures with other distributions was conducted. Finally, the superiority and importance of the TI-HT-OPGW-G family of distributions was illustrated by using four real data sets from different fields. The TI-HT-OPGW-W model as a special case to this new family of distributions was applied to four datasets and from the results it is evident that the new proposed model performs better than several heavy tailed distributions.

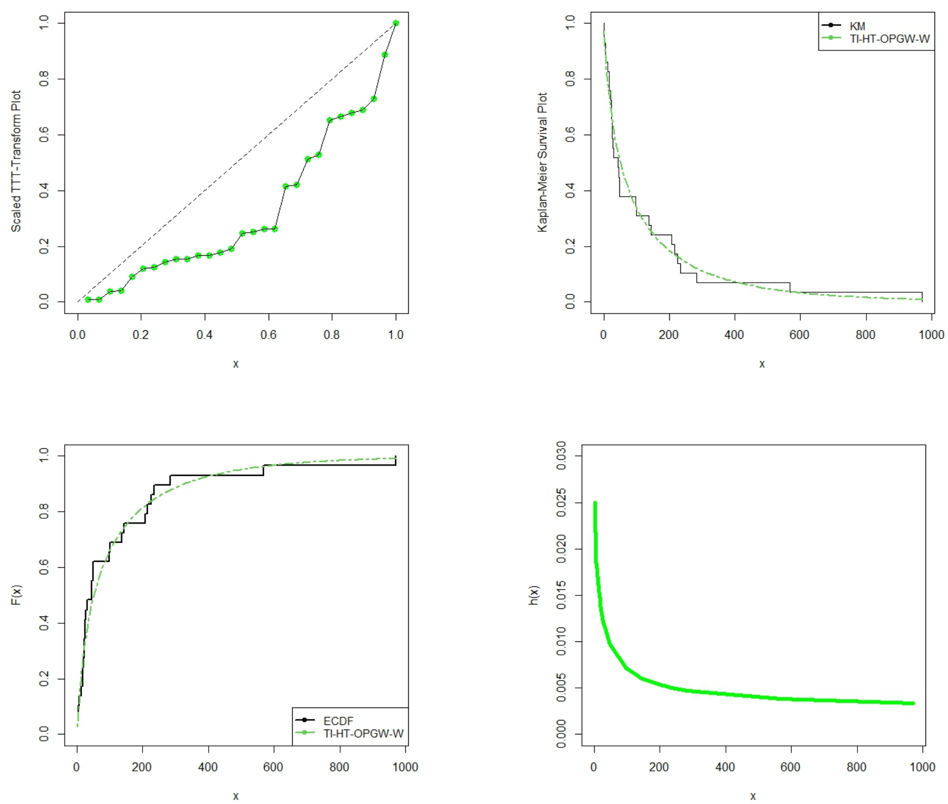

FIGURE 16. Fitted TTT, Kaplan-Meier survival, ECDF and HRF plots for failure times data

APPENDIX

Click the link below to access the appendix.
https://drive.google.com/file/d/124z_Bzokbw7l6cQ-kycxDC-yF_D_mRLn/view?usp=sharing

Author Contribution Statements Conceptualization, Broderick Oluyede; Methodology, Broderick Oluyede; Software, Thatayaone Moakofi; Formal analysis, Thatayaone Moakofi; Writing – review & editing, Broderick Oluyede and Thatayaone Moakofi. All authors have read and agreed to the published version of the manuscript.

Declaration of Competing Interests The authors declare no conflict of interest.

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