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On Higher Order Jacobsthal Hyper Complex Numbers

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Abstract. In this work, we define a new class of hyper complex numbers whose components are higher order Jacobsthal numbers, and call such numbers as the higher order Jacobsthal 2^s-ions. We obtain some algebraic properties of the higher order Jacobsthal 2^s-ions such as recurrence relation, Binet-like formula, generating function, exponential generating function, Vajda's identity, Catalan's identity, Cassini's identity and d'Ocagne's identity. Morever we derive the matrix representation of the higher order Jacobsthal 2^s-ions, and so prove Cassini's identity as a further type.

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1. Introduction

William Rowan Hamilton observed quaternions $\mathbb{Q}$ in 1843, which is a $2^2 = 4$-dimensional algebra over $\mathbb{R}$. Quaternions are associative, non-commutative and extension of complex numbers. John Graves introduced octonions $\mathbb{O}$ in 1843, which is a $2^3 = 8$-dimensional algebra over $\mathbb{R}$. Octonions are non-associative, non-commutative and extension of quaternions. In 1845, Arthur Cayley discovered these algebras again, and called such numbers as Cayley numbers. The Cayley-Dickson doubling process or the Cayley-Dickson process involves passing from $\mathbb{R}$ to $\mathbb{C}$, from $\mathbb{C}$ to $\mathbb{Q}$, and from $\mathbb{Q}$ to $\mathbb{O}$. The sedenions $\mathbb{S}$, which is a $2^4 = 16$-dimensional algebra over $\mathbb{R}$, are produced by the subsequent doubling process applied to $\mathbb{O}$. This doubling process can be extended beyond the sedenions to produce the 2^s-ions.

The real hyper complex algebra $\mathbb{HC}$ (or 2^s-ions) is a 2^s-dimensional $\mathbb{R}$-linear space with basis

$$\{\boldsymbol{e_0}, \boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_{2^s-1}}\}.$$

Here $\boldsymbol{e_0}$ is called the unit, and $\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_{2^s-1}}$ are called imaginaries. Many researchers have investigated the 2^s-ions in a wide range of fields, including coding theory, computer sciences, robotics, physics, navigation and many other areas. For more information, see [1–5, 9, 11, 14].

The Jacobsthal and Jacobsthal-Lucas numbers are defined by the following recurrence relation for $n \geq 2$

$$J_n = J_{n-1} + 2J_{n-2},$$
$$j_n = j_{n-1} + 2j_{n-2},$$

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where $J_0 = 0, J_1 = 1, j_0 = 2$, and $j_0 = 1$, respectively [10]. The Binet's formulas [10] for the Jacobsthal and Jacobsthal-Lucas numbers are given as follows:

$$J_n = \frac{2^n - (-1)^n}{3}, \quad j_n = 2^n + (-1)^n. \tag{1.1}$$

Also for $n \in \mathbb{Z}$, we can give

$$J_{-n} = \frac{(-1)^{n+1} J_n}{2^n}. \tag{1.2}$$

Many mathematicians have generalized Jacobsthal and Jacobsthal-Lucas numbers from ancient times to the present. The higher order Jacobsthal numbers are one of these generalizations. In [16], Özkan and Uysal defined the higher order Jacobsthal numbers for $r \geq 1$ integer, as

$$J_n^{(r)} = \frac{J_{nr}}{J_r} = \frac{2^{nr} - (-1)^{nr}}{2^r - (-1)^r}. \tag{1.3}$$

Since J_{nr} is divisible by J_r, the ratio $\frac{J_{nr}}{J_r}$ is an integer. Hence, all higher order Jacobsthal numbers $J_n^{(r)}$ are integer. For $r = 1$, the higher order Jacobsthal numbers $J_n^{(1)}$ become the known Jacobsthal numbers J_n. We give the higher order Jacobsthal numbers $J_n^{(r)}$ for some n and r in Table 1.

Table 1. The higher order Jacobsthal numbers $J_n^{(r)}$ for some n and r.

$J_n^{(r)}$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
$J_0^{(r)}$	0	0	0	0	0
$J_1^{(r)}$	1	1	1	1	1
$J_2^{(r)}$	1	5	7	17	31
$J_3^{(r)}$	3	21	57	273	993
$J_4^{(r)}$	5	85	455	4369	31775
$J_5^{(r)}$	11	341	3641	69905	1016801
$J_6^{(r)}$	21	1365	29127	1118481	32537631

Many scientists have studied quaternions, octonions, and sedenions whose components are Jacobsthal numbers. Szynal-Liana and Włoch [18] introduced the n-th Jacobsthal quaternions $\mathbb{Q}J_n$ as

$$\mathbb{Q}J_n = J_n \boldsymbol{e_0} + J_{n+1} \boldsymbol{e_1} + J_{n+2} \boldsymbol{e_2} + J_{n+3} \boldsymbol{e_3}.$$

In [19], Torunbalcı-Aydın and Yüce obtained some properties of Jacobsthal quaternions such as Binet-like formula and Cassini's identity. Then, Özkan and Uysal [16] defined the higher order Jacobsthal quaternions as

$$\mathbb{Q}J_n^{(r)} = J_n^{(r)} \boldsymbol{e_0} + J_{n+1}^{(r)} \boldsymbol{e_1} + J_{n+2}^{(r)} \boldsymbol{e_2} + J_{n+3}^{(r)} \boldsymbol{e_3},$$

and presented some identities related to these quaternions.

Çimen and İpek [6, 7] defined the n-th Jacobsthal octonions $\mathbb{O}J_n$ and the n-th Jacobsthal sedenions $\mathbb{S}J_n$ as

$$\mathbb{O}J_n = \sum_{i=0}^{7} J_{n+i} \boldsymbol{e_i},$$

$$\mathbb{S}J_n = \sum_{i=0}^{15} J_{n+i} \boldsymbol{e_i},$$

and they obtained a wide range of identities for these sequences. After, Göcen and Soykan [8] defined the Horadam 2^s-ions, and investigated some properties of these sequences. They are a generalization of Jacobsthal quaternions, octonions and sedenions.

Many researchers have recently focused on higher order numbers. For example, Özvatan [17] introduced the higher order Fibonacci numbers. In [13], Kızılateş and Kone have studied higher order Fibonacci quaternions. Furthermore, they [12] have introduced higher order Fibonacci hyper complex numbers which were defined by using the higher order Fibonacci numbers. Then, they obtained Vajda's identity, Catalan's identity, Cassini's identity and d'Ocagne's identity by Binet-like formula. With the help of the matrix representation of the higher order Fibonacci hyper complex numbers, they proved the another form of Cassini's identity. In [16, 20], Özkan and Uysal defined higher order Jacobsthal and Jacobsthal-Lucas numbers, and higher order Jacobsthal and Jacobsthal-Lucas quaternions, respectively, and gaved some algebraic properties of these numbers. Then, Özimamoğlu [15] defined higher order Pell numbers, and higher order Pell hyper complex numbers and presented some properties related to these hyper complex numbers.

In light of the previous recent studies, we introduce a new family of Jacobsthal 2^s-ions, and give their properties This family is a generalization of the higher order Jacobsthal numbers $J_n^{(r)}$, and call such numbers as the higher order Jacobsthal 2^s-ions $\mathbb{HC}J_n^{(r)}$. We obtain recurrence relation, Binet-like formula, generating function, exponential generating function, Vajda's identity, Catalan's identity, Cassini's identity and d'Ocagne's identity for $\mathbb{HC}J_n^{(r)}$. We define two new matrices $\boldsymbol{T^{(r)}}$ and $\boldsymbol{J^{(r)}}$. By these matrices, we find a matrix whose entries are higher order Jacobsthal 2^s-ions and generate the Cassini's identity.

2. Higher order Jacobsthal 2^s-ions

In this part, we define the higher order Jacobsthal 2^s-ions and obtain several other new identities for them. Throughout this paper, we take

$$\widehat{2} = \sum_{i=0}^{2^s-1} 2^{ir}\boldsymbol{e_i} = \boldsymbol{e_0} + 2^r\boldsymbol{e_1} + 2^{2r}\boldsymbol{e_2} + \cdots + 2^{(2^s-1)r}\boldsymbol{e_{2^s-1}},$$

$$\widehat{(-1)} = \sum_{i=0}^{2^s-1} (-1)^{ir}\boldsymbol{e_i} = \boldsymbol{e_0} + (-1)^r\boldsymbol{e_1} + (-1)^{2r}\boldsymbol{e_2} + \cdots + (-1)^{(2^s-1)r}\boldsymbol{e_{2^s-1}}.$$

Definition 2.1. The higher order Jacobsthal hyper complex numbers $\mathbb{HC}J_n^{(r)}$ (or higher order Jacobsthal 2^s-ions) are defined by

$$\mathbb{HC}J_n^{(r)} = \sum_{i=0}^{2^s-1} J_{n+i}^{(r)}\boldsymbol{e_i} = J_n^{(r)}\boldsymbol{e_0} + J_{n+1}^{(r)}\boldsymbol{e_1} + J_{n+2}^{(r)}\boldsymbol{e_2} + \cdots + J_{n+2^s-1}^{(r)}\boldsymbol{e_{2^s-1}},$$

where $J_n^{(r)}$ is the n-th higher order Jacobsthal number.

Now we can present some special situations of $\mathbb{HC}J_n^{(r)}$ in Definition 2.1 as follows:

(1) For $s = 0$, we have the known higher order Jacobsthal numbers $J_n^{(r)}$ [16].
(2) For $s = 1$, we have the higher order Jacobsthal complex numbers $\mathbb{C}J_n^{(r)}$.
(3) For $s = 2$, we have the higher order Jacobsthal quaternions $\mathbb{Q}J_n^{(r)}$ [16], [19] .
(4) For $s = 3$, we have the higher order Jacobsthal octonions $\mathbb{O}J_n^{(r)}$.
(5) For $s = 4$, we have the higher order Jacobsthal sedenions $\mathbb{S}J_n^{(r)}$.
(6) For $s = 0$ and $r = 1$, we have the known Jacobsthal numbers J_n [10].
(7) For $s = 1$ and $r = 1$, we have the Jacobsthal complex numbers $\mathbb{C}J_n$.
(8) For $s = 2$ and $r = 1$, we have the Jacobsthal quaternions $\mathbb{Q}J_n$ [18].
(9) For $s = 3$ and $r = 1$, we have the Jacobsthal octonions $\mathbb{O}J_n$ [6].
(10) For $s = 4$ and $r = 1$, we have the Jacobsthal sedenions $\mathbb{S}J_n$ [7].

The conjugate of the higher order Jacobsthal 2^s-ions $\mathbb{HC}J_n^{(r)}$ is

$$\left(\mathbb{HC}J_n^{(r)}\right)^* = J_n^{(r)}\boldsymbol{e_0} - \sum_{i=1}^{2^s-1} J_{n+i}^{(r)}\boldsymbol{e_i} = J_n^{(r)}\boldsymbol{e_0} - J_{n+1}^{(r)}\boldsymbol{e_1} - \cdots - J_{n+2^s-1}^{(r)}\boldsymbol{e_{2^s-1}}. \tag{2.1}$$

Proposition 2.2. *For the higher order Jacobsthal 2^s-ions $\mathbb{HC}J_n^{(r)}$, we have*

$$\mathbb{HC}J_n^{(r)} + \left(\mathbb{HC}J_n^{(r)}\right)^* = 2J_n^{(r)}.$$

Proof. From Definition 2.1 and (2.1), we obtain

$$\begin{aligned}\mathbb{HC}J_n^{(r)} + \left(\mathbb{HC}J_n^{(r)}\right)^* &= \left[J_n^{(r)}\boldsymbol{e_0} + J_{n+1}^{(r)}\boldsymbol{e_1} + \cdots + J_{n+2^s-1}^{(r)}\boldsymbol{e_{2^s-1}}\right] + \left[J_n^{(r)}\boldsymbol{e_0} - J_{n+1}^{(r)}\boldsymbol{e_1} - \cdots - J_{n+2^s-1}^{(r)}\boldsymbol{e_{2^s-1}}\right] \\ &= J_n^{(r)}\boldsymbol{e_0} + J_n^{(r)}\boldsymbol{e_0} \\ &= 2J_n^{(r)}\boldsymbol{e_0}.\end{aligned}$$

□

Theorem 2.3. *The Binet-like formula for the higher order Jacobsthal 2^s-ions $\mathbb{HC}J_n^{(r)}$ is*

$$\mathbb{HC}J_n^{(r)} = \frac{2^{nr}\widehat{2} - (-1)^{nr}\widehat{(-1)}}{2^r - (-1)^r}.$$

Proof. From Definition 2.1, we get

$$\mathbb{HC}J_n^{(r)} = J_n^{(r)}\boldsymbol{e_0} + J_{n+1}^{(r)}\boldsymbol{e_1} + J_{n+2}^{(r)}\boldsymbol{e_2} + \cdots + J_{n+2^s-1}^{(r)}\boldsymbol{e_{2^s-1}}$$

and by using the equation (1.3), we obtain

$$\begin{aligned}\mathbb{HC}J_n^{(r)} &= \left[\frac{2^{nr} - (-1)^{nr}}{2^r - (-1)^r}\right]\boldsymbol{e_0} + \left[\frac{2^{(n+1)r} - (-1)^{(n+1)r}}{2^r - (-1)^r}\right]\boldsymbol{e_1} + \left[\frac{2^{(n+2)r} - (-1)^{(n+2)r}}{2^r - (-1)^r}\right]\boldsymbol{e_2} + \cdots \\ &+ \left[\frac{2^{(n+2^s-1)r} - (-1)^{(n+2^s-1)r}}{2^r - (-1)^r}\right]\boldsymbol{e_{2^s-1}} \\ &= \frac{1}{2^r - (-1)^r}\left[2^{nr}\left(\boldsymbol{e_0} + 2^r\boldsymbol{e_1} + 2^{2r}\boldsymbol{e_2} + \cdots + 2^{(2^s-1)r}\boldsymbol{e_{2^s-1}}\right)\right] \\ &+ \frac{1}{2^r - (-1)^r}\left[-(-1)^{nr}\left(\boldsymbol{e_0} + (-1)^r\boldsymbol{e_1} + (-1)^{2r}\boldsymbol{e_2} + \cdots + (-1)^{(2^s-1)r}\boldsymbol{e_{2^s-1}}\right)\right] \\ &= \frac{2^{nr}\widehat{2} - (-1)^{nr}\widehat{(-1)}}{2^r - (-1)^r}.\end{aligned}$$

□

Corollary 2.4. *For some special values s, by Theorem 2.3 the Binet-like formulas of $\mathbb{HC}J_n^{(1)}$ are given as follows:*

(i) *For $s = 1$, we obtain the Binet-like formula for the Jacobsthal complex numbers as*

$$\mathbb{C}J_n = \frac{2^n\widehat{2} - (-1)^n\widehat{(-1)}}{3} \quad (\textit{Jacobsthal } 2^1 - \textit{ions}),$$

where $\widehat{2} = \sum_{i=0}^{1} 2^i\boldsymbol{e_i}$ *and* $\widehat{(-1)} = \sum_{i=0}^{1}(-1)^i\boldsymbol{e_i}$.

(ii) *For $s = 2$, we obtain the Binet-like formula for the Jacobsthal quaternions as*

$$\mathbb{Q}J_n = \frac{2^n\widehat{2} - (-1)^n\widehat{(-1)}}{3} \quad (\textit{Jacobsthal } 2^2 - \textit{ions}),$$

where $\widehat{2} = \sum_{i=0}^{3} 2^i\boldsymbol{e_i}$ *and* $\widehat{(-1)} = \sum_{i=0}^{3}(-1)^i\boldsymbol{e_i}$.

(iii) *For $s = 3$, we obtain the Binet-like formula for the Jacobsthal octonions as*

$$\mathbb{O}J_n = \frac{2^n\widehat{2} - (-1)^n\widehat{(-1)}}{3} \quad (\textit{Jacobsthal } 2^3 - \textit{ions}),$$

where $\widehat{2} = \sum_{i=0}^{7} 2^i\boldsymbol{e_i}$ *and* $\widehat{(-1)} = \sum_{i=0}^{7}(-1)^i\boldsymbol{e_i}$.

(iv) *For $s = 4$, we obtain the Binet-like formula for the Jacobsthal sedenions as*

$$\mathbb{S}J_n = \frac{2^n\widehat{2} - (-1)^n\widehat{(-1)}}{3} \quad (\textit{Jacobsthal } 2^4 - \textit{ions}),$$

where $\widehat{2} = \sum_{i=0}^{15} 2^i\boldsymbol{e_i}$ *and* $\widehat{(-1)} = \sum_{i=0}^{15}(-1)^i\boldsymbol{e_i}$.

(v) *For $s \in \mathbb{Z}^{+}$, we obtain the Binet-like formula for the Jacobsthal 2^s-ions as*

$$\mathbb{HC}J_n^{(1)} = \frac{2^n\widehat{2} - (-1)^n\widehat{(-1)}}{3} \quad (Jacobsthal\ 2^s - ions),$$

where $\widehat{2} = \sum_{i=0}^{2^s-1} 2^i \boldsymbol{e_i}$ and $\widehat{(-1)} = \sum_{i=0}^{2^s-1}(-1)^i\boldsymbol{e_i}$.

Theorem 2.5. *For $n \geq 1$, we have the following recurrence relation.*

$$\mathbb{HC}J_{n+1}^{(r)} = j_r\mathbb{HC}J_n^{(r)} - (-2)^r\mathbb{HC}J_{n-1}^{(r)}.$$

Proof. From Binet-like formula in Theorem 2.3, we get

$$\begin{aligned}
\mathbb{HC}J_{n+1}^{(r)} &= \frac{2^{(n+1)r}\widehat{2} - (-1)^{(n+1)r}\widehat{(-1)}}{2^r - (-1)^r} \\
&= \frac{1}{2^r - (-1)^r}\left[\left(2^{(n+1)r}\widehat{2} - 2^r(-1)^{nr}\widehat{(-1)}\right) + \left(2^r(-1)^{nr}\widehat{(-1)} - (-1)^{(n+1)r}\widehat{(-1)}\right)\right] \\
&= 2^r\left[\frac{1}{2^r - (-1)^r}\left(2^{nr}\widehat{2} - (-1)^{nr}\widehat{(-1)}\right)\right] + \frac{1}{2^r - (-1)^r}\left[2^r(-1)^{nr}\widehat{(-1)} - (-1)^{(n+1)r}\widehat{(-1)}\right] \\
&= 2^r\mathbb{HC}J_n^{(r)} + \frac{1}{2^r - (-1)^r}\left[2^r(-1)^{nr}\widehat{(-1)} - (-1)^{(n+1)r}\widehat{(-1)}\right] \\
&= \left[2^r + (-1)^r\right]\mathbb{HC}J_n^{(r)} - (-1)^r\mathbb{HC}J_n^{(r)} + \frac{1}{2^r - (-1)^r}\left[2^r(-1)^{nr}\widehat{(-1)} - (-1)^{(n+1)r}\widehat{(-1)}\right]
\end{aligned}$$

and by using the equation (1.1), we derive

$$\begin{aligned}
\mathbb{HC}J_{n+1}^{(r)} &= j_r\mathbb{HC}J_n^{(r)} + \frac{1}{2^r - (-1)^r}\left[-(-1)^r2^{nr}\widehat{2} + (-1)^{(n+1)r}\widehat{(-1)} + 2^r(-1)^{nr}\widehat{(-1)} - (-1)^{(n+1)r}\widehat{(-1)}\right] \\
&= j_r\mathbb{HC}J_n^{(r)} + \frac{1}{2^r - (-1)^r}\left[-(-1)^r2^{nr}\widehat{2} + 2^r(-1)^{nr}\widehat{(-1)}\right] \\
&= j_r\mathbb{HC}J_n^{(r)} + \frac{(-2)^r}{2^r - (-1)^r}\left[-2^{nr-r}\widehat{2} + (-1)^{nr-r}\widehat{(-1)}\right] \\
&= j_r\mathbb{HC}J_n^{(r)} - (-2)^r\left[\frac{2^{(n-1)r}\widehat{2} - (-1)^{(n-1)r}\widehat{(-1)}}{2^r - (-1)^r}\right] \\
&= j_r\mathbb{HC}J_n^{(r)} - (-2)^r\mathbb{HC}J_{n-1}^{(r)}.
\end{aligned}$$

□

Theorem 2.6. *The generating function for the higher order Jacobsthal 2^s-ions $\mathbb{HC}J_n^{(r)}$ is*

$$\mathbb{HC}J_n^{(r)}(x) = \frac{\left[\widehat{2} - \widehat{(-1)}\right] - \left[(-1)^r\widehat{2} - 2^r\widehat{(-1)}\right]x}{\left[2^r - (-1)^r\right]\left[1 - j_rx + (-2)^rx^2\right]}.$$

Proof. Let

$$\mathbb{HC}J_n^{(r)}(x) = \sum_{n=0}^{\infty}\mathbb{HC}J_n^{(r)}x^n$$

be the generating function of $\mathbb{HC}J_n^{(r)}$. From Binet-like formula of $\mathbb{HC}J_n^{(r)}$ in Theorem 2.3, we have

$$\begin{aligned}
\mathbb{HC}J_n^{(r)}(x) = \sum_{n=0}^{\infty} \mathbb{HC}J_n^{(r)} x^n &= \sum_{n=0}^{\infty} \left[\frac{2^{nr}\widehat{2} - (-1)^{nr}\widehat{(-1)}}{2^r - (-1)^r} \right] x^n \\
&= \frac{1}{2^r - (-1)^r} \left[\sum_{n=0}^{\infty} (2^r x)^n \widehat{2} - \sum_{n=0}^{\infty} ((-1)^r x)^n \, \widehat{(-1)} \right] \\
&= \left[\frac{\widehat{2}}{2^r - (-1)^r} \right] \left[\frac{1}{1 - 2^r x} \right] - \left[\frac{\widehat{(-1)}}{2^r - (-1)^r} \right] \left[\frac{1}{1 - (-1)^r x} \right] \\
&= \frac{\widehat{2} - (-1)^r \widehat{2} x - \widehat{(-1)} + 2^r \widehat{(-1)} x}{(2^r - (-1)^r)\,(1 - 2^r x)\,(1 - (-1)^r x)}
\end{aligned}$$

and by using the equation (1.1), we obtain

$$\mathbb{HC}J_n^{(r)}(x) = \frac{\left[\widehat{2} - \widehat{(-1)}\right] - \left[(-1)^r \widehat{2} - 2^r \widehat{(-1)}\right] x}{[2^r - (-1)^r]\,[1 - j_r x + (-2)^r x^2]}.$$

□

Corollary 2.7. *For some special values s, by Theorem 2.6 the generating functions of $\mathbb{HC}J_n^{(1)}(x)$ are given as follows:*

(i) *For $s = 1$, we obtain the generating function for the Jacobsthal complex numbers as*

$$\mathbb{C}J_n(x) = \frac{x\boldsymbol{e_0} + \boldsymbol{e_1}}{1 - x - 2x^2} \qquad (Jacobsthal\ 2^1 - ions).$$

(ii) *For $s = 2$, we obtain the generating function for the Jacobsthal quaternions as*

$$\mathbb{Q}J_n(x) = \frac{x\boldsymbol{e_0} + \boldsymbol{e_1} + (1 + 2x)\boldsymbol{e_2} + (3 + 2x)\boldsymbol{e_3}}{1 - x - 2x^2} \qquad (Jacobsthal\ 2^2 - ions).$$

(iii) *For $s = 3$, we obtain the generating function for the Jacobsthal octonions as*

$$\mathbb{O}J_n(x) = \frac{x\boldsymbol{e_0} + \sum_{i=1}^{7} (J_i + 2J_{i-1}x)\, \boldsymbol{e_i}}{1 - x - 2x^2} \qquad (Jacobsthal\ 2^3 - ions).$$

(iv) *For $s = 4$, we obtain the generating function for the Jacobsthal sedenions as*

$$\mathbb{S}J_n(x) = \frac{x\boldsymbol{e_0} + \sum_{i=1}^{15} (J_i + 2J_{i-1}x)\, \boldsymbol{e_i}}{1 - x - 2x^2} \qquad (Jacobsthal\ 2^4 - ions).$$

(v) *For $s \in \mathbb{Z}^+$, we obtain the generating function for the Jacobsthal 2^s-ions as*

$$\mathbb{HC}J_n^{(1)}(x) = \frac{x\boldsymbol{e_0} + \sum_{i=1}^{2^s-1} (J_i + 2J_{i-1}x)\, \boldsymbol{e_i}}{1 - x - 2x^2} \qquad (Jacobsthal\ 2^s - ions).$$

Theorem 2.8. *For $n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$, the generating function for the higher order Jacobsthal 2^s-ions $\mathbb{HC}J_{n+m}^{(r)}$ is*

$$\mathbb{HC}J_{n+m}^{(r)}(x) = \sum_{n=0}^{\infty} \mathbb{HC}J_{n+m}^{(r)} x^n = \frac{\mathbb{HC}J_m^{(r)} - (-2)^r \mathbb{HC}J_{m-1}^{(r)} x}{1 - j_r x + (-2)^r x^2}.$$

Proof. From Binet-like formula in Theorem 2.3, we have

$$\begin{aligned}
\mathbb{HC}J_{n+m}^{(r)}(x) = \sum_{n=0}^{\infty} \mathbb{HC}J_{n+m}^{(r)} x^n &= \sum_{n=0}^{\infty} \left[\frac{2^{(n+m)r}\widehat{2} - (-1)^{(n+m)r}\widehat{(-1)}}{2^r - (-1)^r} \right] x^n \\
&= \frac{1}{2^r - (-1)^r} \left[\sum_{n=0}^{\infty} (2^r x)^n 2^{mr} \widehat{2} - \sum_{n=0}^{\infty} ((-1)^r x)^n \, (-1)^{mr} \widehat{(-1)} \right] \\
&= \frac{1}{2^r - (-1)^r} \left[\frac{2^{mr}\widehat{2}}{1 - 2^r x} - \frac{(-1)^{mr}\widehat{(-1)}}{1 - (-1)^r x} \right] \\
&= \frac{1}{2^r - (-1)^r} \left[\frac{2^{mr}\widehat{2} - 2^{mr}\widehat{2}(-1)^r x - (-1)^{mr}\widehat{(-1)} + (-1)^{mr}\widehat{(-1)} 2^r x}{(1 - 2^r x)\,(1 - (-1)^r x)} \right]
\end{aligned}$$

and by using the equation (1.1), we derive

$$\begin{aligned}\mathbb{HC}J^{(r)}_{n+m}(x) &= \frac{1}{1-j_r x+(-2)^r x^2}\left[\frac{\left(2^{mr}\widehat{2}-(-1)^{mr}\widehat{(-1)}\right)-(-1)^r 2^r x\left(2^{mr-r}\widehat{2}-(-1)^{mr-r}\widehat{(-1)}\right)}{2^r-(-1)^r}\right]\\ &= \frac{\mathbb{HC}J^{(r)}_m-(-2)^r\mathbb{HC}J^{(r)}_{m-1}x}{1-j_rx+(-2)^rx^2}.\end{aligned}$$

□

Theorem 2.9. *The exponential generating function for the higher order Jacobsthal 2^s-ions $\mathbb{HC}J^{(r)}_n$ is*

$$\sum_{n=0}^{\infty}\mathbb{HC}J^{(r)}_n\frac{x^n}{n!} = \frac{e^{2^r x}\widehat{2}-e^{(-1)^r x}\widehat{(-1)}}{2^r-(-1)^r}.$$

Proof. By using Binet-like formula of $\mathbb{HC}J^{(r)}_n$ in Theorem 2.3, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{HC}J^{(r)}_n\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\left[\frac{2^{nr}\widehat{2}-(-1)^{nr}\widehat{(-1)}}{2^r-(-1)^r}\right]\frac{x^n}{n!}\\ &= \frac{1}{2^r-(-1)^r}\left[\sum_{n=0}^{\infty}\frac{(2^r x)^n}{n!}\widehat{2}-\sum_{n=0}^{\infty}\frac{((-1)^r x)^n}{n!}\widehat{(-1)}\right]\\ &= \frac{e^{2^r x}\widehat{2}-e^{(-1)^r x}\widehat{(-1)}}{2^r-(-1)^r}.\end{aligned}$$

□

3. Some Identities for Higher Order Jacobsthal 2^s-ions

In this part, we present some identities for higher order Jacobsthal 2^s-ions.

Theorem 3.1. ***(Vajda's Identity)*** *For any integers n, m and k, we get*

$$\mathbb{HC}J^{(r)}_{n+m}\mathbb{HC}J^{(r)}_{n+k}-\mathbb{HC}J^{(r)}_{n}\mathbb{HC}J^{(r)}_{n+m+k} = \frac{(-2)^{nr}J^{(r)}_m\left[2^{kr}\widehat{(-1)}\widehat{2}-(-1)^{kr}\widehat{2}\widehat{(-1)}\right]}{2^r-(-1)^r}.$$

Proof. From Binet-like formula for the higher order Jacobsthal 2^s-ions in Theorem 2.3, we get

$$\begin{aligned}&\mathbb{HC}J^{(r)}_{n+m}\mathbb{HC}J^{(r)}_{n+k}-\mathbb{HC}J^{(r)}_{n}\mathbb{HC}J^{(r)}_{n+m+k}\\ &= \left[\frac{2^{(n+m)r}\widehat{2}-(-1)^{(n+m)r}\widehat{(-1)}}{2^r-(-1)^r}\right]\left[\frac{2^{(n+k)r}\widehat{2}-(-1)^{(n+k)r}\widehat{(-1)}}{2^r-(-1)^r}\right]-\left[\frac{2^{nr}\widehat{2}-(-1)^{nr}\widehat{(-1)}}{2^r-(-1)^r}\right]\left[\frac{2^{(n+m+k)r}\widehat{2}-(-1)^{(n+m+k)r}\widehat{(-1)}}{2^r-(-1)^r}\right]\\ &= \frac{1}{[2^r-(-1)^r]^2}\left[-2^{(n+m)r}(-1)^{(n+k)r}\widehat{2}\widehat{(-1)}-(-1)^{(n+m)r}2^{(n+k)r}\widehat{(-1)}\widehat{2}+2^{nr}(-1)^{(n+m+k)r}\widehat{2}\widehat{(-1)}+(-1)^{nr}2^{(n+m+k)r}\widehat{(-1)}\widehat{2}\right]\\ &= \frac{2^{nr}(-1)^{nr}}{[2^r-(-1)^r]^2}\left[-2^{mr}(-1)^{kr}\widehat{2}\widehat{(-1)}-(-1)^{mr}2^{kr}\widehat{(-1)}\widehat{2}+(-1)^{(m+k)r}\widehat{2}\widehat{(-1)}+2^{(m+k)r}\widehat{(-1)}\widehat{2}\right]\\ &= \frac{(-2)^{nr}}{[2^r-(-1)^r]^2}\left[-(-1)^{kr}\widehat{2}\widehat{(-1)}\left(2^{mr}-(-1)^{mr}\right)+2^{kr}\widehat{(-1)}\widehat{2}\left(2^{mr}-(-1)^{mr}\right)\right]\\ &= \frac{(-2)^{nr}\left[2^{mr}-(-1)^{mr}\right]\left[2^{kr}\widehat{(-1)}\widehat{2}-(-1)^{kr}\widehat{2}\widehat{(-1)}\right]}{[2^r-(-1)^r]^2}\end{aligned}$$

and by using the equation (1.3), we find that

$$\mathbb{HC}J^{(r)}_{n+m}\mathbb{HC}J^{(r)}_{n+k}-\mathbb{HC}J^{(r)}_{n}\mathbb{HC}J^{(r)}_{n+m+k} = \frac{(-2)^{nr}J^{(r)}_m\left[2^{kr}\widehat{(-1)}\widehat{2}-(-1)^{kr}\widehat{2}\widehat{(-1)}\right]}{2^r-(-1)^r}.$$

□

Proposition 3.2. *For $n \in \mathbb{Z}, r \in \mathbb{Z}^+$ we have*

$$J_{-n}^{(r)} = \frac{(-1)^{nr+1} J_n^{(r)}}{2^{nr}}.$$

Proof. By using (1.3) and (1.2), we can easily obtain

$$J_{-n}^{(r)} = \frac{J_{-nr}}{J_r} = \frac{(-1)^{nr+1} J_{nr}}{2^{nr} J_r} = \frac{(-1)^{nr+1} J_n^{(r)}}{2^{nr}}.$$

□

Now, from the Vajda's identity, we provide the following special situations:

Corollary 3.3. ***(Catalan's Identity)*** *Let $n, k \in \mathbb{Z}^+$ be such that $n \geq k$. Then, the Catalan's identity for the higher order for Jacobsthal 2^s-ions is*

$$\mathbb{HC}J_{n-k}^{(r)} \mathbb{HC}J_{n+k}^{(r)} - \left(\mathbb{HC}J_n^{(r)}\right)^2 = \frac{-(-2)^{(n-k)r} J_k^{(r)} \left[2^{kr}\widehat{(-1)}\widehat{2} - (-1)^{kr}\widehat{2}\widehat{(-1)}\right]}{2^r - (-1)^r}.$$

Proof. For $m = -k$ in Theorem 3.1, we have

$$\mathbb{HC}J_{n-k}^{(r)} \mathbb{HC}J_{n+k}^{(r)} - \left(\mathbb{HC}J_n^{(r)}\right)^2 = \frac{(-2)^{nr} J_{-k}^{(r)} \left[2^{kr}\widehat{(-1)}\widehat{2} - (-1)^{kr}\widehat{2}\widehat{(-1)}\right]}{2^r - (-1)^r}$$

and by using Proposition 3.2, we find that

$$\begin{aligned}\mathbb{HC}J_{n-k}^{(r)} \mathbb{HC}J_{n+k}^{(r)} - \left(\mathbb{HC}J_n^{(r)}\right)^2 &= \frac{2^{(n-k)r}(-1)^{(n+k)r+1} J_k^{(r)} \left[2^{kr}\widehat{(-1)}\widehat{2} - (-1)^{kr}\widehat{2}\widehat{(-1)}\right]}{2^r - (-1)^r} \\ &= \frac{-(-2)^{(n-k)r} J_k^{(r)} \left[2^{kr}\widehat{(-1)}\widehat{2} - (-1)^{kr}\widehat{2}\widehat{(-1)}\right]}{2^r - (-1)^r}.\end{aligned}$$

□

Corollary 3.4. ***(Cassini's Identity)*** *For $n \in \mathbb{Z}^+$, then we get*

$$\mathbb{HC}J_{n-1}^{(r)} \mathbb{HC}J_{n+1}^{(r)} - \left(\mathbb{HC}J_n^{(r)}\right)^2 = \frac{-(-2)^{(n-1)r} \left[2^{r}\widehat{(-1)}\widehat{2} - (-1)^{r}\widehat{2}\widehat{(-1)}\right]}{2^r - (-1)^r}.$$

Proof. For $k = 1$ in Corollary 3.3, we have

$$\mathbb{HC}J_{n-1}^{(r)} \mathbb{HC}J_{n+1}^{(r)} - \left(\mathbb{HC}J_n^{(r)}\right)^2 = \frac{-(-2)^{(n-1)r} J_1^{(r)} \left[2^{r}\widehat{(-1)}\widehat{2} - (-1)^{r}\widehat{2}\widehat{(-1)}\right]}{2^r - (-1)^r}$$

and by using the equation (1.3), we derive

$$\mathbb{HC}J_{n-1}^{(r)} \mathbb{HC}J_{n+1}^{(r)} - \left(\mathbb{HC}J_n^{(r)}\right)^2 = \frac{-(-2)^{(n-1)r} \left[2^{r}\widehat{(-1)}\widehat{2} - (-1)^{r}\widehat{2}\widehat{(-1)}\right]}{2^r - (-1)^r}.$$

□

Corollary 3.5. ***(d'Ocagne's Identity)*** *Let $n \in \mathbb{N}$, $t \in \mathbb{Z}^+$ such that $t > n + 1$. Then, we have*

$$\mathbb{HC}J_{n+1}^{(r)} \mathbb{HC}J_t^{(r)} - \mathbb{HC}J_n^{(r)} \mathbb{HC}J_{t+1}^{(r)} = \frac{(-2)^{nr} \left[2^{(t-n)r}\widehat{(-1)}\widehat{2} - (-1)^{(t-n)r}\widehat{2}\widehat{(-1)}\right]}{2^r - (-1)^r}.$$

Proof. For $m = 1, k = t - n$ in Theorem 3.1, we get

$$\mathbb{HC}J_{n+1}^{(r)} \mathbb{HC}J_t^{(r)} - \mathbb{HC}J_n^{(r)} \mathbb{HC}J_{t+1}^{(r)} = \frac{(-2)^{nr} J_1^{(r)} \left[2^{(t-n)r}\widehat{(-1)}\widehat{2} - (-1)^{(t-n)r}\widehat{2}\widehat{(-1)}\right]}{2^r - (-1)^r}$$

and by using the equation (1.3), we obtain

$$\mathbb{HC}J_{n+1}^{(r)} \mathbb{HC}J_t^{(r)} - \mathbb{HC}J_n^{(r)} \mathbb{HC}J_{t+1}^{(r)} = \frac{(-2)^{nr} \left[2^{(t-n)r}\widehat{(-1)}\widehat{2} - (-1)^{(t-n)r}\widehat{2}\widehat{(-1)}\right]}{2^r - (-1)^r}.$$

□

4. A Matrix Representation for Higher Order Jacobsthal 2^s-ions

In this section of our paper, we generate the matrix representation of the higher order Jacobsthal 2^s-ions. We define two matrices $\boldsymbol{T^{(r)}}$ and $\boldsymbol{J^{(r)}}$ as

$$\boldsymbol{T^{(r)}} = \begin{bmatrix} j_r & -(-2)^r \\ 1 & 0 \end{bmatrix} \quad and \quad \boldsymbol{J^{(r)}} = \begin{bmatrix} \mathbb{HC}J_2^{(r)} & \mathbb{HC}J_1^{(r)} \\ \mathbb{HC}J_1^{(r)} & \mathbb{HC}J_0^{(r)} \end{bmatrix}, \tag{4.1}$$

where j_r is the Jacobsthal-Lucas number. Considering our consequence, we now present the following theorem.

Theorem 4.1. *For $n \in \mathbb{N}$, we have*

$$\left(\boldsymbol{T^{(r)}}\right)^n \boldsymbol{J^{(r)}} = \begin{bmatrix} \mathbb{HC}J_{n+2}^{(r)} & \mathbb{HC}J_{n+1}^{(r)} \\ \mathbb{HC}J_{n+1}^{(r)} & \mathbb{HC}J_n^{(r)} \end{bmatrix}.$$

Proof. We prove the theorem by the induction method on n. For $n = 0$, the equality holds. Suppose that the hypothesis is true for $n = i$. That is,

$$\left(\boldsymbol{T^{(r)}}\right)^i \boldsymbol{J^{(r)}} = \begin{bmatrix} \mathbb{HC}J_{i+2}^{(r)} & \mathbb{HC}J_{i+1}^{(r)} \\ \mathbb{HC}J_{i+1}^{(r)} & \mathbb{HC}J_i^{(r)} \end{bmatrix}. \tag{4.2}$$

For $n = i + 1$, from the equations (4.1) and (4.2), we get

$$\begin{aligned} \left(\boldsymbol{T^{(r)}}\right)^{i+1} \boldsymbol{J^{(r)}} &= \boldsymbol{T^{(r)}} \left(\boldsymbol{T^{(r)}}\right)^i \boldsymbol{J^{(r)}} \\ &= \begin{bmatrix} j_r & -(-2)^r \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{HC}J_{i+2}^{(r)} & \mathbb{HC}J_{i+1}^{(r)} \\ \mathbb{HC}J_{i+1}^{(r)} & \mathbb{HC}J_i^{(r)} \end{bmatrix} \\ &= \begin{bmatrix} j_r\mathbb{HC}J_{i+2}^{(r)} - (-2)^r\mathbb{HC}J_{i+1}^{(r)} & j_r\mathbb{HC}J_{i+1}^{(r)} - (-2)^r\mathbb{HC}J_i^{(r)} \\ \mathbb{HC}J_{i+2}^{(r)} & \mathbb{HC}J_{i+1}^{(r)} \end{bmatrix} \end{aligned}$$

and by using Theorem 2.5, we find that

$$\left(\boldsymbol{T^{(r)}}\right)^{i+1} \boldsymbol{J^{(r)}} = \begin{bmatrix} \mathbb{HC}J_{i+3}^{(r)} & \mathbb{HC}J_{i+2}^{(r)} \\ \mathbb{HC}J_{i+2}^{(r)} & \mathbb{HC}J_{i+1}^{(r)} \end{bmatrix}.$$

Hence, the proof is completed. □

In the following corollary, we also derive Cassini's identity for higher order Jacobsthal 2^s-ions by using the matrices mentioned above.

Corollary 4.2. *For $n \in \mathbb{Z}^+$, then we have*

$$\mathbb{HC}J_{n+1}^{(r)}\mathbb{HC}J_{n-1}^{(r)} - \left(\mathbb{HC}J_n^{(r)}\right)^2 = (-2)^{(n-1)r}\left[\mathbb{HC}J_2^{(r)}\mathbb{HC}J_0^{(r)} - \left(\mathbb{HC}J_1^{(r)}\right)^2\right].$$

Proof. By using (4.1) and Theorem 4.1, we get

$$\begin{bmatrix} j_r & -(-2)^r \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} \mathbb{HC}J_2^{(r)} & \mathbb{HC}J_1^{(r)} \\ \mathbb{HC}J_1^{(r)} & \mathbb{HC}J_0^{(r)} \end{bmatrix} = \begin{bmatrix} \mathbb{HC}J_{n+1}^{(r)} & \mathbb{HC}J_n^{(r)} \\ \mathbb{HC}J_n^{(r)} & \mathbb{HC}J_{n-1}^{(r)} \end{bmatrix}. \tag{4.3}$$

If we take the determinant on both sides of (4.3), then we obtain

$$\mathbb{HC}J_{n+1}^{(r)}\mathbb{HC}J_{n-1}^{(r)} - \left(\mathbb{HC}J_n^{(r)}\right)^2 = (-2)^{(n-1)r}\left[\mathbb{HC}J_2^{(r)}\mathbb{HC}J_0^{(r)} - \left(\mathbb{HC}J_1^{(r)}\right)^2\right].$$

□

5. Conclusions

In this study, we introduce and investigate higher order Jacobsthal 2^s-ions, which are defined by higher order Jacobsthal numbers. Then, we generate numerous structural features of these higher order Jacobsthal 2^s-ions, including recurrence relation, the Binet-like formula, the generating function, and the exponential generating function. Then, we obtain Vajda's identity, Catalan's identity, Cassini's identity, and d'Ocagne's identity by using the Binet-like formula. Finally, we demonstrate as a type Cassini's identity by using matrix representation of the higher order Jacobsthal 2^s-ions.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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