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Research Article

Bilinear Calderón-Zygmund operator and its commutator on some variable exponent spaces of homogeneous type

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Abstract

Let (X, d, μ) be a space of homogeneous type in the sense of Coifman and and Weiss. In this setting, the author proves that a bilinear Calderón-Zygmund operator is bounded from the product of variable exponent Lebesgue spaces $L^{p_1(\cdot)}(X) \times L^{p_2(\cdot)}(X)$ into spaces $L^{p(\cdot)}(X)$, and it is bounded from the product of variable exponent generalized Morrey spaces $\mathcal{L}^{p_1(\cdot),\varphi_1}(X)\times\mathcal{L}^{p_2(\cdot),\varphi_2}(X)$ into spaces $\mathcal{L}^{p(\cdot),\varphi}(X)$, where the Lebesgue measure functions $\varphi(\cdot,\cdot), \varphi_1(\cdot,\cdot)$ and $\varphi_2(\cdot,\cdot)$ satisfy $\varphi_1 \times \varphi_2 = \varphi$, and $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Furthermore, by establishing sharp maximal estimate for the commutator $[b_1, b_2, BT]$ generated by $b_1, b_2 \in \mathrm{BMO}(X)$ and BT, the author shows that the $[b_1, b_2, BT]$ is bounded from the product of spaces $L^{p_1(\cdot)}(X) \times L^{p_2(\cdot)}(X)$ into spaces $L^{p(\cdot)}(X)$, and it is also bounded from product of spaces $\mathcal{L}^{p_1(\cdot),\varphi_1}(X) \times \mathcal{L}^{p_2(\cdot),\varphi_2}(X)$ into spaces $L^{p(\cdot),\varphi}(X)$.

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1. Introduction

It is well known that the classical singular integral theory has been going on for a long period and plays an important role in the fields of harmonic analysis and PDEs. Many scholars have studied the boundedness of following Calderón-Zygmund integral operators

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y)f(y)\mathrm{d}y$$

on various of function spaces (see [2, 6, 18, 26, 32]). In 2002, Grafakos and Torres [9] obtained the definition of multilinear Calderón-Zygmund operators, which generalized the theory of linear operators. Since then, many papers focus on the properties of multilinear Calderón-Zygmund operators on different kinds of spaces; for example, Mirek and Thiele in [27] established a local $T(b)$ theorem for perfect multilinear Calderón-Zygmund operators. In 2005, Hu, Meng and Yang [13] obtained the $L^p(\mu)$-boundedness and some weak type endpoint estimates for the multilinear commutators of Calderón-Zygmund singular integrals with BMO-type functions and Orlicz-type functions. In 2019, Tan [33] showed that bilinear Calderón-Zygmund operators are bounded from product of variable exponent Hardy

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spaces $H^{p_1(\cdot)}(\mathbb{R}^n)\times H^{p_2(\cdot)}(\mathbb{R}^n)$ into variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ or Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$. More researches on the multilinear Calderón-Zygmund operators can be seen [14, 17, 23–25, 31, 34–36] and the corresponding references therein.

On the other hand, to extend the traditional Euclidean space to a general underlying structure for the real harmonic analysis, the following spaces of homogeneous type were introduced by Coifman and Weiss in [1].

Let $X=(X,d,\mu)$ be a space of homogeneous type, i.e., X is a topological space endowed with a quasi-distance $d(\cdot,\cdot)$ defined on $X\times X$ and satisfying the following conditions:

(i) $d(x,y)\geq 0$, $d(x,y)=0$ if and only if $x=y$;
(ii) for all $x,y\in X$, $d(x,y)=d(y,x)$;
(iii) there exists a constant $\kappa\geq 1$ such that, for all $x,y,z\in X$,

$$d(x,y)\leq \kappa(d(x,z)+d(y,z)), \tag{1.1}$$

and a positive measure μ is defined on a σ-algebra of subsets of X which contains the balls $B(x,r):=\{y\in X: d(x,y)<r\}$ for $x\in X$ and $r>0$, and satisfies

$$0<\mu(B(x,2r))\leq C_{\rm dbl}\mu(B(x,r))<\infty, \tag{1.2}$$

where constant $C_{\rm dbl}\geq 1$ is known to play a key role in the quasi-metric measure space. Sometimes we also refer to the following counterpart of this condition:

$$\mu(B(x,2r))\geq \mathbb{K}\times\mu(B(x,r)),\quad \mathbb{K}>1. \tag{1.3}$$

Since then, many papers focus on the mapping properties of integral operators over spaces of homogeneous type. For example, In 2009, Hu *et al.* [16] obtained some sufficient conditions which guarantee the boundedness of operators on spaces of homogeneous type. In 2016, Guliyev and Samko [10] obtained the definition of generalized variable exponent Morrey spaces on (X,d,μ), and established the boundedness of maximal operator on these spaces. Deringoz *et al.* in [4] showed that fractional maximal operator and its commutator associated with BMO functions are bounded on Orlicz spaces and on generalized Orlicz-Morrey spaces. Further research about the spaces of homogeneous type can be seen in [3, 5, 7, 15, 20, 22, 28–30].

However, in this paper, we will manly consider the boundedness of bilinear Calderón-Zygmund operators and their commutators associated with spaces BMO(X) on variable exponent Lebesgue space $L^{p(\cdot)}(X)$ and variable exponent generalized Morrey space $\mathcal{L}^{p(\cdot),\varphi}(X)$ over spaces of homogeneous type.

Before stating the organizations of this paper, we need to recall some necessary notion and definitions. The following notion of the variable exponent Lebesgue space on (X,d,μ) is from [11].

Let $p(\cdot): X\to[1,\infty)$ be a measurable function, and set

$$p_-:=p_-(X)=\operatorname*{ess\,inf}_{x\in X}p(x)\geq 1,\qquad p_+:=p_+(X)=\operatorname*{ess\,sup}_{x\in X}p(x)<\infty.$$

We denote $\mathcal{P}(X)$ the set of all measurable functions $p(\cdot):X\to[1,\infty)$ such that $1\leq p_-\leq p(\cdot)\leq p_+<\infty$.

For $p(\cdot)\in\mathcal{P}(X)$, the variable exponent Lebesgue space $L^{p(\cdot)}(X)$ introduced in [11] denotes the real-valued measurable functions f on $\mathbb{R}^n$ such that, for some $\lambda>0$, the following equation

$$\int_X\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}\mathrm{d}\mu(x)<\infty$$

holds. This becomes a Banach function space with respect to the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(X)}:=\inf\left\{\lambda>0:\int_X\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}\mathrm{d}\mu(x)\leq 1\right\}. \tag{1.4}$$

Now let us recall some classes of variable exponent functions. $\mathcal{P}^{\log}(X)$ is the set of all measurable functions $p(\cdot) \in \mathcal{P}(X)$ satisfying the local Log-Hölder (i.e, LH) condition

$$|p(x) - p(y)| \leq -\frac{A_p}{\ln(d(x,y))}, \quad d(x,y) \leq \frac{1}{2},$$

where $A_p > 0$ does not depend on x and y. Moreover, $\mathbb{P}^{\log}(X)$ represents the set of exponents $p \in \mathcal{P}^{\log}(X)$ with $1 < p_- \leq p_+ < \infty$; for X which may be unbounded, by $\mathcal{P}_\infty(X)$, $\mathcal{P}^{\log}_\infty(X)$, $\mathbb{P}^{\log}_\infty(X)$, we denote the subsets of the above sets of exponents satisfying the decay condition

$$|p(x) - p(\infty)| \leq \frac{A_\infty}{\ln(e + d(x, x_0))}, \quad x \in X,$$

where x_0 is a fixed point in X, A_∞ is a positive constant and $p(\infty) = \lim\limits_{x\to\infty} p(x) > 1$.

For any $x \in X$ and $f \in L^1_{\rm loc}(X)$, the Hardy-Littlewood maximal function $M(f)$ of f (see [20]) is defined by

$$Mf(x) = \sup_{x\in X, r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \mathrm{d}\mu(y), \tag{1.5}$$

where $B(x,r)$ is the open ball centered at $x \in X$ with its radius $r > 0$. The maximal operator M is bounded on spaces $L^{p(\cdot)}(X)$ under the condition $p(\cdot) \in \mathbb{P}^{\log}_\infty(X) \bigcap \mathrm{LH}$. In addition, $\mathcal{B}$ represents the set of $p(\cdot) \in \mathcal{P}(X)$ such that the M is bounded on spaces $L^{p(\cdot)}(X)$.

The following definition of bound mean oscillation spaces ($= \mathrm{BMO}(X)$) is from [1], also see [16].

Definition 1.1. Let $1 \leq q < \infty$. A real-valued function $f \in L^1_{\rm loc}(X)$ is said to be the space $\mathrm{BMO}_q(X)$ if there exists a positive constant C such that, for all balls $B \subset X$,

$$\left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^q \mathrm{d}\mu(x)\right)^{\frac{1}{q}} \leq C, \tag{1.6}$$

where f_B represents the mean value of function f over ball B, namely,

$$f_B := \frac{1}{\mu(B)} \int_B f(y) \mathrm{d}\mu(y).$$

The infimum of the positive constant C in (1.6) is defined to the $\mathrm{BMO}_q(X)$ norm of f and denote by $\|f\|_{\mathrm{BMO}_q(X)}$.

Remark 1.2. From [1], Coifman and Weiss have showed that spaces $\mathrm{BMO}_q(X)$ is independent of the choice of $q \in (1, \infty)$. Hence, space $\mathrm{BMO}_q(X)$ is simply denoted by the space $\mathrm{BMO}(X)$.

Now we state the notion of the bilinear Calderón-Zygmund operator, which was introduced by Grafakos in [8], as follows.

Definition 1.3. A kernel $K(\cdot,\cdot,\cdot) \in L^1_{\rm loc}(X^3 \setminus \{(x,x,x) : x \in X\})$ is called a *bilinear Calderón-Zygmund kernel* if it satisfying the following conditions:

(i) for all $x, y_1, y_2 \in X$ with $x \neq y_i$ for $i \in \{1,2\}$, there exists a positive constant C such that,

$$|K(x, y_1, y_2)| \leq C\left[\sum_{i=1}^{2} \mu(B(x, d(x, y_i)))\right]^{-2}; \tag{1.7}$$

(ii) there exists a positive constant C such that, for all $x, x', y_1, y_2 \in X$ with $d(x,x') \leq \frac{1}{2}\max\{d(x,y_1), d(x,y_2)\}$,

$$|K(x, y_1, y_2) - K(x', y_1, y_2)| \leq C\frac{[d(x,x')]^\delta}{[d(x,y_1) + d(x,y_2)]^\delta}\left[\sum_{i=1}^{2} \mu(B(x, d(x, y_i)))\right]^{-2}; \tag{1.8}$$

(iii) there exists a positive constant C such that, for all $x, y_1, y_1', y_2 \in X$ with $d(y_1, y_1') \leq \frac{1}{2}\max\{d(x,y_1), d(x,y_2)\}$,

$$|K(x,y_1,y_2) - K(x,y_1',y_2)| \leq C\frac{[d(y_1,y_1')]^\delta}{[d(x,y_1)+d(x,y_2)]^\delta}\left[\sum_{i=1}^{2}\mu(B(x,d(x,y_i)))\right]^{-2}; \quad (1.9)$$

(iv) there exists a positive constant C such that, for all $x, y_1, y_2, y_2' \in X$ with $d(y_2, y_2') \leq \frac{1}{2}\max\{d(x,y_1), d(x,y_2)\}$,

$$|K(x,y_1,y_2) - K(x,y_1,y_2')| \leq C\frac{[d(y_2,y_2')]^\delta}{[d(x,y_1)+d(x,y_2)]^\delta}\left[\sum_{i=1}^{2}\mu(B(x,d(x,y_i)))\right]^{-2}. \quad (1.10)$$

Let $L_c^\infty(X)$ be the space of all $L^\infty(X)$ functions with compact support. A bilinear operator BT is called a bilinear Calderón-Zygmund operator with kernel K satisfying (1.7), (1.8), (1.9) and (1.10) if, for all $f_1, f_2 \in L_c^\infty(X)$ and $x \in X \setminus (\mathrm{supp}(f_1)\bigcap \mathrm{supp}(f_2))$,

$$BT(f_1,f_2)(x) = \int_X\int_X K(x,y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}\mu(y_1)\mathrm{d}\mu(y_2). \quad (1.11)$$

Given $b_1, b_2 \in \mathrm{BMO}(X)$, the commutator $[b_1, b_2, BT]$ generated by b_1, b_2 and BT is defined by

$$\begin{aligned}[b_1,b_2,BT](f_1,f_2)(x) &= b_1(x)b_2(x)BT(f_1,f_2)(x) - b_1(x)BT(f_1,b_2f_2)(x)\\ &\quad -b_2(x)BT(b_1f_1,f_2)(x) + BT(b_1f_1,b_2f_2)(x). \quad (1.12)\end{aligned}$$

Also, commutators $[b_1, BT]$ and $[b_2, BT]$ are defined by, respectively,

$$[b_1,BT](f_1,f_2)(x) = b_1(x)BT(f_1,f_2)(x) - BT(b_1f_1,f_2)(x), \quad (1.13)$$

$$[b_2,BT](f_1,f_2)(x) = b_2(x)BT(f_1,f_2)(x) - BT(f_1,b_2f_2)(x). \quad (1.14)$$

Next, we need to recall the following inequality introduced in [10], that is, for any $x \in X$ and $p \in \mathcal{P}^{\log}(X)$, there exists a positive constant C such that

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)} \leq C[\mu(B(x,r))]^{\frac{1}{p_r(x)}} \quad (1.15)$$

holds, where

$$p_r(x) = \begin{cases} p(x), & 0 < r \leq 1,\\ p(\infty), & r > 1,\end{cases}$$

and $p(\infty) = \lim_{x\to\infty} p(x) > 1$.

The following definition of variable exponent generalized Morrey space is from [10].

Definition 1.4. Let $p \in \mathcal{P}(X)$ and $\varphi(\cdot,\cdot)$ be a positive measurable function defined on $X \times (0,\infty)$. Then the generalized variable exponent Morrey space $\mathcal{L}^{p(\cdot),\varphi}(X)$ is defined by

$$\|f\|_{\mathcal{L}^{p(\cdot),\varphi}(X)} = \left\{f \in L^{p(\cdot)}_{\mathrm{loc}}(X) : \|f\|_{\mathcal{L}^{p(\cdot),\varphi}(X)} < \infty\right\},$$

where

$$\|f\|_{\mathcal{L}^{p(\cdot),\varphi}(X)} = \sup_{x\in X, r>0}\frac{1}{\varphi(x,r)}\|f\|_{L^{p(\cdot)}(B(x,r))} = \sup_{x\in X, r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}f\|_{L^{p(\cdot)}(X)}. \quad (1.16)$$

Remark 1.5. Other properties on the spaces $\mathcal{L}^{p(\cdot),\varphi}(X)$ can be seen Section 4.2 in [10].

It is position to state the organization as follows. In Section 2, we should recall and establish some necessary lemmas to prove the main results. In Section 3, we will show that the bilinear Calderón-Zygmund operator BT is bounded from product of variable exponent Lebesgue spaces $L^{p_1(\cdot)}(X) \times L^{p_2(\cdot)}(X)$ into spaces $L^{p(\cdot)}(X)$, and bounded from products of variable exponent generalized Morrey spaces $\mathcal{L}^{p_1(\cdot),\varphi_1}(X) \times \mathcal{L}^{p_2(\cdot),\varphi_2}(X)$ into spaces $\mathcal{L}^{p(\cdot),\varphi}(X)$, where $\varphi_1 \times \varphi_2 = \varphi$ and $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$ with $1 < p_1(\cdot), p_2(\cdot) < \infty$. By establishing sharp maximal estimate for the commutator $[b_1, b_2, BT]$ formed by $b_1, b_2 \in \mathrm{BMO}(X)$ and the BT, the author shows that $[b_1, b_2, BT]$ is bounded from product of spaces $L^{p_1(\cdot)}(X)$ $\times L^{p_2(\cdot)}(X)$ into spaces $L^{p(\cdot)}(X)$, and also bounded from product of spaces $\mathcal{L}^{p_1(\cdot),\varphi_1}(X) \times \mathcal{L}^{p_2(\cdot),\varphi_2}(X)$ into spaces $L^{p(\cdot),\varphi}(X)$ in Section 4.

Finally, we make some conventions on notation. Throughout the paper, C represents a positive constant being independent of the main parameters involved, but it may be different from line to line. For a μ-measurable set E, χ_E denotes its characteristic function. For any variable exponent $p(\cdot)$, we denote by $p'(\cdot)$ its conjugate index, i.e., $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

2. Preliminaries

To prove the main theorems of this paper, in this section, we need to establish and recall some necessary lemmas as follows.

Lemma 2.1. *If $p(\cdot) \in \mathcal{B}$, there exists a positive constant C such that, for all balls $B \subset X$,*

$$\frac{1}{\mu(B(x,r))}\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)}\|\chi_{B(x,r)}\|_{L^{p'(\cdot)}(X)} \le C,$$

where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Proof. By applying (1.15), we have

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)}\|\chi_{B(x,r)}\|_{L^{p'(\cdot)}(X)} \le C[\mu(B(x,r))]^{\frac{1}{p_r(x)}}[\mu(B(x,r))]^{\frac{1}{p'_r(x)}} \le C\mu(B(x,r)).$$

Hence, we complete the proof of Lemma 2.1. □

The following Hölder inequality on the variable exponent on (X, d, μ) is from [10].

Lemma 2.2. *If $p(\cdot) \in \mathcal{P}(X)$, then, for all $f \in L^{p(\cdot)}(X)$ and $g \in L^{p'(\cdot)}(X)$, the following equation*

$$\int_X |f(x)g(x)|\mathrm{d}x \le c_p\|f\|_{L^{p(\cdot)}(X)}\|g\|_{L^{p'(\cdot)}(X)}$$

holds, where $c_p := 1 + \frac{1}{p_-} - \frac{1}{p_+}$.

Lemma 2.3. *Let $1 < p(\cdot), p_1(\cdot), p_2(\cdot) < \infty$ satisfy $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Then there exists some positive constant C, independent of f and g, such that*

$$\|fg\|_{L^{p(\cdot)}(X)} \le C\|f\|_{L^{p_1(\cdot)}(X)}\|g\|_{L^{p_2(\cdot)}(X)}$$

holds for every $f \in L^{p_1(\cdot)}(X)$ and $g \in L^{p_2(\cdot)}(X)$.

Remark 2.4. With an argument similar to that used in the proof of Theorem 2.3 in [17], it is easy to see that the above inequality holds on (X, d, μ).

Also, we need to establish the following characterizations about the $\mathrm{BMO}(X)$ space.

Lemma 2.5. *Let $f \in \mathrm{BMO}(X)$. Then there exists some positive constant C such that, for all $p(\cdot) \in \mathcal{B}$ and $i, j \in \mathbb{Z}_+$ with $j > i$,*

$$C^{-1}\|f\|_{\mathrm{BMO}(X)} \le \sup_{B:\ \mathrm{ball}} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(X)}}\|(f - f_B)\chi_B\|_{L^{p(\cdot)}(X)} \le C\|f\|_{\mathrm{BMO}(X)}, \tag{2.1}$$

$$\|(f - f_{B_i})\chi_{B_j}\|_{L^{p(\cdot)}(X)} \le C(j - i)\|f\|_{\mathrm{BMO}(X)}\|\chi_{B_j}\|_{L^{p(\cdot)}(X)}, \tag{2.2}$$

where B_i represents a ball with the same center to B and radius 2^i times radius of B.

Remark 2.6. By (1.6), the Hölder inequality, Lemmas 2.1 and 2.2, and an argument similar to that used in the proof of Lemma 3 in [19], it is easy to show that Lemma 2.5 holds.

Finally, we recall the following lemma in [12].

Lemma 2.7. *Let $b \in \mathrm{BMO}(X)$. Then there exists some positive constant C being independent of b, x, r and t,*

$$|b_{B(x,r)} - b_{B(x,t)}| \le C \ln\left(\frac{t}{r}\right) \|b\|_{\mathrm{BMO}(X)}, \qquad \text{for } \ 0 < 2r < t. \tag{2.3}$$

3. Estimate for BT on $L^{p(\cdot)}(X)$ and $\mathcal{L}^{p(\cdot),\varphi}(X)$

The main theorems of this section are stated as follows.

Theorem 3.1. *Let $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{B}$ satisfy $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Suppose that BT is defined as in (1.11), and the measure μ satisfies (1.2) and (1.3). Then there exists some positive constant C such that, for all $f_i \in L^{p_i(\cdot)}(X)$ with $i = 1, 2$,*

$$\|BT(f_1, f_2)\|_{L^{p(\cdot)}(X)} \le C \|f_1\|_{L^{p_1(\cdot)}(X)} \|f_2\|_{L^{p_2(\cdot)}(X)},$$

where the constant C does not depend on x and r.

Theorem 3.2. *Let $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{B}$ and $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Suppose that the bilinear Calderón-Zygmund operator BT is defined as in (1.11), $\varphi_i(\cdot,\cdot)(i = 1, 2)$ defined on $X \times (0, \infty)$ are positive measurable functions, and the measure μ satisfies (1.2) and (1.3). If there exists some positive constant C such that, for all $x \in X$ and $r > 0$,*

$$\sum_{k=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)}}{\|\chi_{B(x,2^{k+1}r)}\|_{L^{p(\cdot)}(X)}} \varphi_i(x, 2^{k+1}r) \le C\varphi_i(x, r), \quad i = 1, 2, \tag{3.1}$$

and $\varphi_1\varphi_2 = \varphi$, then

$$\|BT(f_1, f_2)\|_{\mathcal{L}^{p(\cdot),\varphi}(X)} \le C \|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)} \|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)},$$

where $f_i \in \mathcal{L}^{p_i(\cdot),\varphi_i}(X)$ with $i = 1, 2$.

To prove Theorem 3.1, we should recall the following lemmas (see [15,21], respectively).

Lemma 3.3. *Let $p \in (0, \infty)$ and $\omega \in A_\infty(X)$. Then there exists a positive constant C such that*

$$\int_X (M_\delta(f)(x))^p \omega(x) \mathrm{d}\mu(x) \le C \int_X (M_\delta^\sharp(f)(x))^p \omega(x) \mathrm{d}\mu(x),$$

for all functions f when the the left-hand side is finite.

Lemma 3.4. *Let (X, d, μ) be a space of homogeneous type. Suppose that p_0 is a constant such that $1 < p_0 < \infty$. Let $\mathcal{F}$ be a family of pairs of non-negative functions (f, g). Assume that for all $(f, g) \in \mathcal{F}$, and all weights $\omega_0 \in A_{p_0}(X)$, the inequality*

$$\int_X (f(x))^{p_0} \omega_0(x) \mathrm{d}\mu(x) \le C \int_X (g(x))^{p_0} \omega_0(x) \mathrm{d}\mu(x)$$

holds. Then for all $(f, g) \in \mathcal{F}$ and $f \in L^{p(\cdot)}(X)$ with $p(\cdot) \in \mathbb{P}_\infty^{\log}(X)$,

$$\|f\|_{L^{p(\cdot)}(X)} \le C \|g\|_{L^{p(\cdot)}(X)}.$$

Also, we need to establish the following lemma about the operator BT.

Lemma 3.5. *Let $0 < \delta < \frac{1}{2}$, and BT be a bilinear Calderón-Zygmund operator defined as in (1.11). Suppose that the measure μ satisfies (1.2) and (1.3). Then there exists a constant $C > 0$ such that, for all $f_i \in L_c^\infty(X)$ with $i = 1, 2$,*

$$M_\delta^\sharp(BT(f_1, f_2))(x) \le CM(f_1)(x)M(f_2)(x), \tag{3.2}$$

where the sharp maximal operator $M^\sharp$ is defined by, for all $f \in L^1_{\mathrm{loc}}(X)$,

$$M^\sharp(f)(x) = \sup_{x \in X, r > 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \mathrm{d}\mu(y), \tag{3.3}$$

and denote by $M_\delta^\sharp(f) := M_\delta^\sharp(|f|^\delta)^{\frac{1}{\delta}}$.

Proof. To prove (3.2), it only suffices to show that

$$\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}\Big||BT(f_1,f_2)(y)|^\delta-|c_{B(x,r)}|^\delta\Big|\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\le CM(f_1)(x)M(f_2)(x), \quad (3.4)$$

for some constant $c_{B(x,r)}$. In fact, by using the following inequality

$$\Big||\alpha|^r-|\beta|^r\Big|\le|\alpha-\beta|^r,\quad 0<r<1,$$

To estimate (3.4), we only need to show

$$\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT(f_1,f_2)(y)-c_{B(x,r)}|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\le CM(f_1)(x)M(f_2)(x), \quad (3.5)$$

Let $B:=B(x,r)$ be a fixed ball centered at x with radius $r>0$, and decompose f_i as

$$f_i=f_i^1+f_i^\infty=f_i\chi_{2B}+f_i\chi_{X\setminus(2B)},\quad i=1,2,$$

and take $c_{B(x,r)}=m_B(BT(f_1^\infty,f_2^\infty))$ in (3.5).

Then, for any $y\in X$, write

$$\begin{aligned}
&\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT(f_1,f_2)(y)-c_{B(x,r)}|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\le\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT(f_1^1,f_2^1)(y)|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\quad+\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT(f_1^1,f_2^\infty)(y)|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\quad+\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT(f_1^\infty,f_2^1)(y)|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\quad+\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT(f_1^\infty,f_2^\infty)(y)-m_B(BT(f_1^\infty,f_2^\infty))|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&=\mathrm{D}_1+\mathrm{D}_2+\mathrm{D}_3+\mathrm{D}_4.
\end{aligned}$$

To estimate D_1, we will use the following Kolmogorov estimate in [31]: let (X,μ) be a probability measure space and let $0<p<q<\infty$, then there is a constant $C=C_{p,q}$ such that for any measurable function f, the following inequality

$$\|f\|_{L^p(X)}\le C\|f\|_{L^{q,\infty}(X)}$$

holds. From the above Kolmogorov estimate with $p=\delta$ and $q=\frac{1}{2}$, and $(L^1(X)\times L^1(X), L^{\frac{1}{2},\infty}(X))$-boundedness of BT in [22], it then follows that

$$\begin{aligned}
\mathrm{D}_1&=\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT(f_1^1,f_2^1)(y)|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\le C\|BT(f_1^1,f_2^1)\|_{L^{\frac{1}{2},\infty}(B(x,r),\frac{\mathrm{d}\mu(y)}{\mu(B(x,r))})}\\
&\le C\prod_{i=1}^2\frac{1}{\mu(B(x,2r))}\int_{2B}|f_i(z_i)|\mathrm{d}\mu(z_i)\le C\prod_{i=1}^2M(f_i)(x).
\end{aligned}$$

Now let us estimate D_2. For any $y\in B(x,r)$, by (1.3), (1.5) and (1.7), we have

$$\begin{aligned}
&|BT(f_1^1,f_2^\infty)(y)|\\
&\le \int_X\int_X |K(y,z_1,z_2)||f_1^1(z_1)||f_2^\infty(z_2)|\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
&\le C\int_{2B}\int_{X\setminus(2B)} \frac{|f_1(y_1)||f_2(y_2)|}{[\mu(B(y,d(y,z_1)))+\mu(B(y,d(y,z_2)))]^2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
&\le C\int_{2B}|f_1(z_1)|\mathrm{d}\mu(z_1)\int_{X\setminus(2B)}\frac{|f_2(z_2)|}{[\mu(B(y,d(y,z_2)))]^2}\mathrm{d}\mu(z_2)\\
&\le C\int_{2B}|f_1(z_1)|\mathrm{d}\mu(z_1)\left(\sum_{j=1}^\infty\int_{2^{j+1}B\setminus(2^jB)}\frac{|f_2(z_2)|}{[\mu(B(y,d(y,z_2)))]^2}\mathrm{d}\mu(z_2)\right)\\
&\le C\int_{2B}|f_1(z_1)|\mathrm{d}\mu(z_1)\left(\sum_{j=1}^\infty\frac{1}{[\mu(B(x,2^jr))]^2}\int_{2^{j+1}B}|f_2(z_2)|\mathrm{d}\mu(z_2)\right)\\
&\le C\frac{\mu(B(x,2r))}{\mu(B(x,2r))}\int_{2B}|f_1(z_1)|\mathrm{d}\mu(z_1)\\
&\quad\times\left(\sum_{j=1}^\infty\frac{\mu(2^{j+1}B)}{[\mu(B(x,2^jr))]^2}\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B}|f_2(z_2)|\mathrm{d}\mu(z_2)\right)\\
&\le CM(f_1)(x)M(f_2)(x)\left(\sum_{j=1}^\infty\frac{\mu(2B(x,r))}{\mathbb{K}^{j-1}\mu(2B(x,r))}\right)\\
&\le CM(f_1)(x)M(f_2)(x),
\end{aligned}$$

further, we have

$$\begin{aligned}
\mathrm{D}_2&=\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT(f_1^1,f_2^\infty)(y)|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\le CM(f_1)(x)M(f_2)(x).
\end{aligned}$$

With an argument similar to that used in the estimate for D_2, it is easy to see that

$$\mathrm{D}_3\le CM(f_1)(x)M(f_2)(x).$$

To estimate D_4, we first consider $|BT(f_1^\infty,f_2^\infty)(y)-BT(f_1^\infty,f_2^\infty)(x)|$ with $y\in B(x,r)$. By applying (1.2), (1.5) and (1.8), we have

$$\begin{aligned}
&|BT(f_1^\infty,f_2^\infty)(y)-BT(f_1^\infty,f_2^\infty)(x)|\\
&\le\int_X\int_X|K(x,z_1,z_2)-K(y,z_1,z_2)||f_1^\infty(z_1)||f_2^\infty(z_2)|\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)
\end{aligned}$$

$$\begin{aligned}
&\le C\int_{X\setminus(2B)}\int_{X\setminus(2B)}\frac{[d(x,y)]^\delta}{[d(x,z_1)+d(x,z_2)]^\delta}\frac{|f_1(z_1)||f_2(z_2)|}{\left[\sum_{i=1}^2\mu(B(x,d(x,z_i)))\right]^2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
&\le C\prod_{i=1}^2\left[\sum_{j=1}^\infty\frac{r^\delta}{(2^jr)^\delta}\frac{1}{\mu(B(x,2^jr))}\int_{2^{j+1}B}|f_i(z_i)|\mathrm{d}\mu(z_i)\right]\\
&\le CM(f_1)(x)M(f_2)(x),
\end{aligned}$$

further, we have

$$\mathrm{D}_4\le CM(f_1)(x)M(f_2)(x).$$

Which, together with D_{41}, D_2 and D_3, the proof of Lemma 3.5 is finished. □

Now we state the proofs of Theorems 3.1 and 3.2 as follows.

Proof of Theorem 3.1. Since f_1 and f_2 are bounded functions with compact support, and $BT(f_1,f_2)(x) \le M_\delta^\sharp(BT(f_1,f_2))(x)$ for $x \in X$. Then, from Lemmas 2.3, 3.3, 3.4 and 3.5, and the boundedness of M on spaces $L^{p(\cdot)}(X)$, it follows that

$$\begin{aligned}\|BT(f_1,f_2)\|_{L^{p(\cdot)}(X)} &\le C\|M(f_1)M(f_2)\|_{L^{p(\cdot)}(X)} \le C\|M(f_1)\|_{L^{p_1(\cdot)}(X)}\|M(f_2)\|_{L^{p_2(\cdot)}(X)}\\ &\le C\|f_1\|_{L^{p_1(\cdot)}(X)}\|f_2\|_{L^{p_2(\cdot)}(X)}.\end{aligned}$$

Hence, we complete the proof of Theorem 3.1. $\square$

Proof of Theorem 3.2. Let $B := B(x,r)$ be the fixed ball centered at $x \in X$ with radius $r > 0$, and decompose f_i as

$$f_i = f_i^1 + f_i^\infty, \quad i = 1,2,$$

where $f_i^1 = f_i\chi_{2B}$ and $f_i^\infty = f_i\chi_{X\setminus(2B)}$.

Then, by (1.16) and the Minkowski inequality, write

$$\begin{aligned}&\sup_{x\in X, r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}BT(f_1,f_2)\|_{L^{p(\cdot)}(X)}\\ &\le \sup_{x\in X, r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}BT(f_1^1,f_2^1)\|_{L^{p(\cdot)}(X)}\\ &\quad + \sup_{x\in X, r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}BT(f_1^1,f_2^\infty)\|_{L^{p(\cdot)}(X)}\\ &\quad + \sup_{x\in X, r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}BT(f_1^\infty,f_2^1)\|_{L^{p(\cdot)}(X)}\\ &\quad + \sup_{x\in X, r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}BT(f_1^\infty,f_2^\infty)\|_{L^{p(\cdot)}(X)}\\ &= \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3 + \mathrm{E}_4.\end{aligned}$$

From (1.16), Theorem 3.1 and $\varphi_1(x,r)\varphi_2(x,r) = \varphi(x,r)$, it then follows that

$$\begin{aligned}&\sup_{x\in X, r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}BT(f_1^1,f_2^1)\|_{L^{p(\cdot)}(X)}\\ &\le C\sup_{x\in X, r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x,r)}f_2\|_{L^{p_2(\cdot)}(X)}\\ &\le C\sup_{x\in X, r>0}\frac{1}{\varphi(x,r)}\varphi_1(x,r)\varphi_2(x,r)\\ &\qquad\times\frac{1}{\varphi_1(x,r)}\|\chi_{B(x,r)}f_1\|_{L^{p_1(\cdot)}(X)}\frac{1}{\varphi_2(x,r)}\|\chi_{B(x,r)}f_2\|_{L^{p_2(\cdot)}(X)}\\ &\le C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_1}(X)}.\end{aligned}$$

To estimate E_2, we first consider $|BT(f_1^1, f_2^\infty)(y)|$ with $y \in B(x,r)$. By applying (1.3), (1.5), (1.7) and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&|BT(f_1^1, f_2^\infty)(y)| \\
&\leq C \int_X \int_X \frac{|f_1^1(z_1)||f_2^\infty(z_2)|}{[\mu(B(y,d(y,z_1))) + \mu(B(y,d(y,z_2)))]^2} \mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2) \\
&\leq C \int_{2B} \int_{X\setminus(2B)} \frac{|f_1(z_1)||f_2(z_2)|}{[\mu(B(y,d(y,z_2)))]^2} \mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2) \\
&\leq C \int_{2B} |f_1(z_1)|\mathrm{d}\mu(z_1) \left\{ \int_{X\setminus(2B)} \frac{|f_2(z_2)|}{[\mu(B(y,d(y,z_2)))]^2} \mathrm{d}\mu(z_2) \right\} \\
&\leq C \|\chi_{2B}\|_{L^{p_1'(\cdot)}(X)} \|f_1\|_{L^{p_1(\cdot)}(X)} \left\{ \sum_{j=1}^\infty \int_{2^{j+1}B\setminus(2^jB)} \frac{|f_2(z_2)|}{[\mu(B(y,d(y,z_2)))]^2} \mathrm{d}\mu(z_2) \right\} \\
&\leq C \|\chi_{2B}\|_{L^{p_1'(\cdot)}(X)} \|f_1\|_{L^{p_1(\cdot)}(X)} \left\{ \sum_{j=1}^\infty \frac{1}{[\mu(B(x,2^jr))]^2} \int_{2^{j+1}B} |f_2(z_2)|\mathrm{d}\mu(z_2) \right\} \\
&\leq C \|\chi_{2B}\|_{L^{p_1'(\cdot)}(X)} \|\chi_{2B} f_1\|_{L^{p_1(\cdot)}(X)} \left\{ \sum_{j=1}^\infty \frac{\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(X)}}{[\mu(B(x,2^jr))]^2} \|\chi_{2^{j+1}B} f_2\|_{L^{p_2(\cdot)}(X)} \right\} \\
&\leq C \|\chi_{2B}\|_{L^{p_1'(\cdot)}(X)} \frac{\varphi_1(x,2r)}{\varphi_1(x,2r)} \|\chi_{2B} f_1\|_{L^{p_1(\cdot)}(X)} \\
&\quad \times \left\{ \sum_{j=1}^\infty \frac{\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(X)}}{[\mu(B(x,2^{j+1}r))]^2} \frac{\varphi_2(x,2^{j+1}r)}{\varphi_2(x,2^{j+1}r)} \|\chi_{2^{j+1}B} f_2\|_{L^{p_2(\cdot)}(X)} \right\} \\
&\leq C \|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)} \|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)} \|\chi_{2B}\|_{L^{p_1'(\cdot)}(X)} \varphi_1(x,2r) \\
&\quad \times \left\{ \sum_{j=1}^\infty \frac{\|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(X)}}{[\mu(B(x,2^{j+1}r))]^2} \varphi_2(x,2^{j+1}r) \right\} \\
&\leq C \|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)} \|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)} \|\chi_{2B}\|_{L^{p_1'(\cdot)}(X)} \|\chi_{2B}\|_{L^{p_1(\cdot)}(X)} \frac{1}{\mu(2B)} \varphi_1(x,2r) \\
&\quad \times \frac{\mu(2B)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(X)}} \left\{ \sum_{j=1}^\infty \|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(X)} \|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(X)} \frac{1}{\mu(B(x,2^{j+1}r))} \right. \\
&\qquad \left. \times \frac{1}{\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(X)}} \frac{1}{\mu(B(x,2^{j+1}r))} \varphi_2(x,2^{j+1}r) \right\} \\
&\leq C \|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)} \|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)} \varphi_1(x,2r) \\
&\quad \times \frac{\mu(2B)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(X)}} \left\{ \sum_{j=1}^\infty \frac{1}{\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(X)}} \frac{1}{\mu(B(x,2^{j+1}r))} \varphi_2(x,2^{j+1}r) \right\},
\end{aligned}$$

further, by applying Lemma 2.3 and (3.1) and $\varphi_1\varphi_2 = \varphi$, we obtain that

$$\begin{aligned}
\mathrm{E}_2 &= \sup_{x\in X, r>0} \frac{1}{\varphi(x,r)} \|\chi_{B(x,r)} BT(f_1^1, f_2^\infty)\|_{L^{p(\cdot)}(X)} \\
&\leq C \|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)} \|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)} \sup_{x\in X, r>0} \frac{1}{\varphi(x,r)} \|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)} \varphi_1(x,2r)
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{1}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(X)}}\left\{\sum_{j=1}^{\infty}\frac{1}{\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(X)}}\frac{\mu(2B)}{\mu(B(x,2^{j+1}r))}\varphi_2(x,2^{j+1}r)\right\}\\
\le{}& C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)}\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\varphi_1(x,2r)\\
&\times\left\{\sum_{j=1}^{\infty}\frac{\|\chi_{B(x,r)}\|_{L^{p_2(\cdot)}(X)}}{\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(X)}}\varphi_2(x,2^{j+1}r)\right\}\\
\le{}& C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)}\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\varphi_1(x,r)\varphi_2(x,r)\\
\le{}& C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)}
\end{aligned}
$$

With an argument similar to that used in the estimate for E_2, it is easy to get

$$
\mathrm{E}_2 \le C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)}.
$$

Now let us estimate E_4. By applying (1.7), (1.16) and Lemmas 2.1 and 2.2, we have

$$
\begin{aligned}
&|BT(f_1^{\infty},f_2^{\infty})(y)|\\
&\le C\int_X\int_X\frac{|f_1^{\infty}(z_1)||f_2^{\infty}(z_2)|}{[\mu(B(y,d(y,z_1)))+\mu(B(y,d(y,z_2)))]^2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
&\le C\prod_{i=1}^{2}\left(\sum_{j=1}^{\infty}\int_{X\setminus(2B)}\frac{|f_i(z_i)|}{\mu(B(y,d(y,z_i)))}\mathrm{d}\mu(z_i)\right)\\
&\le C\prod_{i=1}^{2}\left(\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}\frac{|f_i(z_i)|}{\mu(B(y,d(y,z_i)))}\mathrm{d}\mu(z_i)\right)\\
&\le C\prod_{i=1}^{2}\left(\sum_{j=1}^{\infty}\frac{1}{\mu(B(x,2^jr))}\int_{2^{j+1}B}|f_i(z_i)|\mathrm{d}\mu(z_i)\right)\\
&\le C\prod_{i=1}^{2}\left(\sum_{j=1}^{\infty}\frac{1}{\mu(B(x,2^jr))}\|\chi_{B(x,2^{j+1}r)}\|_{L^{p_i'(\cdot)}(X)}\|\chi_{B(x,2^{j+1}r)}f_i\|_{L^{p_i(\cdot)}(X)}\right)\\
&\le C\prod_{i=1}^{2}\left(\sum_{j=1}^{\infty}\frac{1}{\mu(B(x,2^jr))}\|\chi_{B(x,2^{j+1}r)}\|_{L^{p_i'(\cdot)}(X)}\|\chi_{B(x,2^{j+1}r)}\|_{L^{p_i(\cdot)}(X)}\right.\\
&\qquad\left.\times\frac{\varphi_i(x,2^{j+1}r)}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p_i(\cdot)}(X)}}\frac{1}{\varphi_i(x,2^{j+1}r)}\|\chi_{B(x,2^{j+1}r)}f_i\|_{L^{p_i(\cdot)}(X)}\right)\\
&\le C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)}\prod_{i=1}^{2}\left(\sum_{j=1}^{\infty}\frac{\varphi_i(x,2^{j+1}r)}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p_i(\cdot)}(X)}}\right),
\end{aligned}
$$

further, via (3.1) and Lemma 2.3, we deduce

$$
\begin{aligned}
\mathrm{E}_4 ={}& \sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}BT(f_1^{\infty},f_2^{\infty})\|_{L^{p(\cdot)}(X)}\\
\le{}& C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)}\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)}\\
&\times\prod_{i=1}^{2}\left(\sum_{j=1}^{\infty}\frac{\varphi_i(x,2^{j+1}r)}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p_i(\cdot)}(X)}}\right)
\end{aligned}
$$

$$\leq C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)} \sup_{x\in X,r>0} \frac{1}{\varphi(x,r)}$$
$$\times \prod_{i=1}^{2}\left(\sum_{j=1}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p_i(\cdot)}(X)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p_i(\cdot)}(X)}}\varphi_i(x,2^{j+1}r)\right)$$
$$\leq C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)} \sup_{x\in X,r>0} \frac{1}{\varphi(x,r)}\varphi_1(x,r)\varphi_2(x,r)$$
$$\leq C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)}.$$

Combining the estimates for E_1, E_2 and E_3, the proof of Theorem 3.2 is finished. □

4. Estimate for $[b_1, b_2, BT]$ on $L^{p(\cdot)}(X)$ and $\mathcal{L}^{p(\cdot),\varphi}(X)$

The main theorems of this section are stated as follows.

Theorem 4.1. *Let $b_1, b_2 \in \mathrm{BMO}(X)$, and $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{B}$ satisfying $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Suppose that the operator BT is defined as in (1.11), and the measure μ satisfies (1.2) and (1.3). Then there exists some constant $C > 0$ such that, for all $f_i \in L^{p_i(\cdot)}(X)$ with $i = 1, 2$,*

$$\|[b_1, b_2, BT](f_1, f_2)\|_{L^{p(\cdot)}(X)} \leq C\|b_1\|_{\mathrm{BMO}(X)}\|b_2\|_{\mathrm{BMO}(X)}\|f_1\|_{L^{p_1(\cdot)}(X)}\|f_2\|_{L^{p_2(\cdot)}(X)}.$$

Theorem 4.2. *Let $b_1, b_2 \in \mathrm{BMO}(X)$, and $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{B}$ satisfying $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Suppose that the operator BT is defined as in (1.11), the measure μ satisfies (1.2) and (1.3), and $\varphi_i(\cdot,\cdot)(i = 1, 2)$ defined on $X \times (0,\infty)$ are positive measurable functions. If there exists some constant $C > 0$ such that, for any $x \in X$ and $r > 0$,*

$$\sum_{k=0}^{\infty}(k+1)\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)}}{\|\chi_{B(x,2^{k+1}r)}\|_{L^{p(\cdot)}(X)}}\varphi_i(x,2^{k+1}r) \leq C\varphi_i(x,r), \quad i = 1, 2, \tag{4.1}$$

and $\varphi_1\varphi_2 = \varphi$, then

$$\|[b_1, b_2, BT](f_1, f_2)\|_{\mathcal{L}^{p(\cdot),\varphi}(X)} \leq C\|b_1\|_{\mathrm{BMO}(X)}\|b_2\|_{\mathrm{BMO}(X)}\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)},$$

where $f_i \in \mathcal{L}^{p_i(\cdot),\varphi_i}(X)$ with $i = 1, 2$.

To prove the above theorems, we should establish the following lemma.

Lemma 4.3. *Let $0 < \delta < \tau < \frac{1}{2}$, $1 < s_1, s_2 < \infty$ and $b_1, b_2 \in \mathrm{BMO}(X)$. Suppose that the operator BT defined as in (1.11) is bounded from product spaces $L^1(X)\times L^1(X)$ into spaces $L^{\frac{1}{2}}(X)$, and the measure μ satisfies (1.2) and (1.3). Then there exists some positive constant C such that, for all $f_1, f_2 \in L^{\infty}_{\mathrm{loc}}(X)$,*

$$\begin{aligned}M^{\sharp}_{\delta}([b_1, b_2, BT](f_1, f_2))(x) \leq{}& C\|b_1\|_{\mathrm{BMO}(X)}M_{\tau}([b_2, BT](f_1, f_2))(x)\\ &+C\|b_2\|_{\mathrm{BMO}(X)}M_{\tau}([b_1, BT](f_1, f_2))(x)\\ &+C\|b_1\|_{\mathrm{BMO}(X)}\|b_2\|_{\mathrm{BMO}(X)}\Big[M_{\tau}(BT(f_1, f_2))(x) + M_{s_1}f_1(x)M_{s_2}f_2(x)\Big].\end{aligned} \tag{4.2}$$

$$M^{\sharp}_{\delta}([b_1, BT](f_1, f_2))(x) \leq C\|b_1\|_{\mathrm{BMO}(X)}\Big[M_{\tau}(BT(f_1, f_2))(x) + M_{s_1}f_1(x)M_{s_2}f_2(x)\Big] \tag{4.3}$$

and

$$M^{\sharp}_{\delta}([b_2, BT](f_1, f_2))(x) \leq C\|b_2\|_{\mathrm{BMO}(X)}\Big[M_{\tau}(BT(f_1, f_2))(x) + M_{s_1}f_1(x)M_{s_2}f_2(x)\Big]. \tag{4.4}$$

Proof. Without loss of generality, we may assume that $\|b_1\|_{\mathrm{BMO}(X)} = \|b_2\|_{\mathrm{BMO}(X)} = 1$. Moreover, because of the methods for the estimates of (4.2), (4.3) and (4.4) are the same, therefore, here we only show (4.2).

To show (4.2), by (3.3), it suffices to prove that

$$\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}\Big||[b_1,b_2,BT](f_1,f_2)(y)|^\delta-|h_{B(x,r)}|^\delta\Big|\mathrm{d}\mu(y)\right)^\delta \le C\Big[M_\tau(BT(f_1,f_2))(x)$$

$$+M_\tau([b_2,BT](f_1,f_2))(x)+M_{s_1}f_1(x)M_{s_2}f_2(x)+M_\tau([b_1,BT](f_1,f_2))(x)\Big] \tag{4.5}$$

where $h_{B(x,r)}$ is defined by

$$h_{B(x,r)} = m_B(BT((b_1(\cdot)-(b_1)_B)f_1\chi_{X\setminus(2B)},(b_2(\cdot)-(b_2)_B)f_2\chi_{X\setminus(2B)})).$$

Further, we will show that

$$\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|[b_1,b_2,BT](f_1,f_2)(y)-h_{B(x,r)}|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}} \le C\Big[M_\tau(BT(f_1,f_2))(x)$$

$$+M_\tau([b_2,BT](f_1,f_2))(x)+M_{s_1}f_1(x)M_{s_2}f_2(x)+M_\tau([b_1,BT](f_1,f_2))(x)\Big], \tag{4.6}$$

To prove (4.6), decompose the functions f_i as

$$f_i = f_i\chi_{2B}+f_i\chi_{X\setminus(2B)} = f_i^1+f_i^\infty, \quad i=1,2,$$

where $B := B(x,r)$ is the fixed ball centered at x with radius r. And then write

$$\begin{aligned}
&\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|[b_1,b_2,BT](f_1,f_2)(y)-h_{B(x,r)}|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\le C\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|b_1(y)-(b_1)_B|^\delta|b_2(y)-(b_2)_B|^\delta|BT(f_1,f_2)(y)|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\quad+C\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|b_1(y)-(b_1)_B|^\delta|BT(f_1,(b_2(\cdot)-(b_2)_B)f_2)(y)|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\quad+C\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|b_2(y)-(b_2)_B|^\delta|BT((b_1(\cdot)-(b_1)_B)f_1,f_2)(y)|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\quad+C\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT((b_1(\cdot)-(b_1)_B)f_1^1,(b_2(\cdot)-(b_2)_B)f_2^1)(y)|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\quad+C\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT((b_1(\cdot)-(b_1)_B)f_1^1,(b_2(\cdot)-(b_2)_B)f_2^\infty)(y)|^\delta\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}
\end{aligned}$$

$$+ C\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT((b_1(\cdot)-(b_1)_B)f_1^{\infty},(b_2(\cdot)-(b_2)_B)f_2^{1})(y)|^{\delta}\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}$$

$$+ C\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT((b_1(\cdot)-(b_1)_B)f_1^{\infty},(b_2(\cdot)-(b_2)_B)f_2^{\infty})(y) - h_{B(x,r)}|^{\delta}\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}$$

$$= \mathrm{F}_1+\mathrm{F}_2+\mathrm{F}_3+\mathrm{F}_4+\mathrm{F}_5+\mathrm{F}_6+\mathrm{F}_7.$$

Taking $s_1, s_2 > 1$ such that $\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{\tau}=\frac{1}{\delta}$. By applying the Hölder inequality, (1.5) and (1.6), we deduce

$$\begin{aligned}
\mathrm{F}_1 &\le C\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|b_1(y)-(b_1)_B|^{\delta}|b_1(y)-(b_1)_B|^{\delta}|BT(f_1,f_2)(y)|^{\delta}\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\le \frac{C}{[\mu(B(x,r))]^{\frac{1}{\delta}}}\left(\int_{B(x,r)}|b_1(y)-(b_1)_B|^{s_1}\mathrm{d}\mu(y)\right)^{\frac{1}{s_1}}\\
&\quad\times\left(\int_{B(x,r)}|b_2(y)-(b_2)_B|^{s_2}\mathrm{d}\mu(y)\right)^{\frac{1}{s_2}}\left(\int_{B(x,r)}|BT(f_1,f_2)(y)|^{\tau}\mathrm{d}\mu(y)\right)^{\frac{1}{\tau}}\\
&\le CM_{\tau}(BT(f_1,f_2))(x).
\end{aligned}$$

Let $s > 1$ with satisfying $\frac{1}{s}+\frac{1}{\tau}=\frac{1}{\delta}$. Via the Hölder inequality, (1.5) and (1.6), we have

$$\begin{aligned}
\mathrm{F}_2 &\le C\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|b_1(y)-(b_1)_B|^{\delta}|BT(f_1,(b_2(\cdot)-(b_2)_B)f_2)(y)|^{\delta}\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\le \frac{C}{[\mu(B(x,r))]^{\frac{1}{\delta}}}\left(\int_{B(x,r)}|b_1(y)-(b_1)_B|^{s}\mathrm{d}\mu(y)\right)^{\frac{1}{s}}\\
&\quad\times\left(\int_{B(x,r)}|BT(f_1,(b_2(\cdot)-(b_2)_B)f_2)(y)|^{\tau}\mathrm{d}\mu(y)\right)^{\frac{1}{\tau}}\\
&\le CM_{\tau}(BT(f_1,(b_2(\cdot)-(b_2)_B)f_2))(x).
\end{aligned}$$

Similarly, we also have $\mathrm{F}_3 \le CM_{\tau}(BT((b_1(\cdot)-(b_1)_B)f_1,f_2))(x)$.

Let $p=\delta$ and $q=\frac{1}{2}$. By using the Kolmogorov estimate and the $(L^1(X)\times L^1(X), L^{\frac{1}{2},\infty}(X))$-boundedness of BT, we have

$$\begin{aligned}
\mathrm{F}_4 &\le C\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT((b_1(\cdot)-(b_1)_B)f_1^{1},(b_2(\cdot)-(b_2)_B)f_2^{1})(y)|^{\delta}\mathrm{d}\mu(y)\right)^{\frac{1}{\delta}}\\
&\le C\|BT((b_1(\cdot)-(b_1)_B)f_1^{1},(b_2(\cdot)-(b_2)_B)f_2^{1})\|_{L^{\frac{1}{2}}(B(x,r),\frac{\mathrm{d}\mu(y)}{\mu(B(x,r))})}\\
&\le C\frac{1}{\mu(B(x,r))}\int_{2B(x,r)}|b_1(z_1)-(b_1)_B||f_1(z_1)|\mathrm{d}\mu(z_1)\\
&\quad\times\frac{1}{\mu(B(x,r))}\int_{2B(x,r)}|b_2(z_2)-(b_2)_B||f_2(z_2)|\mathrm{d}\mu(z_2)
\end{aligned}$$

$$
\begin{aligned}
&\le C\frac{1}{\mu(2B(x,r))}\int_{2B(x,r)}\Big(|b_1(z_1)-(b_1)_{2B}|+|(b_1)_{2B}-(b_1)_B|\Big)|f_1(z_1)|\mathrm{d}\mu(z_1)\\
&\quad\times\frac{1}{\mu(2B(x,r))}\int_{2B(x,r)}\Big(|b_2(z_2)-(b_2)_{2B}|+|(b_2)_{2B}-(b_2)_B|\Big)|f_2(z_2)|\mathrm{d}\mu(z_2)\\
&\le C\frac{1}{\mu(2B(x,r))}\bigg\{\bigg(\int_{2B(x,r)}|f_1(z_1)|^{s_1}\mathrm{d}\mu(z_1)\bigg)^{\frac{1}{s_1}}[\mu(2B(x,r))]^{1-\frac{1}{s_1}}\\
&\qquad+\bigg(\int_{2B(x,r)}|f_1(z_1)|^{s_1}\mathrm{d}\mu(z_1)\bigg)^{\frac{1}{s_1}}\bigg(\int_{2B(x,r)}|b_1(z_1)-(b_1)_{2B}|^{s_1'})\mathrm{d}\mu(z_1)\bigg)^{\frac{1}{s_1'}}\bigg\}\\
&\quad\times\frac{1}{\mu(2B(x,r))}\bigg\{\bigg(\int_{2B(x,r)}|f_2(z_2)|^{s_2}\mathrm{d}\mu(z_2)\bigg)^{\frac{1}{s_2}}[\mu(2B(x,r))]^{1-\frac{1}{s_2}}\\
&\qquad+\bigg(\int_{2B(x,r)}|f_2(z_2)|^{s_2}\mathrm{d}\mu(z_1)\bigg)^{\frac{1}{s_2}}\bigg(\int_{2B(x,r)}|b_2(z_2)-(b_2)_{2B}|^{s_2'})\mathrm{d}\mu(z_2)\bigg)^{\frac{1}{s_2'}}\bigg\}\\
&\le CM_{s_1}(f_1)(x)M_{s_2}(f_2)(x).
\end{aligned}
$$

To show F_5, we consider $|BT((b_1(\cdot)-(b_1)_B)f_1^1,(b_2(\cdot)-(b_2)_B)f_2^\infty)(y)|$ with $y\in B(x,r)$. By applying (1.6), (1.7) and the Hölder inequality, we have

$$
\begin{aligned}
&|BT((b_1(\cdot)-(b_1)_B)f_1^1,(b_2(\cdot)-(b_2)_B)f_2^\infty)(y)|\\
&\quad\le C\int_{2B}\int_{X\setminus(2B)}\frac{|b_1(z_1)-(b_1)_B||b_2(z_2)-(b_2)_B|}{[\mu(B(y,d(y,z_1)))+\mu(B(y,d(y,z_2)))]^2}|f_1(z_1)||f_2(z_2)|\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
&\quad\le C\int_{2B}|b_1(z_1)-(b_1)_B||f_1(z_1)|\mathrm{d}\mu(z_1)\\
&\qquad\qquad\times\bigg(\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus(2^kB)}\frac{|b_2(z_2)-(b_2)_B|}{[\mu(B(x,d(x,z_2)))]^2}|f_2(z_2)|\mathrm{d}\mu(z_2)\bigg)\\
&\quad\le C\bigg(|(b_1)_{2B}-(b_1)_B|\int_{2B}|f_1(z_1)|\mathrm{d}\mu(z_1)+\int_{2B}|b_1(z_1)-(b_1)_{2B}||f_1(z_1)|\mathrm{d}\mu(z_1)\bigg)\\
&\qquad\qquad\times\bigg(\sum_{k=1}^{\infty}\frac{1}{[\mu(B(x,2^kr))]^2}\int_{2^{k+1}B}|b_2(z_2)-(b_2)_B||f_2(z_2)|\mathrm{d}\mu(z_2)\bigg)\\
&\quad\le C\bigg\{\bigg(\int_{2B}|f_1(z_1)|^{s_1}\mathrm{d}\mu(z_1)\bigg)^{\frac{1}{s_1}}\bigg(\int_{2B}|b_1(z_1)-(b_1)_{2B}|^{s_1'}\mathrm{d}\mu(z_1)\bigg)^{\frac{1}{s_1'}}\\
&\qquad\qquad+\bigg(\int_{2B}|f_1(z_1)|^{s_1}\mathrm{d}\mu(z_1)\bigg)^{\frac{1}{s_1}}[\mu(2B)]^{1-\frac{1}{s_1}}\bigg\}\\
&\qquad\qquad\times\bigg[\sum_{k=1}^{\infty}\frac{1}{[\mu(B(x,2^kr))]^2}\bigg(\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2^{k+1}B}||f_2(z_2)|\mathrm{d}\mu(z_2)\\
&\qquad\qquad\qquad+|(b_2)_{2^{k+1}B}-(b_2)_B|\int_{2^{k+1}B}|f_2(z_2)|\mathrm{d}\mu(z_2)\bigg)\bigg]\\
&\quad\le C\mu(2B)M_{s_1}(f_1)(x)\bigg\{\sum_{k=1}^{\infty}\frac{1}{[\mu(B(x,2^kr))]^2}
\end{aligned}
$$

$$\times\Bigg[\bigg(\int_{2^{k+1}B}|f_2(z_2)|^{s_2}\mathrm{d}\mu(z_2)\bigg)^{\frac{1}{s_2}}\bigg(\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2^{k+1}B}|^{s_2'}\mathrm{d}\mu(z_2)\bigg)^{\frac{1}{s_2'}}$$
$$+k\bigg(\int_{2^{k+1}B}|f_2(z_2)|^{s_2}\mathrm{d}\mu(z_2)\bigg)^{\frac{1}{s_2}}[\mu(2^{k+1}B)]^{1-\frac{1}{s_2}}\Bigg]\Bigg\}$$
$$\leq C\mu(2B)M_{s_1}(f_1)(x)M_{s_2}(f_2)(x)\Bigg\{\sum_{k=1}^{\infty}\frac{k\mu(2^{k+1}B)}{[\mu(B(x,2^kr))]^2}\Bigg\}$$
$$\leq CM_{s_1}(f_1)(x)M_{s_2}(f_2)(x)\Bigg\{\sum_{k=1}^{\infty}\frac{k\mu(2B)}{\mu(B(x,2^kr))}\Bigg\}$$
$$\leq CM_{s_1}(f_1)(x)M_{s_2}(f_2)(x)\Bigg\{\sum_{k=1}^{\infty}\frac{k\mu(2B)}{\mathbb{K}^{k-1}\mu(B(x,2r))}\Bigg\}$$
$$\leq CM_{s_1}(f_1)(x)M_{s_2}(f_2)(x),$$

thus, we obtain that $\mathrm{F}_5\leq CM_{s_1}(f_1)(x)M_{s_2}(f_2)(x)$.

With an argument similar to that used in the estimate of F_5, it is easy to see that

$$\mathrm{F}_6\leq CM_{s_1}(f_1)(x)M_{s_2}(f_2)(x).$$

Since

$$\bigg(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|BT((b_1(\cdot)-(b_1)_B)f_1^{\infty},(b_2(\cdot)-(b_2)_B)f_2^{\infty})(y)-h_{B(x,r)}|^{\delta}\mathrm{d}\mu(y)\bigg)^{\frac{1}{\delta}}$$
$$\leq\bigg(\frac{1}{\mu(B(x,r))}\frac{1}{\mu(B(x,r))}\int_{B(x,r)}\int_{B(x,r)}|BT((b_1(\cdot)-(b_1)_B)f_1^{\infty},(b_2(\cdot)-(b_2)_B)f_2^{\infty})(y)$$
$$-BT((b_1(\cdot)-(b_1)_B)f_1^{\infty},(b_2(\cdot)-(b_2)_B)f_2^{\infty})(\omega)|^{\delta}\mathrm{d}\mu(y)\mathrm{d}\mu(\omega)\bigg)^{\frac{1}{\delta}}.$$

Thus, to show F_7, we only need to consider $|BT((b_1(\cdot)-(b_1)_B)f_1^{\infty},(b_2(\cdot)-(b_2)_B)f_2^{\infty})(y)-BT((b_1(\cdot)-(b_1)_B)f_1^{\infty},(b_2(\cdot)-(b_2)_B)f_2^{\infty})(\omega)|$ with $y,w\in B(x,r)$. From (1.6), (1.8) and the Hölder inequality, it then follows that

$$|BT((b_1(\cdot)-(b_1)_B)f_1^{\infty},(b_2(\cdot)-(b_2)_B)f_2^{\infty})(y)$$
$$-BT((b_1(\cdot)-(b_1)_B)f_1^{\infty},(b_2(\cdot)-(b_2)_B)f_2^{\infty})(\omega)|$$
$$\leq\int_X\int_X|K(y,z_1,z_2)-K(\omega,z_1,z_2)||b_1(z_1)-(b_1)_B||b_2(z_2)-(b_2)_B|$$
$$\times|f_1^{\infty}(z_1)||f_2^{\infty}(z_2)|\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)$$
$$\leq C\int_{X\setminus(2B)}\int_{X\setminus(2B)}\frac{[d(y,\omega)]^{\delta}}{[d(y,z_1)+d(y,z_2)]^{\delta}}\frac{|b_1(z_1)-(b_1)_B||b_2(z_2)-(b_2)_B|}{\Big[\sum_{i=1}^{2}\mu(B(y,d(y,z_i)))\Big]^2}$$
$$\times|f_1(z_1)||f_2(z_2)|\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)$$
$$\leq C\prod_{i=1}^{2}\int_{X\setminus(2B)}\frac{[d(y,\omega)]^{\delta}}{[d(y,z_i)]^{\delta}}\frac{|b_i(z_i)-(b_i)_B|}{\mu(B(y,d(y,z_i)))}|f_i(z_i)|\mathrm{d}\mu(z_i)$$
$$\leq C\prod_{i=1}^{2}\bigg(\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus(2^kB)}\frac{[d(y,\omega)]^{\delta}}{[d(y,z_i)]^{\delta}}\frac{|b_i(z_i)-(b_i)_B|}{\mu(B(y,d(x,z_i)))}|f_i(z_i)|\mathrm{d}\mu(z_i)\bigg)$$

$$
\begin{aligned}
&\le C\prod_{i=1}^{2}\bigg(\sum_{k=1}^{\infty}\frac{r^{\delta}}{(2^k r)^{\delta}}\frac{1}{\mu(B(x,2^k r))}\int_{2^{k+1}B}|b_i(z_i)-(b_i)_B||f_i(z_i)|\mathrm{d}\mu(z_i)\bigg)\\
&\le C\prod_{i=1}^{2}\bigg[\sum_{k=1}^{\infty}\frac{2^{-k\delta}}{\mu(B(x,2^k r))}\bigg(|(b_i)_{2^{k+1}B}-(b_i)_B|\int_{2^{k+1}B}|f_i(z_i)|\mathrm{d}\mu(z_i)\\
&\qquad\qquad+\int_{2^{k+1}B}|b_i(z_i)-(b_i)_{2^{k+1}B})||f_i(z_i)|\mathrm{d}\mu(z_i)\bigg)\bigg]\\
&\le C\prod_{i=1}^{2}\bigg\{\sum_{k=1}^{\infty}\frac{2^{-k\delta}}{\mu(B(x,2^k r))}\bigg[k\bigg(\int_{2^{k+1}B}|f_i(z_i)|^{s_i}\mathrm{d}\mu(z_i)\bigg)^{\frac{1}{s_i}}[\mu(2^{k+1}B)]^{1-\frac{1}{s_i}}\\
&\qquad+\bigg(\int_{2^{k+1}B}|f_i(z_i)|^{s_i}\mathrm{d}\mu(z_i)\bigg)^{\frac{1}{s_i}}\bigg(\int_{2^{k+1}B}|b_i(z_i)-(b_i)_{2^{k+1}B})|^{s_i'}\mathrm{d}\mu(z_i)\bigg)^{\frac{1}{s_i'}}\bigg]\bigg\}\\
&\le CM_{s_1}(f_1)(x)M_{s_2}(f_2)(x)\bigg(\sum_{k=1}^{\infty}\frac{k}{2^{k\delta}}\bigg)\\
&\le CM_{s_1}(f_1)(x)M_{s_2}(f_2)(x),
\end{aligned}
$$

therefore, we obtain that $\mathrm{F}_7\le CM_{s_1}(f_1)(x)M_{s_2}(f_2)(x)$. Which, combining the estimates for $\mathrm{F}_1-\mathrm{F}_6$, implies (4.6). □

Now let us state the proofs of the Theorems 4.1 and 4.2 as follows.

Proof of Theorem 4.1. By Lemmas 3.3 and 4.3, and the $(L^{p_1}(X)\times L^{p_2}(X),L^p(X))$-boundedness of operators BT, we have

$$
\begin{aligned}
&\int_X([b_1,b_2,BT](f_1,f_2)(x))^{p_0}\omega_0(x)\mathrm{d}\mu(x)\\
&\quad\le\int_X(M_\delta([b_1,b_2,BT](f_1,f_2))(x))^{p_0}\omega_0(x)\mathrm{d}\mu(x)\\
&\quad\le C\int_X(M_\delta^{\sharp}([b_1,b_2,BT](f_1,f_2))(x))^{p_0}\omega_0(x)\mathrm{d}\mu(x)\\
&\quad\le C\int_X(\|b_1\|_{\mathrm{BMO}(X)}\|b_2\|_{\mathrm{BMO}(X)}[M_\tau(BT(f_1,f_2))(x)]^{p_0}\omega_0(x)\mathrm{d}\mu(x)\\
&\qquad+C\int_X(\|b_1\|_{\mathrm{BMO}(X)}\|b_2\|_{\mathrm{BMO}(X)}[M_{s_1}f_1(x)M_{s_2}f_2(x)]^{p_0}\omega_0(x)\mathrm{d}\mu(x)\\
&\quad\le C\int_X(\|b_1\|_{\mathrm{BMO}(X)}\|b_2\|_{\mathrm{BMO}(X)}M_{s_1}f_1(x)M_{s_2}f_2(x))^{p_0}\omega_0(x)\mathrm{d}\mu(x),
\end{aligned}
$$

further, by Lemmas 2.3 and 3.4, we deduce

$$
\begin{aligned}
\|[b_1,b_2,BT](f_1,f_2)\|_{L^{p(\cdot)}(X)}&\le C\|b_1\|_{\mathrm{BMO}(X)}\|b_2\|_{\mathrm{BMO}(X)}\|M_{s_1}f_1M_{s_2}f_2\|_{L^{p(\cdot)}(X)}\\
&\le C\|b_1\|_{\mathrm{BMO}(X)}\|b_2\|_{\mathrm{BMO}(X)}\|f_1\|_{L^{p_1(\cdot)}(X)}\|f_2\|_{L^{p_2(\cdot)}(X)}.
\end{aligned}
$$

Hence, the proof of Theorem 4.1 is completed. □

Proof of Theorem 4.2. Without loss of generality, we may assume that $\|b_1\|_{\mathrm{BMO}(X)}=\|b_1\|_{\mathrm{BMO}(X)}=1$. And decompose the function f_i as

$$
f_i=f_i^1+f_i^{\infty}=f_i\chi_{2B}+f_i\chi_{X\setminus(2B)},\quad i=1,2,
$$

where $B:=B(x,r)$ is a fixed ball centered at $x\in X$ with radius $r>0$. Then, by applying (1.16) and the Minkowski inequality, write

$$
\begin{aligned}
&\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}([b_1,b_2,BT](f_1,f_2))\|_{L^{p(\cdot)}(X)}\\
&\le \sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}([b_1,b_2,BT](f_1^1,f_2^1))\|_{L^{p(\cdot)}(X)}\\
&\quad+\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}([b_1,b_2,BT](f_1^1,f_2^\infty))\|_{L^{p(\cdot)}(X)}\\
&\quad+\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}([b_1,b_2,BT](f_1^\infty,f_2^1))\|_{L^{p(\cdot)}(X)}\\
&\quad+\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\|\chi_{B(x,r)}([b_1,b_2,BT](f_1^\infty,f_2^\infty))\|_{L^{p(\cdot)}(X)}\\
&= \mathrm{G}_1+\mathrm{G}_2+\mathrm{G}_3+\mathrm{G}_4.
\end{aligned}
$$

From (1.16), Lemma 2.3, Theorem 4.1 and $\varphi_1\varphi_2=\varphi$, it then follows that

$$
\begin{aligned}
\mathrm{G}_1&=\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\|[b_1,b_2,BT](f_1^1,f_2^1)\|_{L^{p(\cdot)}(B(x,r))}\\
&\le C\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\|f_1^1\|_{L^{p_1(\cdot)}(B(x,r))}\|f_2^1\|_{L^{p_2(\cdot)}(B(x,r))}\\
&\le C\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\|f_1\|_{L^{p_1(\cdot)}(B(x,r))}\|f_2\|_{L^{p_2(\cdot)}(B(x,r))}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{\mathcal{L}^{p_2(\cdot),\varphi_2}(X)},
\end{aligned}
$$

where $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$.

To estimate G_2, we only need to consider $|[b_1,b_2,BT](f_1^1,f_2^\infty)(y)|$ with $y\in B(x,r)$. By applying (1.7), Lemmas 2.1, 2.2, 2.5, 2.7 and (4.1), we have

$$
\begin{aligned}
&|[b_1,b_2,BT](f_1^1,f_2^\infty)(y)|\\
&\quad\le \int_X\int_X|b_1(y)-b_1(z_1)||b_2(y)-b_2(z_2)||K(y,z_1,z_2)||f_1^1(z_1)||f_2^\infty(z_2)|\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
&\quad\le C\int_{2B}\int_{X\setminus(2B)}\frac{|b_1(y)-b_1(z_1)||b_2(y)-b_2(z_2)|}{[\mu(B(y,d(y,z_1)))+\mu(B(y,d(y,z_2)))]^2}|f_1(z_1)||f_2(z_2)|\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
&\quad\le C\int_{2B}|b_1(y)-b_1(z_1)||f_1(z_1)|\mathrm{d}\mu(z_1)\int_{X\setminus(2B)}\frac{|b_2(y)-b_2(z_2)|}{[\mu(B(y,d(y,z_2)))]^2}|f_2(z_2)|\mathrm{d}\mu(z_2)\\
&\quad\le C\left(|b_1(y)-(b_1)_{2B}|\int_{2B}|f_1(z_1)|\mathrm{d}\mu(z_1)+\int_{2B}|b_1(z_1)-(b_1)_{2B}||f_1(z_1)|\mathrm{d}\mu(z_1)\right)\\
&\qquad\times\left(|b_2(y)-(b_2)_{2B}|\sum_{k=1}^\infty\int_{(2^{k+1}B)\setminus(2^kB)}\frac{|f_2(z_2)|}{[\mu(B(y,d(y,z_2)))]^2}\mathrm{d}\mu(z_2)\right.\\
&\qquad\left.+\sum_{k=1}^\infty\int_{(2^{k+1}B)\setminus(2^kB)}\frac{|b_2(z_2)-(b_2)_{2B}|}{[\mu(B(y,d(y,z_2)))]^2}|f_2(z_2)|\mathrm{d}\mu(z_2)\right)\\
&\quad\le C\Bigg\{|b_1(y)-(b_1)_{2B}|\|\chi_{2B}f_1\|_{L^{p_1(\cdot)}(X)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(X)}\\
&\qquad+\|\chi_{2B}f_1\|_{L^{p_1(\cdot)}(X)}\|\chi_{2B}(b_1-(b_1)_{2B})\|_{L^{p_1'(\cdot)}(X)}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Bigg(|b_2(y)-(b_2)_{2B}| \sum_{k=1}^{\infty} \frac{1}{[\mu(B(x,2^k r))]^2} \int_{2^{k+1}B} |f_2(z_2)| \mathrm{d}\mu(z_2) \\
&\qquad + \sum_{k=1}^{\infty} \frac{1}{[\mu(B(x,2^k r))]^2} \int_{2^{k+1}B} |b_2(z_2)-(b_2)_{2B}| |f_2(z_2)| \mathrm{d}\mu(z_2) \Bigg) \\
&\le C \|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)} |b_1(y)-(b_1)_{2B}| \|\chi_{2B}\|_{L^{p_1'(\cdot)}(X)} \varphi_1(x,2r) \\
&\times \Bigg\{ |b_2(y)-(b_2)_{2B}| \sum_{k=1}^{\infty} \frac{1}{[\mu(B(x,2^k r))]^2} \|\chi_{2^{k+1}B} f_2\|_{L^{p_2(\cdot)}(X)} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(X)} \\
&\qquad + \sum_{k=1}^{\infty} \frac{|(b_2)_{2B}-(b_2)_{2^{k+1}B}|}{[\mu(B(x,2^k r))]^2} \int_{2^{k+1}B} |f_2(z_2)| \mathrm{d}\mu(z_2) \\
&\qquad + \sum_{k=1}^{\infty} \frac{1}{[\mu(B(x,2^k r))]^2} \int_{2^{k+1}B} |b_2(z_2)-(b_2)_{2^{k+1}B}| |f_2(z_2)| \mathrm{d}\mu(z_2) \Bigg\} \\
&\le C \|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)} |b_1(y)-(b_1)_{2B}| \|\chi_{2B}\|_{L^{p_1'(\cdot)}(X)} \varphi_1(x,2r) \\
&\times \Bigg\{ |b_2(y)-(b_2)_{2B}| \sum_{k=1}^{\infty} \frac{1}{[\mu(B(x,2^k r))]^2} \|\chi_{2^{k+1}B} f_2\|_{L^{p_2(\cdot)}(X)} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(X)} \\
&\qquad + \sum_{k=1}^{\infty} k \frac{1}{[\mu(B(x,2^k r))]^2} \|\chi_{2^{k+1}B} f_2\|_{L^{p_2(\cdot)}(X)} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(X)} \\
&\quad + \sum_{k=1}^{\infty} \frac{1}{[\mu(B(x,2^k r))]^2} \|\chi_{2^{k+1}B} f_2\|_{L^{p_2(\cdot)}(X)} \|\chi_{2^{k+1}B}(b_2(z_2)-(b_2)_{2^{k+1}B})\|_{L^{p_2'(\cdot)}(X)} \Bigg\} \\
&\le C \|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)} \|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)} |b_1(y)-(b_1)_{2B}| \|\chi_{2B}\|_{L^{p_1'(\cdot)}(X)} \varphi_1(x,2r) \\
&\times \Bigg\{ |b_2(y)-(b_2)_{2B}| \sum_{k=1}^{\infty} \frac{\varphi_2(x,2^{k+1}r)}{[\mu(B(x,2^k r))]^2} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(X)} \\
&\qquad + \sum_{k=1}^{\infty} (k+1) \frac{\varphi_2(x,2^{k+1}r)}{[\mu(B(x,2^k r))]^2} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(X)} \Bigg\} \\
&\le C \|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)} \|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)} |b_1(y)-(b_1)_{2B}| \mu(2B) \frac{\varphi_1(x,2r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(X)}} \\
&\times \Bigg\{ |b_2(y)-(b_2)_{2B}| \sum_{k=1}^{\infty} \frac{1}{\mu(B(x,2^k r))} \frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(X)}} \\
&\qquad + \sum_{k=1}^{\infty} \frac{(k+1)}{\mu(B(x,2^k r))} \frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(X)}} \Bigg\} \\
&\le C \|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)} \|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)} |b_1(y)-(b_1)_{2B}| \frac{\varphi_1(x,2r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(X)}} \\
&\times \Bigg\{ |b_2(y)-(b_2)_{2B}| \sum_{k=1}^{\infty} \frac{\mu(2B)}{\mu(B(x,2^k r))} \frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(X)}} \\
&\qquad + \sum_{k=1}^{\infty} (k+1) \frac{\mu(2B)}{\mu(B(x,2^k r))} \frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(X)}} \Bigg\} \\
&\le C \|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)} \|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)} |b_1(y)-(b_1)_{2B}| \frac{\varphi_1(x,2r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(X)}} \\
&\times \Bigg\{ |b_2(y)-(b_2)_{2B}| \sum_{k=1}^{\infty} \frac{\mu(2B)}{\mathbb{K}^{k-1}\mu(B(x,2r))} \frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(X)}}
\end{aligned}
$$

$$+\sum_{k=1}^{\infty}(k+1)\frac{\mu(2B)}{\mathbb{K}^{k-1}\mu(B(x,2r))}\frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(X)}}\Bigg\}$$
$$\leq C\|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)}|b_1(y)-(b_1)_{2B}|\frac{\varphi_1(x,2r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(X)}}$$
$$\times\left\{|b_2(y)-(b_2)_{2B}|\sum_{k=1}^{\infty}(k+1)\frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(X)}}\right\},$$

further, by Lemmas 2.3 and 2.5, we obtain

$$\begin{aligned}
\mathrm{G}_2 &\leq C\|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)}\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\frac{\varphi_1(x,2r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(X)}}\\
&\quad\times\left\{\sum_{k=1}^{\infty}(k+1)\frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(X)}}\right\}\\
&\qquad\times\|\chi_{B(x,r)}(b_1(\cdot)-(b_1)_{2B})(b_2(\cdot)-(b_2)_{2B})\|_{L^{p(\cdot)}(X)}\\
&\leq C\|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)}\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\frac{\varphi_1(x,2r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(X)}}\\
&\quad\times\|\chi_{B(x,r)}(b_1(\cdot)-(b_1)_{2B})\|_{L^{p_1(\cdot)}(X)}\\
&\quad\times\left\{\sum_{k=1}^{\infty}(k+1)\frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(X)}}\right\}\|\chi_{B(x,r)}(b_2(\cdot)-(b_2)_{2B})\|_{L^{p_2(\cdot)}(X)}\\
&\leq C\|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)}\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\frac{\varphi_1(x,2r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(X)}}\\
&\quad\times\Big(\|\chi_{B(x,r)}\|_{L^{p_1(\cdot)}(X)}+\|\chi_{B(x,r)}(b_1(\cdot)-(b_1)_B)\|_{L^{p_1(\cdot)}(X)}\Big)\\
&\quad\times\left\{\sum_{k=1}^{\infty}(k+1)\frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(X)}}\right\}\\
&\qquad\times\Big(\|\chi_{B(x,r)}\|_{L^{p_2(\cdot)}(X)}+\|\chi_{B(x,r)}(b_2(\cdot)-(b_2)_B)\|_{L^{p_2(\cdot)}(X)}\Big)\\
&\leq C\|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)}\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\frac{\varphi_1(x,2r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(X)}}\|\chi_{B(x,r)}\|_{L^{p_1(\cdot)}(X)}\\
&\quad\times\left\{\sum_{k=1}^{\infty}(k+1)\frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(X)}}\|\chi_{B(x,r)}\|_{L^{p_2(\cdot)}(X)}\right\}\\
&\leq C\|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)}\sup_{x\in X,r>0}\frac{1}{\varphi(x,r)}\varphi_1(x,r)\varphi_2(x,r)\\
&\leq C\|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)}.
\end{aligned}$$

With an argument similar to that used in the estimate for G_2, it is easy to obtain that

$$\mathrm{G}_3\leq C\|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)}\|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)}.$$

Now let us turn G_4. For any $y\in B(x,r)$, by (1.7), Lemmas 2.1, 2.2, 2.5 and 2.7, we deduce

$$|[b_1,b_2,BT](f_1^{\infty},f_2^{\infty})(y)|$$

$$
\begin{aligned}
&\leq \int_X \int_X |b_1(y)-b_1(z_1)||b_2(y)-b_2(z_2)||K(y,z_1,z_2)||f_1^{\infty}(z_1)||f_2^{\infty}(z_2)|\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2) \\
&\leq C\int_{X\setminus(2B)}\int_{X\setminus(2B)} \frac{|b_1(y)-b_1(z_1)||b_2(y)-b_2(z_2)||f_1(z_1)||f_2(z_2)|}{[\mu(B(y,d(y,z_1)))+\mu(B(y,d(y,z_2)))]^2}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2) \\
&\leq C\int_{X\setminus(2B)}\int_{X\setminus(2B)} \frac{|b_1(y)-b_1(z_1)||b_2(y)-b_2(z_2)||f_1(z_1)||f_2(z_2)|}{\mu(B(y,d(y,z_1)))\mu(B(y,d(y,z_2)))}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2) \\
&\leq C\prod_{i=1}^{2}\int_{X\setminus(2B)} \frac{|b_i(y)-b_i(z_i)||f_i(z_i)|}{\mu(B(y,d(y,z_i)))}\mathrm{d}\mu(z_i) \\
&\leq C\prod_{i=1}^{2}\Bigg(|b_i(y)-(b_i)_{2B}|\int_{X\setminus(2B)}\frac{|f_i(z_i)|}{\mu(B(y,d(y,z_i)))}\mathrm{d}\mu(z_i) \\
&\qquad +\int_{X\setminus(2B)}\frac{|b_i(z_i)-(b_i)_{2B}||f_i(z_i)|}{\mu(B(y,d(y,z_i)))}\mathrm{d}\mu(z_i)\Bigg) \\
&\leq C\prod_{i=1}^{2}\Bigg(|b_i(y)-(b_i)_{2B}|\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus(2^kB)}\frac{|f_i(z_i)|}{\mu(B(y,d(y,z_i)))}\mathrm{d}\mu(z_i) \\
&\qquad +\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus(2^kB)}\frac{|b_i(z_i)-(b_i)_{2B}||f_i(z_i)|}{\mu(B(y,d(y,z_i)))}\mathrm{d}\mu(z_i)\Bigg) \\
&\leq C\prod_{i=1}^{2}\Bigg(|b_i(y)-(b_i)_{2B}|\sum_{k=1}^{\infty}\frac{1}{\mu(B(x,2^kr))}\int_{2^{k+1}B}|f_i(z_i)|\mathrm{d}\mu(z_i) \\
&\qquad +\sum_{k=1}^{\infty}\frac{1}{\mu(B(x,2^kr))}\int_{2^{k+1}B}|b_i(z_i)-(b_i)_{2B}||f_i(z_i)|\mathrm{d}\mu(z_i)\Bigg) \\
&\leq C\prod_{i=1}^{2}\Bigg(|b_i(y)-(b_i)_{2B}|\sum_{k=1}^{\infty}\frac{1}{\mu(B(x,2^kr))}\int_{2^{k+1}B}|f_i(z_i)|\mathrm{d}\mu(z_i) \\
&\qquad +\sum_{k=1}^{\infty}\frac{1}{\mu(B(x,2^kr))}\int_{2^{k+1}B}|b_i(z_i)-(b_i)_{2^{k+1}B}||f_i(z_i)|\mathrm{d}\mu(z_i) \\
&\qquad\quad +\sum_{k=1}^{\infty}\frac{|(b_i)_{2^{k+1}B}-(b_i)_{2B}|}{\mu(B(x,2^kr))}\int_{2^{k+1}B}|f_i(z_i)|\mathrm{d}\mu(z_i)\Bigg) \\
&\leq C\prod_{i=1}^{2}\Bigg(|b_i(y)-(b_i)_{2B}|\sum_{k=1}^{\infty}\frac{1}{\mu(B(x,2^kr))}\|\chi_{2^{k+1}B}f_i\|_{L^{p_i(\cdot)}(X)}\|\chi_{2^{k+1}B}\|_{L^{p_i'(\cdot)}(X)} \\
&\qquad +\sum_{k=1}^{\infty}\frac{1}{\mu(B(x,2^kr))}\|\chi_{2^{k+1}B}f_i\|_{L^{p_i(\cdot)}(X)}\|\chi_{2^{k+1}B}(b_i(\cdot)-(b_i)_{2^{k+1}B})\|_{L^{p_i'(\cdot)}(X)} \\
&\qquad\quad +\sum_{k=1}^{\infty}\frac{k}{\mu(B(x,2^kr))}\|\chi_{2^{k+1}B}f_i\|_{L^{p_i(\cdot)}(X)}\|\chi_{2^{k+1}B}\|_{L^{p_i'(\cdot)}(X)}\Bigg) \\
&\leq C\prod_{i=1}^{2}\|f_i\|_{L^{p_i(\cdot),\varphi_i}(X)}\Bigg(|b_i(y)-(b_i)_{2B}|\sum_{k=1}^{\infty}\frac{\varphi_i(x,2^{k+1}r)}{\mu(B(x,2^kr))}\|\chi_{2^{k+1}B}\|_{L^{p_i'(\cdot)}(X)} \\
&\qquad +\sum_{k=1}^{\infty}\frac{1}{\mu(B(x,2^kr))}\varphi_i(x,2^{k+1}r)\|\chi_{2^{k+1}B}\|_{L^{p_i'(\cdot)}(X)}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \sum_{k=1}^{\infty} \frac{k}{\mu(B(x,2^k r))} \varphi_i(x,2^{k+1}r) \|\chi_{2^{k+1}B}\|_{L^{p_i'(\cdot)}(X)} \Bigg) \\
&\le C \prod_{i=1}^{2} \|f_i\|_{L^{p_i(\cdot),\varphi_i}(X)} \Bigg(|b_i(y) - (b_i)_{2B}| \sum_{k=1}^{\infty} \frac{\varphi_i(x,2^{k+1}r)}{\mu(B(x,2^k r))} \|\chi_{2^{k+1}B}\|_{L^{p_i'(\cdot)}(X)} \\
&\qquad + \sum_{k=1}^{\infty} \frac{k+1}{\mu(B(x,2^k r))} \varphi_i(x,2^{k+1}r) \|\chi_{2^{k+1}B}\|_{L^{p_i'(\cdot)}(X)} \Bigg) \\
&\le C \prod_{i=1}^{2} \|f_i\|_{L^{p_i(\cdot),\varphi_i}(X)} \Bigg(|b_i(y) - (b_i)_{2B}| \sum_{k=1}^{\infty} \frac{\varphi_i(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_i(\cdot)}(X)}} \\
&\qquad + \sum_{k=1}^{\infty} (k+1) \frac{\varphi_i(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_i(\cdot)}(X)}} \Bigg),
\end{aligned}
$$

further, by Lemma 2.3, Lemma 2.5 and (4.1), we obtain that

$$
\begin{aligned}
\mathrm{G}_4 &= \sup_{x\in X, r>0} \frac{1}{\varphi(x,r)} \|\chi_{B(x,r)}([b_1,b_2,BT](f_1^\infty, f_2^\infty))\|_{L^{p(\cdot)}(X)} \\
&\le C \prod_{i=1}^{2} \|f_i\|_{L^{p_i(\cdot),\varphi_i}(X)} \sup_{x\in X, r>0} \frac{1}{\varphi(x,r)} \\
&\quad \times \Bigg(\sum_{k=1}^{\infty} \frac{\varphi_i(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_i(\cdot)}(X)}} \|\chi_{B(x,r)} \prod_{i=1}^{2} (b_i(\cdot) - (b_i)_{2B})\|_{L^{p(\cdot)}(X)} \\
&\qquad + \sum_{k=1}^{\infty} (k+1) \frac{\varphi_i(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_i(\cdot)}(X)}} \|\chi_{B(x,r)}\|_{L^{p(\cdot)}(X)} \Bigg) \\
&\le C \prod_{i=1}^{2} \|f_i\|_{L^{p_i(\cdot),\varphi_i}(X)} \sup_{x\in X, r>0} \frac{1}{\varphi(x,r)} \\
&\quad \times \Bigg(\sum_{k=1}^{\infty} \frac{\varphi_i(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_i(\cdot)}(X)}} \|\chi_{B(x,r)} (b_i(\cdot) - (b_i)_{B})\|_{L^{p_i(\cdot)}(X)} \\
&\qquad + \sum_{k=1}^{\infty} (k+1) \frac{\varphi_i(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_i(\cdot)}(X)}} \|\chi_{B(x,r)}\|_{L^{p_i(\cdot)}(X)} \Bigg) \\
&\le C \prod_{i=1}^{2} \|f_i\|_{L^{p_i(\cdot),\varphi_i}(X)} \sup_{x\in X, r>0} \frac{1}{\varphi(x,r)} \\
&\quad \times \Bigg(\sum_{k=1}^{\infty} \frac{\|\chi_B\|_{L^{p_i(\cdot)}(X)}}{\|\chi_{2^{k+1}B}\|_{L^{p_i(\cdot)}(X)}} \varphi_i(x,2^{k+1}r) \\
&\qquad + \sum_{k=1}^{\infty} (k+1) \frac{\varphi_i(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_i(\cdot)}(X)}} \|\chi_{B(x,r)}\|_{L^{p_i(\cdot)}(X)} \Bigg) \\
&\le C \|f_1\|_{L^{p_1(\cdot),\varphi_1}(X)} \|f_2\|_{L^{p_2(\cdot),\varphi_2}(X)}.
\end{aligned}
$$

Which, together with G_1, G_2 and G_3, the proof of Theorem 4.2 is completed. □

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