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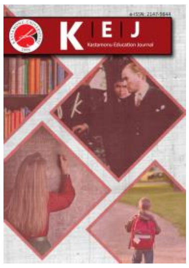

| Research Article / Araştırma Makalesi |

What Is Mathematical Understanding and How Is It Achieved? Preservice Middle School Mathematics Teachers' Views and Reflections in Their Teaching Practices

Matematiksel Anlama Nedir ve Nasıl Gerçekleştirilebilir? Ortaokul Matematik Öğretmen Adaylarının Görüşleri ve Öğretim Uygulamalarından Yansımalar

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Keywords

1.Mathematics education
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Abstract

This research was designed to determine the views of preservice middle school mathematics teachers on what mathematical understanding is and how to achieve it, as well as the reflections of these views in their teaching practices. The participants included six preservice teachers studying in the final year of the Middle School Mathematics Teaching Program at a state university in Turkey. The research was carried out during the Teaching Practice course. The data consisted of the lesson plans prepared by the preservice teachers, video recordings of the lessons in which they put those plans into practice, field notes taken during lesson observations, and transcripts of one-on-one interviews with the preservice teachers after the lessons. Participants related mathematical understanding to interpreting situations or problems encountered in daily life mathematically, representing mathematical ideas, answering every question related to the subject, and developing a positive attitude toward mathematics. They began their lessons with daily life examples and then reminded the students of their prior knowledge, presented definitions of concepts, and worked with students on sample exercises using different representations during the lessons. These views and practices are discussed in the context of preservice teachers' designs of learning environments that promote mathematical understanding.

Öz

Bu araştırmanın amacı, ilköğretim matematik öğretmen adaylarının matematiksel anlamanın ne olduğuna ve matematiksel anlamanın nasıl gerçekleşeceğine yönelik görüşlerini ve bu görüşlerin öğretim uygulamalarına yansımalarını belirlemektir. Araştırmanın katılımcıları, bir devlet üniversitesinde İlköğretim Matematik Öğretmenliği Programı'nın son sınıfında öğrenim gören altı öğretmen adayıdır. Araştırma, Öğretmenlik Uygulaması dersi kapsamında gerçekleştirilmiştir. Araştırmanın verilerini adayların hazırlamış oldukları ders planları, bu planları uygulamaya geçirdikleri derslerin video kayıtları ve ders gözlemleri sırasında tutulan alan notları ile derslerden sonra adaylarla gerçekleştirilen birebir görüşmelerin transkript edilmesiyle elde edilen notlar oluşturmaktadır. Katılımcılar, matematiksel anlamayı ile günlük hayatta karşılaşılan durum veya problemleri matematiksel olarak yorumlamak, matematiksel fikirleri temsil etmek, konu ile ilgili her soruyu yanıtlamak ve matematiğe karşı olumlu bir tutum geliştirmekle ilişkilendirmiştir. Derslerine günlük hayattan örneklerle başlamışlar ve sonra öğrencilere ön bilgilerini hatırlatmışlardır, kavramların tanımlarını sunmuşlar ve öğrencilerle farklı temsilleri kullanarak örnek alıştırmalar üzerinde çalışmışlardır. Bu görüş ve uygulamalar, öğretmen adaylarının matematiksel anlamayı teşvik eden öğrenme ortamları tasarlamaları bağlamında tartışılmıştır.

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INTRODUCTION

One of the main ideas that mathematics education researchers and curricula commonly emphasize is that students should learn mathematics with understanding. For example, Cai and Ding (2017) noted that researchers have been conducting extensive studies on mathematical understanding for many years, while Simon (2017) stated that mathematics education researchers and instructional designers take mathematical understanding as a goal of mathematics instruction. In the curricula applied in recent years in Turkey, the importance of including practices that will enable students to create meaning in mathematics lessons was emphasized (see Ministry of National Education [MEB], 2013, 2017). Current middle and high school mathematics curricula are also designed to provide "meaningful and permanent learning" (MEB, 2018a, 2018b, p. 4).

Teachers are the first to implement these ideas, which are underlined in both the literature and curricula. Since designing meaningful activities imposes a significant workload on teachers, teaching mathematics for understanding requires much effort (Van de Walle et al., 2012). Therefore, it is crucial to determine what mathematics teachers understand by designing instruction and implementing practices to promote understanding. Mathematical understanding is at the center of studies in critical areas such as the training of mathematics teachers and curriculum development or interactions in the classroom (Mousley, 2005). In this study, the views of preservice middle school mathematics teachers (henceforth PTs), expected to become mathematics teachers soon, on mathematical understanding and reflections of those views in their teaching experiences will be examined.

Mathematical Understanding

The efforts of researchers to explain understanding have been ongoing for many years (Meel, 2003). For example, Richard R. Skemp (1987), one of the most cited researchers in the literature on mathematical understanding, discussed mathematical understanding in three different categories: instrumental, relational, and logical understanding. He described instrumental understanding as performing a mathematical action only by following rules, relational understanding as performing a mathematical action while knowing why it is done, and logical understanding as presenting the action as "a valid sequence of logical inferences" (p. 170). In this context, instrumental understanding includes the routine use of mathematical rules or algorithms without knowing their reasons, while relational understanding consists of associating mathematical actions with appropriate schemas (Skemp, 1978). Although relational understanding puts essential responsibilities on the shoulders of teachers, it saves students from rote learning and offers them the advantages of permanent learning and better problem-solving skills (Van de Walle et al., 2012).

Researchers generally evaluate mathematical understanding in two ways: as a process or as a product obtained as a result of a process (Cai & Ding, 2017). Some researchers (e.g., Davis, 1992; Hiebert & Carpenter, 1992) have explained mathematical understanding as a network of mental representations developed for mathematical concepts or features that is expanded with every new piece of information. According to this approach, mathematical understanding, more than simply having mathematical knowledge or knowing a mathematical property, "can be defined as a measure of the quality and quantity of connections that an idea has with existing ideas" (Van de Walle et al., 2012, p. 23). On the other hand, some researchers such as Pirie and Kieren (1994) have described mathematical understanding as a "whole, dynamic, levelled but non-linear process of growth" (p. 166).

Sierpinska (1994) suggested three different ways of understanding in mathematics: the act of understanding, understanding, and the processes of understanding. Duffin and Simpson (2000), who stated that they were impressed by Sierpinska's ideas about understanding, indicated that understanding in mathematics education should include three different perspectives: building understanding, having understanding, and enacting understanding. They described having and using understanding as follows:

> In imagining a ball with potential energy, we note that we cannot directly see that energy. However, when the ball is falling, *while we still cannot see the energy*, we can infer from our observation of the ball moving and our theoretical perspective on the meaning of energy that the ball both has some kinetic energy and began with some potential energy. Similarly, we suggest that when a learner has some understanding, we have no way of directly seeing it. However, when they enact that understanding, while we still cannot directly see it, we can begin to infer from our observations of the physical actions the learner performs, and our theoretical perspective on the meaning of understanding, something about the form of understanding the learner previously had. (pp. 419-420)

Usiskin (2015) developed a multidimensional approach to mathematical understanding and stated that understanding should be considered within the five different dimensions of skill-algorithm, property-proof, use-application, representation-metaphor, and history-culture. Accordingly, while students understand mathematical procedures and how to use algorithms fluently in the skill-algorithm dimension, they understand that they are aware of the mathematical properties and can explain why their answers are correct in the property-proof dimension. The use-application dimension corresponds to the understanding of when and where mathematical concepts or procedures can be applied in real-life situations, while the representation-metaphor dimension corresponds to understanding regarding different representations of mathematical concepts. Finally, the dimension of history-culture is related to the historical development of a concept and the knowledge that must be possessed about how different cultures use that concept. Görgüt and Dede (2020) conducted a study in which the views of mathematics teachers about evaluating students' mathematical understanding were determined, and they found that teachers partially included the indicators of the first four dimensions in their evaluations; furthermore, they primarily focused on the indicators in the skill-algorithm dimension.

Many national and international studies have shown that students face various difficulties because they do not learn mathematics with understanding (Cai, 2004; Richland et al., 2012; Sengul & Argat, 2015). How can it be explained that students' mathematical understanding is below expectations even though its importance is clearly emphasized in mathematics curricula? The first reasons that come to mind are related to components of the teaching process. For the learning of mathematics in a meaningful way, discussions and research are ongoing, especially regarding teachers' roles (Cai & Ding, 2017). Hiebert and Grouws (2007) compiled the ideas and results presented in the literature on how students learn mathematics with understanding and stated that two main features can guide teachers in classroom mathematics discussions. Accordingly, learning with understanding can occur when mathematics instruction includes "explicit attention to connections among ideas, facts and procedures, and engagement of students in struggling with important mathematics" (p. 391).

In the field of mathematics education, a need still remains for robust and useful teaching theories that will guide instruction (Hiebert & Grouws, 2007). Most mathematics and preservice teachers have problems bringing the results or ideas that researchers have proposed about students' mathematical understanding into the classroom (Goodell, 2000; Weiss et al., 2003). To achieve mathematical understanding, students need to experience many mathematical thinking processes, such as problem-solving, searching for patterns, making assumptions, interpreting data, and verifying (Stein et al., 1996). The present study focuses on how mathematical understanding is understood by PTs and what they do in their lessons so that students can learn mathematics with understanding. Mathematical understanding is generally accepted to have the following characteristics described by Cai and Ding (2017):

> (1) Understanding is both a process of understanding (or knowing) and a result of the act of understanding (sometimes called "knowledge"); (2) understanding is both the making of connections and a result of making connections; (3) understanding is a dynamic and continual process; (4) understanding may have different levels and different kinds; and (5) the goal is to reach a deep mathematical understanding. (p. 8)

The purpose of this research is to determine the views of PTs on what mathematical understanding is and what needs to be done to achieve it, as well as the reflections of those views on their teaching practices. This research is important in determining how preservice teachers' education in undergraduate courses, including the ideas mentioned in mathematics education literature and predominantly theoretical discourses, is reflected in practice. Goodell (2000) stated that preservice mathematics teachers should be supported while designing classroom applications that will provide mathematical understanding. Determinations of what kind of support should be provided to preservice mathematics teachers depend primarily on determining how they describe mathematical understanding and how they design their lessons to promote such understanding. Therefore, the views and practices of PTs about mathematical understanding were taken as the basis of this study. Although it does not contain generalizable results, the study's findings may help identify the teacher training practices to be provided to preservice teachers who will teach mathematics. In order to discuss the research findings more consistently, the practices of PTs for mathematical understanding are limited to the subject of transformation geometry. In this context, the research problems were as follows:

According to PTs, what is mathematical understanding and how is it achieved by students?

How do PTs design their lessons to teach mathematics for understanding?

METHOD

This research, which was conducted to determine PTs' views of mathematical understanding and how students achieve it, and to examine how they teach in their classes to promote mathematical understanding, is a case study. Case studies involve "an in-depth exploration of a bounded system (e.g., an activity, an event, a process, or an individual) based on extensive data collection" (Creswell, 2007, as cited in Creswell, 2012, p. 617).

Participants

The research participants were six preservice teachers studying in the final year of the Middle School Mathematics Teaching Program at a state university in Turkey. While selecting participants, special attention was paid to the fact that these PTs had completed the previous mathematics and mathematics education courses in the undergraduate program. The PTs made observations in middle schools within the scope of the Teaching Practice course in the first semester of the academic year and successfully completed that course.

Research Context and Data Collection

The research was carried out during the Teaching Practice course that the PTs took in the second semester of their senior year. Within the scope of this course, the PTs made observations and conducted practices in mathematics lessons in a public middle school during the first eight weeks of the semester. A meeting was held with the PTs every week, and interviews were conducted about their practices in the previous and upcoming weeks. In this context, discussions were held about the observations and practices they carried out in previous weeks, and they were given feedback about the lesson plans and practices they had prepared. At the end of eight weeks, each of the PTs was asked to prepare lesson plans that would enable students to learn transformation geometry concepts with understanding and apply them in the classroom. While designing or implementing their plans, the PTs were not interfered with or guided. The first author observed the practices performed by each PT in the classroom. When all the PTs had finished implementing their lessons, semi-structured individual interviews were conducted with them. The interviews took approximately one hour, and all interviews were recorded. The PTs were asked questions such as "What is mathematical understanding?" "How do you think mathematical understanding is achieved?" and "How do you determine that a student is learning mathematics with understanding?"

Data Analysis

The data consisted of the lesson plans prepared by the PTs for certain learning outcomes related to transformation geometry, video recordings of the lessons in which they put those plans into practice, field notes taken during lesson observations, and notes obtained by transcribing the one-on-one interviews with the PTs after their lessons.

The data were analyzed in two stages. First, the sentences used by the PTs while explaining what mathematical understanding is and how it is achieved were analyzed in detail. At this stage, if the coding for each PT's answers matched existing categories, they were included in those categories. If not, new categories were created. The views of the PTs about what mathematical understanding is and how it is achieved were grouped within four different categories: i) interpreting a situation/problem encountered in daily life mathematically, ii) representing mathematical ideas, iii) being able to answer every question related to the subject, and iv) developing a positive attitude toward mathematics.

The findings obtained by coding the contents of the PTs' lesson plans and their processes of applying those plans in the classroom were brought together within three different categories: i) giving examples from daily life, ii) using previous knowledge and presenting definitions, and iii) applications with different representations. The data obtained from the interviews were coded separately by the researchers. The researchers regularly came together during the coding process and discussed their results until they agreed on the codes.

After the interview data were coded, the recordings of each PT's lessons were transcribed and analyzed by the first researcher. All of the PTs' activities that were considered to have been done to promote mathematical understanding were divided into sections and codes were created for those sections. For example, when a PT used concrete material during a lesson to help students understand how the distance to a reflection line would be determined while reflecting a figure, that section was coded as *making use of different representations*. Similarly, if a PT requested or gave examples of congruent figures in daily life in order to attract students' attention to the subject, it was coded as *connecting the subject with daily life*. Two other researchers also coded 35% of the lesson data and the reliability of the codes was checked. In this process, the observation notes of the first author and the written lesson plans prepared by the PTs were also used to support the coding of lesson data. In addition, the answers to interview questions in which the PTs explained the purposes and contents of the practices they applied in their lessons were also used to support the data.

FINDINGS

The findings are presented below under sub-headings reflecting the categories that emerged from the analysis of the data obtained for each research problem. In this context, the ideas that shaped the PTs' views about what mathematical understanding is and how students can achieve it are provided first, and then the teaching practices that they applied while conducting their lessons in the classroom will be addressed.

What is mathematical understanding and how is it achieved?

When the PTs' views on mathematical understanding and what needs to be done for students to achieve it were analyzed, four categories arose. These were i) interpreting a situation/problem encountered in daily life mathematically, ii) representing mathematical ideas, iii) being able to answer every question related to the subject, and iv) developing a positive attitude toward mathematics. Details of these categories are respectively presented below using quotations that reveal the PTs' views.

Interpreting a situation/problem encountered in daily life mathematically

While explaining mathematical understanding, all of the PTs talked about the ability to interpret situations or problems encountered in daily life by expressing them mathematically. Statements given by Safiye[4] and Kumru in this regard were as follows:

Safiye: [It is] to explain any more complex problem in daily life or business life, for example, in the field of engineering, using mathematical expressions. These problems can be advanced problems, or they can be done in a way that anyone can do; I mean, they can be effortless.

[4] All names used in this study are pseudonyms.

Kumru: Hmm, mathematical understanding. It's being able to see and think about anything in daily life mathematically.

Other PTs used similar expressions. Dilay described mathematical understanding as "the ability to interpret any event we see mathematically," while Serra explained it as "translating problems into mathematics and solving them." Sena stated that mathematical understanding is "thinking about things in daily life mathematically" and Sude indicated that it is "evaluating a situation mathematically." The PTs seemed to view acts of mathematically interpreting any situation encountered in daily life or using mathematical knowledge and skills while solving a problem in any field other than mathematics within the scope of mathematical understanding.

Representing mathematical ideas

When the expressions used by the PTs to explain mathematical understanding were examined, one of the emerging categories was related to mathematical representations. For example, Safiye, Sena, and Serra mentioned mathematical language and transitions between representations while expressing their views of mathematical understanding. Serra's statement in this regard was as follows:

Serra: I mean, [it is] expressing what is given verbally with a little more mathematical things, expressing it mathematically. We call it mathematical language; in fact, [mathematical understanding is] to comprehend it. I immediately think of problems. The problems are verbal. We solve them by mathematical modeling. That means we understand it mathematically, transform it into mathematics, and solve it. In no way do we think of [the problem] in the form of a paragraph. But, for example, according to a Turkish teacher, [the problem] is just a paragraph. We, for example, transform it into a shape. We express with numbers, with unknowns.

Similar to Serra, Sena described it as "[being] able to process a verbal expression numerically or translate a numerical expression into a verbal expression." Safiye indicated that "[it is] to think of any problem mathematically, to set up the equation according to it, to explain it with algebraic expressions." The PTs expressed verbal and symbolic representations and the transitions between those representations as indicators of mathematical understanding.

In addition, Safiye, Serra, Kumru, and Dilay pointed to the use of different representations as one of the necessary conditions for achieving mathematical understanding. The statements of Safiye and Dilay related to this category were as follows:

Safiye: Using materials that will attract students' attention further increases their understanding. It is more effective in their understanding.

Dilay: Middle school subjects are better understood if we use daily life, concrete objects, or visualization.

For students to achieve mathematical understanding, the PTs emphasized the use of different representations. As can be seen from their statements above, they focused on transitions between verbal and symbolic representations while discussing mathematical understanding. They also focused on visual representations and concrete materials to help students understand mathematical subjects.

Being able to answer every question related to the subject

The PTs stated that answering every question related to a mathematical subject is an indicator of mathematical understanding of that subject and a way of achieving mathematical understanding. Relevant statements by Dilay and Sena were as follows:

Dilay: If a student can correctly answer when you ask a question, it means she understands.

Sena: How do I know that I understand [something] mathematically? I answer the questions. When I come across a question about that subject… if I can answer that question in a really comfortable way… I mean, if I can solve not just a single example but several examples for that subject, then I have really understood the subject.

According to Dilay and Sena, being able to answer questions about a subject shows that the person understands that mathematical subject. In addition, Safiye and Serra stated that answering many different questions would help students understand mathematical subjects. They respectively said: "For example, while I am working on the subject, I can understand more easily if I answer a lot of questions and see different types of questions" and "She also needs to answer questions. Students can become good at subjects by solving many."

Developing a positive attitude toward mathematics

According to Sude, Safiye, Serra, and Sena, one of the conditions for achieving mathematical understanding is to have a positive attitude toward mathematics. Relevant statements by Sude and Serra in this regard were as follows:

Sude: Understanding mathematics [is], valuing it, working on it. In other words, I think that whatever subject the student enjoys both facilitates her understanding and increases her effort to understand it. … She should give it value. We always try to pay attention to this at the beginning of the lesson plans. … The lesson and students' interest will be good if we can make a striking beginning while creating the plans. By striking, I mean attracting students' attention, saying things that will push them to work in terms of motivation, talking about things that will work for them, or using language that will appeal to them.

Serra: I think the teacher should be able to endear herself first. Then she should be able to make students love mathematics. It actually seems like math is a terrifying thing, but if a student realizes that she is actually using mathematics, she doesn't look at it as something foreign.

As can be seen from these statements by Sude and Serra, PTs used expressions such as "enjoying," "valuing," "attracting attention," "liking the teacher," and "usefulness" to emphasize that students should first develop a positive attitude toward mathematics in order to achieve mathematical understanding. Similarly, Safiye stated that "students should be interested in and love mathematics" to learn mathematical concepts with understanding.

What do preservice teachers do to promote mathematical understanding in their teaching practices?

Lesson plans and the ways in which the PTs implemented those plans in the classroom were analyzed to address the second research problem. As a result of the analysis, three categories were determined: i) giving examples of daily life, ii) using previous knowledge and presenting definitions, and iii) applications with different representations. Details about these categories and sample images from the practices of the PTs are given below.

Giving examples of daily life

The PTs started their lessons by asking students how they might encounter the concepts of transformation geometry in daily life. For example, Sude, who dealt with the concept of congruence, reflected visuals of the game known as Find the Differences on the board (Figure 1-a) and asked the students questions such as "What is the logic of the Find the Differences game?" and "How do we find the different ones?" Then she stated that the visuals should be congruent so that there would be no difference between the shapes she reflected on the board. Afterward, Sude asked for the students' opinions on tangram applications and Escher tessellations and asked them to think about where the word congruence is used in daily life. Based on the various answers from the students, she talked about cakes made with the same cake molds, the congruence of the tiles on the floor of the classroom, and the congruent shapes used in architecture and jewelry design. Another PT, Serra, presented examples of Escher tessellations at the beginning of her lesson so that students could work on creating patterns (Figure 1-b). In addition to verbal or static visual examples, it was observed that PTs also used some dynamic elements in the introductions of their lessons. For example, while discussing the concept of translation, Dilay and Sena performed brief dances with volunteering students, in which they used figures they associated with translation (Figure 1-c).

a b c d

Figure 1. Examples of daily life used by PTs at the beginning of the lesson

Safiye and Kumru, who dealt with the learning outcomes related to reflection, tried to connect the concept with the students' daily life experiences by using concrete materials at the beginning of their lessons. Both of these PTs talked about mirrors and the images in mirrors. For example, Safiye defined reflections with a mirror she brought to class and then explained the properties of reflection in the mirror. Safiye invited a student (S1) to the board and asked some questions about her reflection in the mirror (Figure 1-d). Safiye also included the other students in the discussion and the following dialogue took place:

Safiye: Now, reflection has some properties. We will use the mirror to see these properties. [*Turning to S1*] What do you see when you look in the mirror?

S1: Myself.

Safiye: Yourself. [*Turning back toward the class*] So, what has happened? Exactly the same as your friend. Well, what do we call figures that are exactly the same?

Students: Congruent figures.

Safiye: Then what can we say about the figure and its image? They become congruent. Well, the mirror looks like this. Let your friend take a step back. Please get away from the mirror [says to S1]. When our friend moves away from the mirror, does her image also move away from the mirror?

Students: Yes.

Safiye: How can we explain this?

Students: The further the person moves away from the mirror, the further away the image gets in the mirror.

Safiye: What else? For example, we see the exact same thing again. There is no shrinking; that is, the shape and the size of the shape don't change. At the same time, what else can we say here? Now our friend is here, and the mirror is here. Her image is there. What can we say about these three things?

Students: There is a distance inside the mirror as well as the distance to the mirror.

Safiye: Exactly.

As can be seen from the dialogue above, Safiye presented the mirror as a daily life example for the concept of reflection along with the questions she posed to the students in the classroom. The PTs tended to remind students of their prior knowledge about the concepts they were working on and presented the definitions after giving the daily life examples they used while starting their lessons. The following section will detail how PTs used prior knowledge and definitions during their lessons.

Using previous knowledge and presenting definitions

Serra tried to create a learning environment that considered the students' prior knowledge after connecting the concept of congruence with daily life examples in her lesson. For this, she encouraged her students to think about the congruent angles that they had worked on before. Asking students to remember how congruent angles can be defined, Serra created some examples of congruent angles on the board with a volunteering student (Figure 2-a). Then, within the scope of the learning outcomes in the curriculum, she questioned the properties of the basic elements (sides, angles) of polygons and asked the students what the necessary conditions would be for two polygons to be congruent. Evaluating the answers from the students by drawing two rectangles and two trapezoids on the board, Serra continued to discuss whether these polygons were congruent or not (Figures 2-b and 2-c). In order to better understand the teaching environment in the classroom during these actions, the following dialogue can be considered:

Serra: Now, guys, the directions these angles [*See Figure 2-a*] look toward are actually different, right? It would be difficult to say that they are congruent if the angle measurements were not given. In other words, for us to say [that they are] congruent, the measures of the angles must be equal. Okay, let's go [from here] to congruent polygons. What is a polygon?

Student 1: Closed shapes with three or more sides.

Serra: What should I do if I want to draw two congruent polygons? [*A short pause*] What elements do polygons have? We said that for angles to be congruent, their measures must be the same. So, what else do you think needs to happen for polygons to be congruent?

Student 2: Sides.

Student 3: Interior angles.

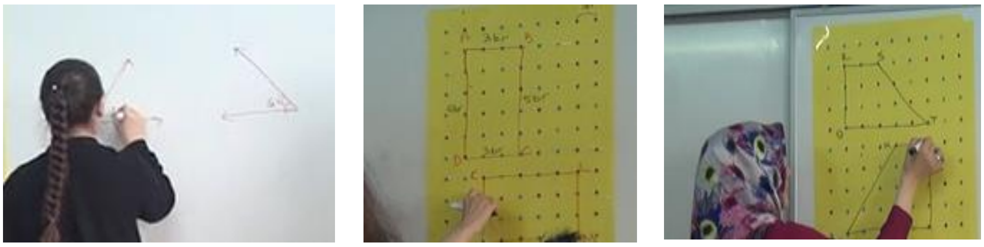
Serra: Well, let me first draw two rectangles for you. [*See Figure 2-b*] How do we decide if these two rectangles are congruent?

Student 4: The corner numbers are the same.

Serra: Yes, why? Because they are both rectangular. What can we say more decisively?

Student 5: If we count the units, the sides of both are congruent.

Serra: Yes, from here, we call polygons or shapes that have the same side lengths and corresponding angles congruent polygons.



a b c

Figure 2. Drawings made while using previous knowledge in Serra's lesson

Serra strove to make the conditions necessary for two polygons to be congruent meaningful for students based on the students' prior knowledge about congruent angles. It was also found important that Serra emphasized prior knowledge during her interview. She expressed her ideas about mathematical understanding and how students can learn mathematics with understanding as follows: "From the very basics, it's necessary to go on top of it by adding to it. The curriculum is already trying to do that. [A student] must learn to progress by adding in this way and also have a solid foundation. It is very difficult [for the student] to learn a second thing without learning the first thing." However, Serra was the only PT who made an effort to design her lesson with the use of the students' prior knowledge. The other PTs were content to restate the learning outcomes of the previous lessons in certain parts of the lessons they began with examples from daily life. All PTs continued their lessons by giving the definitions of the concepts included in the learning outcomes directly after the processes mentioned above. PTs, who stated that they used the textbook or the internet for these definitions, began working on exercises that contained different representations of the relevant learning outcomes.

Application with different representations

While designing their lessons, the PTs benefited from teaching tools offering different representations. For example, Kumru, Sude, Sena, and Safiye preferred to use simple concrete materials that they prepared to be erased and reused to better represent the planar figures they dealt with. Their materials included the axes, origin, and unit squares of a coordinate system (Figure 3-a). Kumru and Sude also used the GeoGebra program to represent the mathematical concepts and features they discussed (Figure 3-b).

a b

Figure 3. Examples of PTs' representation use

In addition to drawing on their materials, Serra, Safiye, and Dilay used previously prepared objects to represent some planar figures. For example, Serra pasted the concrete objects seen in Figure 4 onto the board and said the following to identify the congruent objects among these objects together with the students:

Serra: Now, if I ask which ones are the same? [*A short pause*] Does anyone want to come and do it on the board? We'll attach the same shapes on top of each other. [*A student is trying to identify congruent objects among the objects Serra presented; see Figure 4-a*] When identifying congruent figures, we were looking at the side lengths, but when I have the opportunity, I can take the shapes and put them on top of each other to decide if they're congruent. Does anyone else want to do it? [*A student she has called to the board does another exercise; see Figure 4-b*] Now, how was the pink figure actually located here? Is there a connection between the figure's location and whether it is congruent to another figure? It was located like this. [*See Figure 4-c*] I took it to see if it was congruent [to the other figure], turned it, and put them right on top of each other. Since they overlapped exactly, they were congruent figures.[5]

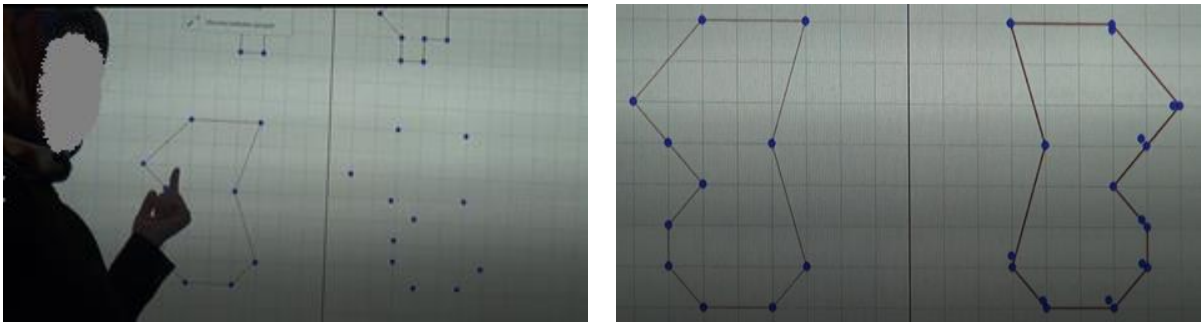
[5] The color of the figures was ignored.

a b c

Figure 4. Concrete materials that Serra used to explain congruent figures

During their lessons, the PTs also used other concrete materials to present different representations of the concepts they were working on. For example, Serra handed out isometric paper, and Safiye dotted papers for the students to practice with. Kumru and Safiye used mirrors or symmetry mirrors to address the properties of reflection.

The PTs all used different representations during their lessons, and they used them for different purposes. For example, Kumru, one of the PTs using GeoGebra, directly showed the reflection of certain points about a line by using the relevant tool of the program. Later, she used the program to check what she had done with the students. For this, she asked the students to determine the corner points of the image formed as a result of the reflection of the given planar figure on the screen (Figure 5-a). After a discussion about the location of points, she checked the answers using the related software tool (Figure 5-b).



a b

Figure 5. Examples used by Kumru in reflection about a line

Sude, on the other hand, actively used the dragging feature in GeoGebra while addressing the concept of congruence. As can be seen in Figure 6, she created a polygon and stated that all its sides should be taken into account to create a polygon congruent to the initial polygon. Later, when the red line segment that she created was to be considered as a sideof this polygon, she asked how the endpoints of the blue line segment should be dragged to be congruent to this line segment (Figure 6).

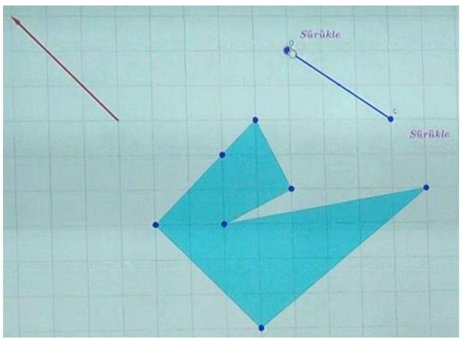
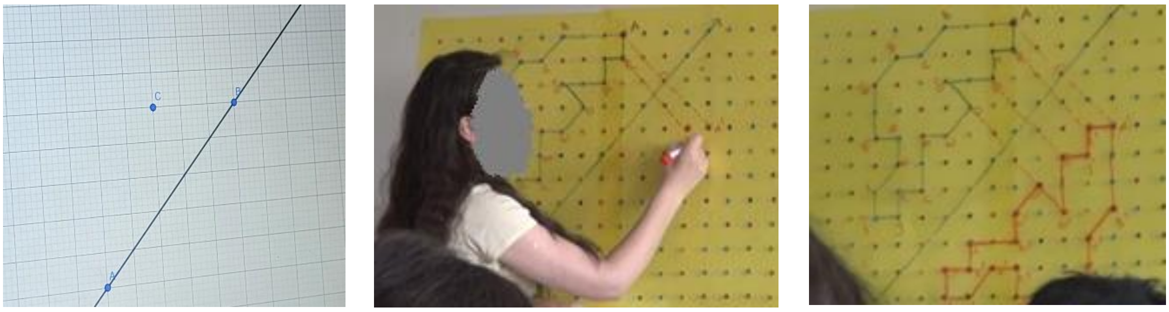


Figure 6. The drawings that Sude used while discussing congruent figures

In addition to the teaching tools mentioned above, PTs also carried out various activities based on students' movements in the classroom. For example, Kumru drew a reflection line on the ground and invited a student to come forward. She asked the student to act as her image about this line. Then she moved randomly and waited for the student to adjust his position according to the line at the end of each movement.[6] Dilay, on the other hand, asked two students to stand in different places in the classroom, placing them at the board, and asked the other students in the class to bring these two students together at a common point with translations.

While presenting the definitions and properties of the concepts in their lessons, the PTs used examples with different difficulty levels with the representations mentioned above in the tasks that they chose. For example, Kumru, who dealt with the properties of a reflection about an oblique line, was limited to examples in which a point was reflected about the line (Figure 7-a). At the same time, Safiye used examples that included more complex planar figures (Figures 7-b and 7-c).



a b c

Figure 7. Examples that PTs used in reflection

DISCUSSION

Although there are many studies on what mathematical understanding is and the ways to achieve it (e.g., Pirie & Kieren, 1994; Skemp, 1978; Usiskin, 2015), studies conducted in different countries show that students still struggle to learn mathematics with understanding (Cai, 2004; Li et al., 2008; Torbeyns et al., 2009). The present study aimed to determine the views of PTs on what mathematical understanding is and how to achieve it, as well as identifying the reflections of those views on their teaching practices. The PTs were first asked to prepare lesson plans in which they would teach about the learning outcomes related to geometric transformations and apply them in the classroom. After implementing their lessons, the PTs were asked about their views of mathematical understanding and how students achieve it.

When the statements of the PTs about what mathematical understanding is were analyzed, one of the most prominent categories included statements about interpreting a situation or problem encountered in daily life in a mathematical way. This has common points with the use-application dimension that Usiskin (2015) considered as a component of mathematical understanding. It can also be said that this understanding, which corresponds to the understanding of when mathematical concepts or procedures can be applied in real-life situations (Usiskin, 2015), is evaluated within the scope of mathematical modeling or literacy in the current mathematics curriculum in Turkey (MEB, 2018a). In making connections, which is one of the mathematical process standards, it is stated that students should connect mathematical concepts with daily life and recognize their applications in different contexts outside of mathematics (National Council of Teachers of Mathematics [NCTM], 2000). Although the PTs clearly described these skills while explaining mathematical understanding, it was observed that they were generally content with only mentioning daily life examples for the concepts they discussed during their lessons. In other words, they preferred to present examples from daily life about the concepts rather than creating a discussion environment by considering the concepts in the context of daily life or a learning environment in which they would discuss the features of the concepts in light of real-life problems. For example, Kumru and Safiye talked about mirrors as an example while discussing reflections, while Sude mentioned cake molds and tiles on the classroom floor for the concept of congruence. Similar to these findings, Coşkun (2013) reported that teachers used verbal examples to connect mathematical concepts with real life but neglected to deal with the related concepts contextually.

On the other hand, it can be said that these PTs aimed to attract the attention of their students by giving examples such as the Find the Differences game and Escher tessellations in the introductory parts of their lessons. One of the components that each PT emphasized while explaining mathematical understanding was the importance of affective variables of students in dealing with mathematics. The PTs noted that students should develop positive attitudes toward mathematics to achieve mathematical understanding, using expressions such as “enjoying,” “valuing,” “attracting attention,” “liking the teacher,” and “usefulness.” These views may be the underlying reason why the PTs used various examples to connect mathematical concepts with daily life. It is known that affective factors play a role in cognitive efforts while learning mathematics (McLeod, 1992). These PTs seemed to use daily life examples related to the concepts mostly for attracting attention.

[6] The fact that reflections create congruent figures was ignored.

Students' prior knowledge plays a critical role in the development of mathematical understanding (Pirie & Kieren, 1994). Therefore, when designing mathematics lessons, teachers should design their instructions by considering students' prior knowledge for understanding mathematics (Kilpatrick et al., 2001). When the PTs' views on mathematical understanding were examined, except for Serra, no ideas were found that correlated mathematical understanding with students' prior knowledge. In their lessons, the PTs were content to tell the students the relevant subjects they had encountered in previous lessons rather than teaching them based on prior knowledge. Only Serra revisited the prior knowledge of congruent angles for students to make sense of congruent polygons. Other PTs directly presented the definitions of the concepts after giving daily life examples and stating the previous relevant topics.

Another category emerging from the expressions of these PTs regarding mathematical understanding was directly related to representations, which is one of the mathematical process standards of the NCTM (2000). Many researchers explaining mathematical understanding (e.g. Goldin & Shteingold, 2001; Lesh et al., 1987; Pape & Tchoshanov, 2001; Usiskin, 2015) have noted the role of students' experiences with representations. These researchers have stated that students' ability to recognize a mathematical concept with its different representations and switch between representations while making applications relevant to that concept is a requirement and an indicator of mathematical understanding. Contrary to the findings about relating daily life to mathematical understanding, these PTs' views on the role of representations in mathematical understanding and their classroom practices overlapped in certain regards. PTs who used different concrete materials and visual representations in their practices preferred to have students work on tasks with varying degrees of difficulty that included those representations. For example, Safiye asked questions about reflected figures using a mirror and Serra guided a discussion about the conditions of congruent polygons with applications using a material she had prepared. Kumru was limited to GeoGebra examples where a point was reflected about an oblique line, and Safiye used examples involving more complex planar figures on concrete material. However, the PTs generally preferred to utilize these representations themselves during their instruction. They asked the students questions that they expected to be answered quickly while showing and telling. In other words, although the students saw different representations, they did not experience a learning process in which they would discover similarities, differences, or relationships between those representations. The PTs mostly established relationships between concrete material/visual and verbal representations. Students were not offered rich opportunities to work with symbolic representations or establish connections among multiple representations. Although the PTs said that mathematical understanding meant expressing encountered situations or ideas mathematically, they did not reflect those views sufficiently in their lessons.

Mathematical understanding emerges from mental activities in which students establish relationships, expand and apply mathematical knowledge, think about their experiences, express what they know clearly, and transform mathematical knowledge into their own knowledge (Carpenter & Lehrer, 1999). Considering the lesson plans and practices of these PTs, it was seen that they did not actively engage their students with mathematical ideas, as emphasized by Hiebert and Grouws (2007). During the PTs' lessons, the students did not have opportunities to experience the mathematical thinking processes that Stein et al. (1996) stated are necessary for achieving mathematical understanding.

CONCLUSION

This research has determined what PTs think about mathematical understanding and what they do while teaching to promote mathematical understanding. According to the PTs who participated in this study, mathematical understanding is related to interpreting a situation or problem encountered in daily life mathematically, representing mathematical ideas, answering every question related to the subject, and developing a positive attitude toward mathematics. The PTs started their lessons with daily life examples and then reminded the students of their prior knowledge, presented the definitions of the concepts, and worked with students on sample exercises using different representations during the lessons they implemented according to their plans. Therefore, it can be said that there was only superficial similarity between the PTs' views and practices regarding mathematical understanding. Comparative analysis of their views and practices showed that the PTs' practices were inefficient in creating learning environments that would support mathematical understanding. The situations and problems encountered in daily life were limited to examples aimed at attracting students' attention to the lesson. Students were not provided with enough opportunities to connect different representations, and activities to support conceptual understanding were not offered during the lessons. In other words, the practices of the PTs were far from problem-based teaching in which students would model daily life situations or an approach based on exploration that would lead to a discussion environment where students could observe the similarities, differences, and relationships between representations. The PTs prioritized students' affective characteristics to achieve mathematical understanding and used different strategies that would enable students to develop positive attitudes toward the lessons. Mathematics teacher educators can benefit from these views and practices in supporting PTs in promoting mathematical understanding.

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Statements of publication ethics

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Researchers' contribution rate

The study was conducted and reported with equal collaboration of the researchers.

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