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Research Article

Hom-Gel'fand-Dorfman conformal superbialgebras

Taoufik Chtioui

University of Gabes, Faculty of Sciences Gabes, City Riadh 6072 Zrig, Gabes, Tunisia.

Abstract

Gel'fand Dorfman superbialgebra, which is both a Lie superalgebra and a (left) Novikov superalgebra with some compatibility condition, appears in the study of Hamiltonian pairs in completely integrable systems and a class of special Lie conformal superalgebras called quadratic Lie conformal superalgebras. In the present paper, we generalize this algebraic structure to the Hom-conformal case. We introduce first, Hom-Novikov conformal superalgebras and exihibit several properties. Then we introduce Hom-Gel'fand Dorfman superbialgebra and provide some construction results.

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1. Introduction

Lie conformal superalgebras were introduced by Kac in [8], he gave an axiomatic description of the singular part of the operator product expansion of chiral fields in conformal field theory. It is an useful tool to study vertex algebras and has many applications in the theory of Lie superalgebras and integrable systems. The main examples of Lie conformal superalgebras are Virasoro conformal algebra, current conformal superalgebras, Neveu-Schwarz conformal superalgebra and so on. Conformal algebras are quite intriguing subjects in the purely algebraic viewpoint. Structure theory and representation theory of Lie and associative conformal algebras are studied in a series of papers (see for example [1], [2], [9], [19] and [13]). In particular, associative conformal algebras naturally appear in the representation theory of Lie conformal algebras. Left-symmetric conformal algebras and Novikov algebras were introduced in [7]. Leibniz conformal algebras were given in [2].

Gelfand-Dorfman bialgebra (see [14]) is a vector space A endowed with a Lie bracket $[\cdot,\cdot]$ and a Novikov algebraic structure $\circ$ (the term was proposed in [12]) with some compatibility condition. It corresponds to a Hamiltonian pair, which plays fundamental roles in completely integrable systems (see [3]). Moreover, there are also close relationships between Gelfand-Dorfman bialgebras and Lie conformal algebras (see [6] and [15], for instance). In [5], Y. Homg, studied a class of Lie conformal superalgebras in high dimension named r-dim i-linear Lie conformal superalgebras. He gave a equivalent characterization

Email address: chtioui.taoufik@yahoo.fr
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of this class by means of r-dim Gel'fand Dorfman conformal superbialgebras. The topic of this paper is about Hom-Gel'fand Dorman conformal superbialgebras.

The Hom-type algebras arose firstly in quasi-deformation of Lie algebras of vector fields. Discrete modifications of vector fields via twisted derivations lead to Hom-Lie algebras, which were introduced in the context of studying deformations of Witt and Virasoro algebras [4]. The Hom-analogue of associative, alternative, Jordan and Novikov algebras were introduced in [10, 11, 16]. Hom-Gel'fand Dorfman (super)bialgebras were introduced in [17, 18].

Motived by all these works, this paper is organized as follows: In Section 2, we recall some definitions and introduce the notion of Hom-Novikov conformal superalgebras. In Section 3, we give an equivalent characterization of both Hom-left symmetric and Hom-Novikov conformal superalgebras. In Section 4, we introduce the notion of Hom-Gel'fand Dorfman conformal superbialgebras and exhibit some construction results.

2. Hom-Novikov conformal superalgebras

In this section, we introduce the notion of Hom-Novikov conformal superalgebras. But, we begin first, by recalling some definitions which we need in the sequel.

Throughout this paper, all vector spaces, linear maps and tensor products are assumed to be over complex field $\mathbb{C}$. We denote by $\mathbb{C}^*$, $\mathbb{Z}$ and $\mathbb{Z}_+$ the sets of nonzero complex numbers, integers and nonnegative integers, respectively. Moreover, if A is a vector space, the space of polynomials of λ with coefficients in A is denoted by $A[\lambda]$.

We denote $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. A vector space U is called $\mathbb{Z}_2$-graded if $U = U_{\bar{0}} \oplus U_{\bar{1}}$, and $u \in U_{\bar{i}}$ is called $\mathbb{Z}_2$-homogenous and write $|u| = \bar{i}$.

Next, we introduce the definitions of Hom-Novikov superalgebra and Hom-Novikov-Poisson superalgebra.

Definition 2.1. A (left) Hom-Novikov superalgebra is a $\mathbb{Z}_2$-graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with an operation "$\circ$" and an even linear map $\phi : A \to A$ satisfying the following axioms: for $a \in A_\alpha$, $b \in A_\beta$, $c \in A_\gamma$,

$$(a \circ b) \circ \phi(c) = (-1)^{\beta\gamma}(a \circ c) \circ \phi(b), \tag{2.1}$$

$$(a \circ b) \circ \phi(c) - \phi(a) \circ (b \circ c) = (-1)^{\alpha\beta}[(b \circ a) \circ \phi(c) - \phi(b) \circ (a \circ c)]. \tag{2.2}$$

When $A_{\bar{1}} = 0$, we call A a (left) Hom-Novikov algebra.

A Hom-Novikov-Poisson superalgebra is a $\mathbb{Z}_2$-graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with two operations "$\circ$" and "$\cdot$" and an even linear map $\phi : A \to A$ where $(A, \circ, \phi)$ is a Hom-Novikov superalgebra, $(A, \cdot, \phi)$ is a commutative Hom-associative superalgebra, and they satisfy the following axioms:

$$(a \circ b) \cdot \phi(c) - \phi(a) \circ (b \cdot c) = (-1)^{\alpha\beta}((b \circ a) \cdot \phi(c) - \phi(b) \circ (a \cdot c)), \tag{2.3}$$

$$(a \cdot b) \circ \phi(c) = \phi(a) \cdot (b \circ c), \tag{2.4}$$

for $a \in A_\alpha$, $b \in A_\beta$. When $A_{\bar{1}} = \{0\}$, it is called Hom-Novikov-Poisson algebra.

Example 2.1. Let $(A, \cdot, \phi)$ be a commutayive Hom-associative superalgebra and $D : A \to A$ be a derivation with $D\phi = \phi D$. Then $(A, \circ, \cdot, \phi)$ is a Hom-Novikov-Poisson superalgebra, where

$$a \circ b = a \cdot D(b).$$

Definition 2.2. A Hom-Lie conformal superalgebra $S = S_{\bar{0}} \oplus S_{\bar{1}}$ is a $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module endowed with an even linear map $\phi : S \to S$ such that $\phi\partial = \partial\phi$ and a λ-bracket $[\cdot_\lambda \cdot]$ which defines a linear map $S_\alpha \otimes S_\beta \to \mathbb{C}[\lambda] \otimes S_{\alpha+\beta}$, where λ is an indeterminate, and satisfy the following axioms:

$$[\partial a\,{}_\lambda\, b] = -\lambda[a\,{}_\lambda\, b],\ \ [a\,{}_\lambda\, \partial b] = (\partial+\lambda)[a\,{}_\lambda\, b], \tag{2.5}$$

$$[a\,{}_\lambda\, b] = -(-1)^{|a||b|}[b\,{}_{-\lambda-\partial}\, a], \tag{2.6}$$

$$[\phi(a)\,{}_\lambda\,[b\,{}_\mu\, c]] = [[a\,{}_\lambda\, b]\,{}_{\lambda+\mu}\,\phi(c)] + (-1)^{|a||b|}[\phi(b)\,{}_\mu\,[a\,{}_\lambda\, c]] \tag{2.7}$$

for all $c \in S$, $\mathbb{Z}_2$-homogenous elements a, b in S, and α, $\beta \in \mathbb{Z}_2$.

Example 2.2. Let $(A, [\cdot_\lambda\cdot])$ be Lie conformal superalgebra and $\phi : A \to A$ a morphism of conformal superalgebras. Then $(A, \{\cdot_\lambda\cdot\}, \phi)$ is a Hom-Lie conformal superalgebra, where

$$\{a_\lambda b\} = [\phi(a)_\lambda \phi(b)], \quad \forall a, b \in A.$$

Example 2.3. Let $S = S_{\bar{0}} \oplus S_{\bar{1}}$ be a complex Hom-Lie superalgebra with Lie bracket $[-,-]$ and a morphism ϕ. Let $(Curg)_\theta := \mathbb{C}[\partial] \otimes S_\theta$ be the free $\mathbb{C}[\partial]$-module. Then $CurS = (CurS)_{\bar{0}} \oplus (CurS)_{\bar{1}}$ is a Hom-Lie conformal superalgebra with λ-bracket given by

$$\phi(f(\partial)\otimes a) = f(\partial)\otimes\phi(a),$$
$$[(f(\partial)\otimes a)_\lambda(g(\partial)\otimes b)] = f(-\lambda)g(\partial+\lambda)\otimes[a,b], \forall a, b \in S.$$

Example 2.4. Let $S = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]E$ be a free $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module. Define

$$\phi(L) = f(\partial)L, \phi(E) = g(\partial)E,$$
$$[L_\lambda L] = (\partial+2\lambda)L, [L_\lambda E] = (\partial+\frac{3}{2}\lambda)E,$$
$$[E_\lambda L] = (\frac{1}{2}\partial+\frac{3}{2}\lambda)E, [E_\lambda E] = 0,$$

where $S_{\bar{0}} = \mathbb{C}[\partial]L$ and $S_{\bar{1}} = \mathbb{C}[\partial]E$. Then $(S, [\cdot_\lambda\cdot], \phi)$ is a Hom-Lie conformal superalgebra.

Now, we introduce the notion of Hom-Novikov conformal superalgebras.

Definition 2.3. A (left) Hom-Novikov conformal superalgebra is a left $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module $A = A_{\bar{0}} \oplus A_{\bar{1}}$ endowed with an even linear map $\phi : A \to A$ such that $\partial\phi = \phi\partial$ and a λ-product $a_\lambda b$ which defines a $\mathbb{C}$-bilinear map $A \otimes A \to A[\lambda]$, subject to the following axioms:

$$(\partial a_\lambda b) = -\lambda(a_\lambda b), \qquad (a_\lambda \partial b) = (\partial+\lambda)(a_\lambda b), \tag{2.8}$$

$$(a_\lambda b)_{\lambda+\mu}\phi(c) - \phi(a)_\lambda(b_\mu c) = (-1)^{\alpha\beta}((b_\mu a)_{\lambda+\mu}\phi(c) - \phi(b)_\mu(a_\lambda c)), \tag{2.9}$$

$$(a_\lambda b)_{\lambda+\mu}\phi(c) = (-1)^{\beta\gamma}(a_\lambda c)_{-\mu-\partial}\phi(b), \tag{2.10}$$

for $a \in A_\alpha$, $b \in A_\beta$ and $c \in A_\gamma$. When $A_{\bar{1}} = \{0\}$, A is called (left) Hom-Novikov conformal algebra.

A right Hom-Novikov conformal superalgebra is a left $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module $A = A_{\bar{0}} \oplus A_{\bar{1}}$ endowed with an even linear map $\phi : A \to A$ such that $\partial\phi = \phi\partial$ and a λ-product $a_\lambda b$ which defines a $\mathbb{C}$-bilinear map $A \otimes A \to A[\lambda]$, subject to the following axioms:

$$(\partial a_\lambda b) = -\lambda_(a_\lambda b), \qquad (a_\lambda \partial b) = (\partial+\lambda)(a_\lambda b), \tag{2.11}$$

$$\phi(a)_\lambda(b_\mu c) - (a_\lambda b)_{\lambda+\mu}\phi(c) = (-1)^{\beta\gamma}(\phi(a)_\lambda(c_{-\mu-\partial}b) - (a_\lambda c)_{-\mu-\partial}\phi(b)), \tag{2.12}$$

$$\phi(a)_\lambda(b_\mu c) = (-1)^{\alpha\beta}\phi(b)_\mu(a_\lambda c). \tag{2.13}$$

for $a \in A_\alpha$, $b \in A_\beta$ and $c \in A_\gamma$. When $A_{\bar{1}} = \{0\}$, A is called right Hom-Novikov conformal algebra.

Remark 2.1. Obviously, if $(A, \cdot_\lambda\cdot, \phi)$ is a left (resp. right) Hom-Novikov conformal superalgebra, then $(A, \circ_\lambda, \phi)$ is a right (resp. left) Hom-Novikov conformal superalgebra with $a \circ_\lambda b = (-1)^{\alpha\beta} b_{-\lambda-\partial} a$ for any $a \in A_\alpha$, $b \in A_\beta$.

Remark 2.2. A Hom-superalgebra satisfying conditions (2.8) and (2.9) is called a Hom-left symmetric conformal superalgebra.

The following result is straightforward.

Proposition 2.3. Let $(A, \cdot_\lambda)$ be Novikov conformal superalgebra and $\phi : A \to A$ a morphism of conformal superalgebras. Then $(A, \circ_\lambda, \phi)$ is a Hom-Novikov conformal superalgebra, where

$$a \circ_\lambda b = \phi(a)_\lambda \phi(b), \quad \forall a, b \in A.$$

Example 2.5. A simplest example of a Hom-Novikov conformal (super)algebra is provided by the current functor. Namely, if $(A, \circ, \phi)$ is a Hom-Novkiov superalgebra, then Cur $A = \mathbb{C}[\partial] \otimes A \simeq A[\partial]$ is a Hom-Novikov conformal superalgebra with the following λ-product:

$$f(\partial)a_\lambda g(\partial)b = f(-\lambda)g(\partial+\lambda)a \circ b,$$

and $\phi(f(\partial) \otimes a) = f(\partial)\phi(a)$, for any $a \in A$

Now, we introduce a construction of Hom-Novkiov conformal superalgebras from Hom-Novkov-Poisson superalgebras

Proposition 2.4. Let $(V, \circ, \cdot, \phi)$ be a Hom-Novikov-Poisson superalgebra, then $A = \mathbb{C}[\partial]V$ is a Hom-Novikov conformal superalgebra with the following λ-product:

$$a_\lambda b = (\lambda+\partial)(a \cdot b) + a \circ b, \quad a \in V_\alpha, \quad b \in V_\beta. \tag{2.14}$$

Proof. For any $a \in V_\alpha, b \in V_\beta, c \in V_\gamma$, we have

$$\begin{aligned}
&(a_\lambda b)_{\lambda+\mu}\phi(c) = ((\lambda+\partial)(a \cdot b) + a \circ b)_{\lambda+\mu}\phi(c)\\
=& -\mu(\lambda+\mu+\partial)(a \cdot b) \cdot \phi(c) - \mu(a \cdot b) \circ \phi(c) + (\lambda+\mu+\partial)(a \circ b) \cdot \phi(c) + (a \circ b) \circ \phi(c)\\
=&(-1)^{\beta\gamma}\big(-\mu(\lambda+\mu+\partial)(a \cdot c) \cdot \phi(b) - \mu(a \circ c) \cdot \phi(b)\\
&+ (\lambda+\mu+\partial)(a \cdot c) \circ \phi(b) + (a \circ c) \circ \phi(b)\big)
\end{aligned}$$

and

$$\begin{aligned}
&(a_\lambda c)_{-\mu-\partial}\phi(b) = ((\lambda+\partial)(a \cdot c) + a \circ c)_{-\mu-\partial}\phi(b)\\
=&\lambda(a \cdot c)_{-\mu-\partial}\phi(b) + \partial(a \cdot c)_{-\mu-\partial}\phi(b) + (a \circ c)_{-\mu-\partial}\phi(b)\\
=&\lambda(-\mu(a \cdot c) \cdot \phi(b) + (a \cdot c) \circ \phi(b)) + (\mu+\partial)(-\mu(a \cdot c) \cdot \phi(b) + (a \cdot c) \circ \phi(b))\\
&\qquad - \mu(a \circ c) \cdot \phi(b) + (a \circ c) \circ \phi(b)\\
=& -\mu(\lambda+\mu+\partial)(a \cdot c) \cdot \phi(b) - \mu(a \circ c) \cdot \phi(b) + (\lambda+\mu+\partial)(a \cdot c) \circ \phi(b) + (a \circ c) \circ \phi(b),
\end{aligned}$$

which implies that $(a_\lambda b)_{\lambda+\mu}\phi(c) = (-1)^{\beta\gamma}(a_\lambda c)_{-\mu-\partial}\phi(b)$. On the other hand, we have

$$\begin{aligned}
&(a_\lambda b)_{\lambda+\mu}\phi(c) = ((\lambda+\partial)(a \cdot b) + a \circ b)_{\lambda+\mu}\phi(c)\\
=&\lambda(a \cdot b)_{\lambda+\mu}\phi(c) - (\lambda+\mu)(a \cdot b)_{\lambda+\mu}\phi(c) + (a \circ b)_{\lambda+\mu}\phi(c)\\
=& -\mu((\lambda+\mu+\partial)(a \cdot b) \cdot \phi(c) + (a \cdot b) \circ \phi(c)) + (\lambda+\mu+\partial)(a \circ b) \cdot \phi(c) + (a \circ b) \circ \phi(c)\\
=& -\lambda\mu(a \cdot b) \cdot \phi(c) - \mu^2(a \cdot b) \cdot \phi(c) - \mu\partial(a \cdot b) \cdot \phi(c) + \lambda(a \circ b) \cdot \phi(c)\\
&\quad + \mu((a \circ b) \cdot \phi(c) - (a \cdot b) \circ \phi(c)) + \partial(a \circ b) \cdot \phi(c) + (a \circ b) \circ \phi(c),
\end{aligned}$$

$$
\begin{aligned}
&\phi(a)_\lambda(b_\mu c) = \phi(a)_\lambda((\mu+\partial)(b\cdot c) + b\circ c)\\
=&\mu\phi(a)_\lambda(b\cdot c) + (\lambda+\partial)(\phi(a)_\lambda(b\cdot c)) + \phi(a)_\lambda(b\circ c)\\
=&(\lambda+\partial)(\lambda+\mu+\partial)\phi(a)\cdot(b\cdot c) + (\lambda+\mu+\partial)\phi(a)\circ(b\cdot c) + (\lambda+\partial)\phi(a)\cdot(b\circ c)\\
&+\phi(a)\circ(b\circ c)\\
=&\lambda\mu\phi(a)\cdot(b\cdot c) + \mu\partial\phi(a)\cdot(b\cdot c) + \mu\phi(a)\circ(b\cdot c) + \lambda^2\phi(a)\cdot(b\cdot c) + 2\lambda\partial\phi(a)\cdot(b\cdot c)\\
+&\partial^2\phi(a)\cdot(b\cdot c) + \lambda(\phi(a)\circ(b\cdot c) + \phi(a)\cdot(b\circ c)) + \partial(\phi(a)\circ(b\cdot c) + \phi(a)\cdot(b\circ c))\\
&+\phi(a)\circ(b\circ c),
\end{aligned}
$$

$$
\begin{aligned}
&(b_\mu a)_{\lambda+\mu}\phi(c) = -\lambda\mu(b\cdot a)\cdot\phi(c) - \lambda^2(b\cdot a)\cdot\phi(c) - \lambda\partial(b\cdot a)\cdot\phi(c) + +\mu(b\circ b)\cdot\phi(c)\\
&\quad+\lambda((b\circ a)\cdot\phi(c) - (b\cdot a)\circ\phi(c)) + \partial(b\circ a)\cdot\phi(c) + (b\circ a)\circ\phi(c)
\end{aligned}
$$

and

$$
\begin{aligned}
&\phi(b)_\mu(a_\lambda c) = \lambda\mu\phi(b)\cdot(a\cdot c) + \lambda\partial\phi(b)\cdot(a\cdot c) + \lambda\phi(b)\circ(a\cdot c) + \mu^2\phi(b)\cdot(a\cdot c)\\
&+2\mu\partial\phi(b)\cdot(a\cdot c) + \partial^2\phi(b)\cdot(a\cdot c) + \mu(\phi(b)\circ(a\cdot c) + \phi(b)\cdot(a\circ c)) + \partial(\phi(b)\circ(a\cdot c)\\
&+\phi(b)\cdot(a\circ c)) + \phi(b)\circ(a\circ c).
\end{aligned}
$$

Now, using Eqs. (2.1)-(2.2)-(2.3) and (2.4) and according to the above computations, we can deduce that the coefficients of $\lambda^2, \mu^2, \partial^2, \lambda\mu, \lambda\partial, \mu\partial, \lambda, \mu$ and ∂ in the expression $(a_\lambda b)_{\lambda+\mu}\phi(c) - \phi(a)_\lambda(b_\mu c) - (-1)^{\alpha\beta}(b_\mu a)_{\lambda+\mu}\phi(c) + (-1)^{\alpha\beta}\phi(b)_\mu(a_\lambda c)$ vanish. Therefore, $(A, \cdot_\lambda)$ is a Hom-Novikov conformal superalgebra. □

It is easy to check the following proposition.

Proposition 2.5. Let $(A, \cdot_\lambda\cdot, \phi)$ be a left (resp. right) Hom-Novikov conformal superalgebra, then $(A, [\cdot_\lambda\cdot], \phi)$ is a Hom-Lie conformal superalgebra with the following λ-bracket:

$$[a_\lambda b] = a_\lambda b - (-1)^{\alpha\beta}(b_{-\lambda-\partial}a), \qquad a\in A_\alpha, \quad b\in A_\beta.$$

3. Equivalent characterization of quadratic Hom-Novikov conformal superalgebras

In this section, we give an equivalent characterization of a special class of Hom-Novikov conformal superalgebras called quadratic Hom-Novikov conformal superalgebras. We begin by characterizing quadratic Hom-left symmetric conformal superalgebras since they consist a particular case.

Definition 3.1. Let $(A, \cdot_\lambda, \phi)$ be a Hom-Novikov (or Hom-left symmetric) conformal superalgebra. If there exists a vector space V such that $A = \mathbb{C}[\partial]V$ is a free $\mathbb{C}[\partial]V$-module over V and for all $a, b\in V$, the corresponding λ-product is of the following form:

$$a_\lambda b = \partial u + \lambda v + w, \quad u, v, w\in V. \tag{3.1}$$

Then A is called quadratic.

Theorem 3.1. A quadratic Hom-left-symmetric conformal superalgebra $A = \mathbb{C}[\partial]V$ is equivalent to the quadruple $(V, *_1, \circ, *_2, \phi)$ where$(V, \circ, \phi)$ is a Hom-left symmetric superalgebra and $*_1$ and $*_2$ are two operations on V satisfying the following identities:

$$\phi(a) *_1 (b *_1 c) = (-1)^{|a||b|}\phi(b) *_1 (a *_1 c), \tag{3.2}$$

$$\begin{aligned}(a *_1 b) *_1 \phi(c) - (a *_2 b) *_1 \phi(c) + \phi(a) *_1 (b *_1 c) + \phi(a) *_2 (b *_1 c)\\ = (-1)^{|a||b|}\big((b *_1 a) *_1 \phi(c) + \phi(b) *_1 (a *_2 c)\big),\end{aligned} \tag{3.3}$$

$$\begin{aligned}(a *_1 b) *_1 \phi(c) + \phi(a) *_1 (b *_2 c) = (-1)^{|a||b|}\big((b *_1 a) *_1 \phi(c)\\ - (b *_2 a) *_1 \phi(c) + \phi(b) *_1 (a *_1 c) + \phi(b) *_2 (a * -1c), \big)\end{aligned} \tag{3.4}$$

$$\begin{aligned}(a \circ b) *_1 \phi(c) - \phi(a) \circ (b *_1 c) - \phi(a) *_1 (b \circ c)\\ = (-1)^{|a||b|}\big((b \circ a) *_1 \phi(c) - \phi(b) \circ (a *_1 c) - \phi(b) *_1 (a \circ c)\big),\end{aligned} \tag{3.5}$$

$$(a *_1 b) *_2 \phi(c) - (a *_2 b) *_2 \phi(c) + \phi(a) *_2 (b *_1 c) = (-1)^{|a||b|}(b *_1 a) *_2 \phi(c), \tag{3.6}$$

$$\begin{aligned}2(a *_1 b) *_2 \phi(c) - (a *_2 b) *_2 \phi(c) + \phi(a) *_2 (b *_2 c)\\ = 2(b *_1 a) *_2 \phi(c) - (b *_2 a) *_2 \phi(c) + \phi(b) *_2 (a *_2 c),\end{aligned} \tag{3.7}$$

$$(a *_1 b) *_2 \phi(c) = (-1)^{|a||b|}\big((b *_1 a) *_2 \phi(c) - (b *_2 a) *_2 \phi(c) + \phi(b) *_2 (a *_1 c)\big), \tag{3.8}$$

$$\begin{aligned}(a *_1 b) \circ \phi(c) - (a \circ b) *_2 \phi(c) - (a *_2 b) \circ \phi(c) + \phi(a) \circ (b *_1 c) + \phi(a) *_2 (b \circ c)\\ = (-1)^{|a||b|}\big((b *_1 a) \circ \phi(c) - (b \circ a) *_2 \phi(c) + \phi(b) \circ (a *_2 c)\big),\end{aligned} \tag{3.9}$$

$$\begin{aligned}(a *_1 b) \circ \phi(c) - (a \circ b) *_2 \phi(c) + \phi(a) \circ (b *_2 c) = (-1)^{|a||b|}\big((b *_1 a) \circ \phi(c) - (b \circ a) *_2 \phi(c)\\ - (b *_2 a) \circ \phi(c) + \phi(b) \circ (a *_1 c) + \phi(b) *_2 (a \circ c)\big).\end{aligned} \tag{3.10}$$

Proof. Let $A = \mathbb{C}[\partial]V$ be a quadratic Hom-left-symmetric conformal superalgebra. Then by (3.1), we set

$$a_\lambda b = \partial(a *_1 b) + a \circ b + \lambda(a *_2 b), \tag{3.11}$$

where $*_1$, $\circ$ and $*_2$ are three operations on V. Then, since A is a free $\mathbb{C}[\partial]$-module over V , using (2.8) taking (3.1)into (2.9), by comparing coefficients of $\lambda^2, \mu^2, \partial^2, \lambda\mu, \lambda\partial, \mu\partial, \lambda, \mu, \partial$ and ∂^0 we can obtain that (2.9) is equivalent to (3.2)–(3.10) and $\circ$ being a Hom-left symmetric superalgebraic structure. □

Theorem 3.2. A quadratic Hom-Novikov conformal superalgebra $A = \mathbb{C}[\partial]V$ is equivalent to the quadruple $(V, *_1, \circ, *_2, \phi)$ where$(V, \circ, \phi)$ is a Hom-Novikov superalgebra and $*_1$ and $*_2$ are two operations on V satisfying identities (3.2)–(3.10) and obeying to the following conditions:

$$\begin{aligned}(a *_1 b) *_1 \phi(c) = (a *_1 b) *_2 \phi(c) = (a *_2 b) *_2 \phi(c)\\ = (-1)^{|c||b|}(a *_1 c) *_1 \phi(b) = (-1)^{|c||b|}(a *_1 c) *_2 \phi(b) = (-1)^{|c||b|}(a *_2 c) *_2 \phi(b),\end{aligned} \tag{3.12}$$

$$(a *_2 b) *_1 \phi(c) = (-1)^{|c||b|}(a *_2 c) *_1 \phi(b), \tag{3.13}$$

$$(a \circ b) *_2 \phi(c) + (a *_2 b) \circ \phi(c) = (a *_1 b) \circ \phi(c) + (-1)^{|c||b|}(a *_2 c) \circ \phi(b), \tag{3.14}$$

$$(a \circ b) *_2 \phi(c) - (a *_1 b) \circ \phi(c) = (-1)^{|c||b|}\big((a *_1 c) \circ \phi(b) - (a \circ c) *_2 \phi(b)\big), \tag{3.15}$$

$$(a \circ b) *_1 \phi(c) + (-1)^{|c||b|}(a \circ c) *_2 \phi(b) = (-1)^{|c||b|}\big((a *_1 c) \circ \phi(b) + (a \circ c) *_1 \phi(b)\big) \tag{3.16}$$

Proof. Since A is a free $\mathbb{C}[\partial]$-module over V , using (2.8) taking (3.1)into (2.10), by comparing coefficients of $\lambda^2, \mu^2, \partial^2, \lambda\mu, \lambda\partial, \mu\partial, \lambda, \mu, \partial$ and ∂^0 we can obtain that (2.10) is equivalent to (3.12) –(3.16) and $\circ$ is a Hom-Novikov superalgebraic structure. Then by Theorem 3.1, this theorem is obtained. □

4. Hom-Gelfand-Dorfman conformal superbialgebras

In this section we introduce the notion of Hom-Gelfand-Dorfman conformal superbialgebra. Then, we give some construction results involving Hom-Novikov conformal superalgebras and Hom-Novikov Poisson algebras

4.1. Definition and construction results

Definition 4.1. A Hom-Gel'fand-Dorfman conformal superbialgebra is a $\mathbb{Z}_2$-graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with two algebraic operations $[\cdot_\lambda \cdot]$ and $\cdot_\lambda$ and an even linear map ϕ satisfying $\phi\partial = \partial\phi$ such that $(A, [\cdot_\lambda \cdot], \phi)$ forms a Hom-Lie conformal superalgebra, $(A, \cdot_\lambda, \phi)$ forms a Hom-Novikov conformal superalgebra and the following compatibility condition holds:

$$\begin{aligned}&[(a_{-\mu-\partial} b)_{-\lambda-\partial}\phi(c)] + [a_{-\mu-\partial} b]_{-\lambda-\partial}\phi(c) - \phi(a)_{-\lambda-\mu-\partial}[b_{-\lambda-\partial} c]\\ &-(-1)^{\beta\gamma}[(a_{-\lambda-\partial} c)_{-\mu-\partial}\phi(b)] - (-1)^{\beta\gamma}[a_{-\lambda-\partial} c]_{-\mu-\partial}\phi(b) = 0,\end{aligned} \tag{4.1}$$

for $a \in A$, $b \in A_\beta$, $c \in A_\gamma$.

Remark 4.1. If $\alpha = id$, then A is a Gel'fand Dorfman conformal superbialgebra and if $A_{\bar{1}} = 0$, A will be a Hom-Gel'fand Dorfman conformal algebra.

Remark 4.2. In fact, there are some relations between Hom-Gel'fand-Dorfman conformal superbialgebra and Hom-Gel'fand-Dorfman superbialgebra. If $(A = \mathbb{C}[\partial]V, [\cdot_\lambda \cdot], \cdot_\lambda \cdot, \phi)$ is Hom-Gel'fand-Dorfman conformal superbialgebra, then $(\mathrm{Lie}(A), [\cdot,\cdot], \circ, \phi)$ is a Hom-Gel'fand-Dorfman superbialgebra with the Lie bracket and the Novikov algebraic operation as follows:

$$[a_m, b_n] = \sum_{j \in \mathbb{Z}_+^r} \begin{pmatrix} m \\ j \end{pmatrix} (a_{(j)} b)_{m+n-j}, \tag{4.2}$$

$$a_m \circ b_n = \sum_{j \in \mathbb{Z}_+^r} \begin{pmatrix} m \\ j \end{pmatrix} (a_{(j)} b)_{m+n-j}, \tag{4.3}$$

where $[a_\lambda b] = \sum_{m \in \mathbb{Z}_+^r} \lambda^{(m)} a_{(m)} b$ and $a_\lambda b = \sum_{m \in \mathbb{Z}_+^r} \lambda^{(m)} a_m b$.

Example 4.1. Let $(A, \cdot[\cdot,\cdot], \phi)$ is a Hom-Gel'fand Dorfman superalgebra, then Cur $A = \mathbb{C}[\partial] \otimes A \simeq A[\partial]$ is a Hom-Gel'fand Dorfman conformal superalgebra with the following λ-product $\cdot_\lambda$ and λ-bracket $[\cdot_\lambda \cdot]$:

$$\begin{aligned}&f(\partial)a_\lambda g(\partial)b = f(-\lambda)g(\partial+\lambda)a \circ b,\\ &[(f(\partial) \otimes a)_\lambda (g(\partial) \otimes b)] = f(-\lambda)g(\partial+\lambda) \otimes [a,b],\\ &\phi(f(\partial) \otimes a) = f(\partial)\phi(a),\end{aligned}$$

for any $a, b \in A$.

Proposition 4.3. Let $(A, \cdot_\lambda, [\cdot_\lambda \cdot])$ be a Gel'fand Dorfman conformal superalgebra and $\phi : A \to A$ a morphism on A, namely a morphism of Lie conformal superalgebras and a morphism of Novikov conformal superalgebras. Then $(A, \circ_\lambda, \{\cdot_\lambda \cdot\}, \phi)$ is a Hom-Gel'fand Dorfman conformal superalgebra, where

$$a \circ_\lambda b = \phi(a)_\lambda \phi(b), \quad \{a_\lambda b\} = [\phi(a)_\lambda \phi(b)],$$

for any $a \in A_\alpha, b \in A_\beta$.

Proof. It can be checked by a direct computations. □

In what follows, a Hom-Gel'fand Dorfman superbialgebras will be called a Hom-GD superbialgebra, for short.

Theorem 4.4. Let $(A, \cdot_\lambda\cdot, \phi)$ be a Hom-Novikov conformal superalgebra. Then, $(A, \cdot_\lambda\cdot, [\cdot_\lambda\cdot], \phi)$ is a Hom-GD conformal superbialgebra with the λ-bracket given as follows:

$$[a_\lambda b] = a_\lambda b - (-1)^{\alpha\beta} b_{-\lambda-\partial} a, \quad a \in A_\alpha, \quad b \in A_\beta. \tag{4.4}$$

Proof. By Proposition 2.5, we only need to check that (4.1) hold. By (4.4), the right hand side of (4.1) is the following:

$$\begin{aligned}
& (a_{-\mu-\partial}b)_{-\lambda-\partial}\phi(c) - (-1)^{(\alpha+\beta)\gamma}\phi(c)_\lambda(a_{-\mu-\partial}b) + (a_{-\mu-\partial}b)_{-\lambda-\partial}\phi(c) \\
& -(-1)^{\alpha\beta}(b_\mu a)_{-\lambda-\partial}\phi(c) - \phi(a)_{-\lambda-\mu-\partial}(b_{-\lambda-\partial}c) + (-1)^{\beta\gamma}\phi(a)_{-\lambda-\mu-\partial}(c_\lambda b) \\
& -(-1)^{\beta\gamma}(a_{-\lambda-\partial}c)_{-\mu-\partial}\phi(b) + (-1)^{\alpha\beta}\phi(b)_\mu(a_{-\lambda-\partial}c) \\
& -(-1)^{\beta\gamma}(a_{-\lambda-\partial}c)_{-\mu-\partial}\phi(b) + (-1)^{(\alpha+\beta)\gamma}(c_\lambda a)_{-\mu-\partial}\phi(b) \\
= \; & ((a_{-\mu-\partial}b)_{-\lambda-\partial}\phi(c) - \phi(a)_{-\lambda-\mu-\partial}(b_{-\lambda-\partial}c) - (-1)^{\alpha\beta}(b_\mu a)_{-\lambda-\partial}\phi(c) \\
& +(-1)^{\alpha\beta}\phi(b)_\mu(a_{-\lambda-\partial}c)) - (-1)^{\beta\gamma}((a_{-\lambda-\partial}c)_{-\mu-\partial}\phi(b) - \phi(a)_{-\lambda-\mu-\partial}(c_\lambda b) \\
& +(-1)^{\alpha\gamma}\phi(c)_\lambda(a_{-\mu-\partial}b) - (-1)^{\alpha\gamma}(c_\lambda a)_{-\mu-\partial}\phi(b)) \\
& +((a_{-\mu-\partial}b)_{-\lambda-\partial}\phi(c) - (-1)^{\beta\gamma}(a_{-\lambda-\partial}c)_{-\mu-\partial}\phi(b)) \\
= \; & 0.
\end{aligned}$$

Therefore, $(A, \cdot_\lambda, [\cdot_\lambda\cdot], \phi)$ is a Hom-Gel'fand-Dorfman conformal superbialgebra. □

Using Theorem 4.4 and Proposition 2.4, we can easily obtain the following result.

Proposition 4.5. Let $(V, \circ, \cdot, \phi)$ be a Hom-Novikov-Poisson superalgebra, then $A = \mathbb{C}[\partial]V$ is a Hom-GD conformal superbialgebra with the following λ-product and λ-bracket $(a \in V_\alpha, b \in V_\beta)$:

$$a_\lambda b = (\lambda + \partial)(a \cdot b) + a \circ b, \tag{4.5}$$

$$[a_\lambda b] = (\partial + 2\lambda)(a \cdot b) + (a \circ b - (-1)^{\alpha\beta} b \circ a). \tag{4.6}$$

Proposition 4.6. If $(V, [\cdot,\cdot], \circ, \phi)$ is a Hom-GD superbialgebra, $(V, \circ, \cdot, \phi)$ is Hom-Novikov-Poisson superalgebra and $(V, [\cdot,\cdot], \cdot, \phi)$ is a Hom-Poisson superalgebra, then $A = \mathbb{C}[\partial]V$ is a Hom-GD conformal superbialgebra with the following λ-product and λ-bracket $(a \in V_\alpha, b \in V_\beta)$:

$$a_\lambda b = (\lambda + \partial)(a \cdot b) + a \circ b,$$
$$[a_\lambda b] = (-1)^{\alpha\beta}\partial(b \circ a) + \lambda(a \circ b + (-1)^{\alpha\beta} b \circ a) + (-1)^{\alpha\beta}[b, a].$$

The quadruple $(V, [\cdot,\cdot], \circ, \cdot, \phi)$ is called Hom-GD Novikov-Poisson superalgebra.

Proof. Straightforward. □

Remark 4.7. If the operation $\cdot$ in the above proposition is trival, then we can obtain a construction of Hom-GD conformal superbialgebras from Hom-GD superbialgebras. In fact, if $(V, [\cdot,\cdot], \circ, \phi)$ is a Hom-GD superbialgebra, then $A = \mathbb{C}[\partial]V$ is a Hom-GD conformal superbialgebra with the following λ-product and λ-bracket $(a \in V_\alpha, b \in V_\beta)$

$$a_\lambda b = a \circ b,$$
$$[a_\lambda b] = (-1)^{\alpha\beta}\partial(b \circ a) + \lambda(a \circ b + (-1)^{\alpha\beta} b \circ a) + (-1)^{\alpha\beta}[b, a].$$

4.2. Further discussion

It is known that a Hom-Novikov (resp. Hom-GD) conformal superalgebra is a $\mathbb{C}[\partial]$-module. Therefore, there is a natural generalization if we replace $\mathbb{C}[\partial]$ by $\mathbb{C}[\partial_1, \cdots, \partial_r]$. This is just the r-dim Hom-Novikov (resp. Hom-GD) conformal superalgebras which is a twisted version of r-dim Novikov (resp. GD) conformal superalgebras introduced in [5]. In a next work, we will introduce the Hom-version of Lie conformal superalgebras in high dimension and characterize a special class of them. Hom-GD conformal superalgebras characterize, in fact, 2-dim-linear Hom-Lie conformal superalgebras, that is when we replace ∂ by the vector $\partial = (\partial_1, \partial_2)$.

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