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The Unreasonable Effectiveness of the Chaotic Tent Map in Engineering Applications

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ABSTRACT From decimal expansion of real numbers to complex behaviour in physical, biological and human-made systems, deterministic *chaos* is ubiquitous. One of the simplest examples of a nonlinear dynamical system that exhibits chaos is the well known 1-dimensional piecewise linear Tent map. The Tent map (and their skewed cousins) are instances of a larger family of maps namely Generalized Lüroth Series (GLS) which are studied for their rich number theoretic and ergodic properties. In this work, we discuss the unreasonable effectiveness of the Tent map and their generalizations (GLS maps) in a number of applications in electronics, communication and computer engineering. To list a few of these applications: (a) GLS-coding: a lossless data compression algorithm for i.i.d sources is Shannon optimal and is in fact a generalization of the popular Arithmetic Coding algorithm used in the image compression standard JPEG2000; (b) GLS maps are used as neurons in the recently proposed Neurochaos Learning architecture which delivers state-of-the-art performance in classification tasks; (c) GLS maps are ideal candidates for chaos-based computing since they can simulate XOR, NAND and other gates and for dense storage of information for efficient search and retrieval; (d) Noise-resistant versions of GLS maps are useful for signal multiplexing in the presence of noise and error detection; (e) GLS maps are known to be useful in a number of cryptographic protocols - for joint compression and encryption, and also in generating pseudo-random numbers. The unique properties and rich features of the Tent Map (GLS maps) that enable these wide variety of engineering applications will be investigated. A list of open problems are indicated as well.

KEYWORDS
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Compression
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INTRODUCTION

Deterministic Chaos refers to the seemingly random-like complicated (and often strange) behaviour of simple dynamical systems (Alligood *et al.* 2000; Devaney 2018). From decimal expansion of real numbers to complex behaviour in physical, biological and human-made systems, deterministic chaos is ubiquitous (Strogatz 2018). One of the simplest examples of a nonlinear dynamical system that exhibits chaos is the well known 1-dimensional piecewise linear Tent map (Alligood *et al.* 2000). The Tent map is topologically conjugate to the Logistic Map (the other popular 1D chaotic map) and finds numerous engineering applications in electronics, communications, compression, coding, computing and cryptography.

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The Tent map and their skewed cousins (Skew Tent map) are instances of a larger family of maps namely Generalized Lüroth Series (GLS) which are studied for their rich number theoretic and ergodic properties (Dajani and Kraaikamp 2002; Barrera and Robert 2022). In this work, we discuss the unreasonable effectiveness of the Tent map and their generalizations (GLS maps) in a number of applications in electronics, communication and computer engineering. To list a few of these applications: (a) GLS-coding (Nagaraj *et al.* 2009): a lossless data compression algorithm for independent and identically distributed (i.i.d) sources is Shannon optimal and is in fact a generalization of the popular Arithmetic Coding (Rissanen and Langdon 1979) algorithm used in the image compression standard JPEG2000; (b) GLS maps are used as neurons in a recently proposed novel *Neurochaos Learning* (Balakrishnan *et al.* 2019; Harikrishnan and Nagaraj 2021; Harikrishnan *et al.* 2022b) architecture which delivers state-of-the-art performance in classification tasks; (c) GLS maps are ideal candidates for chaos-based computing since they can simulate XOR, NAND and other gates (Sinha and Ditto 1998; Ditto *et al.* 2010; Jaimes-Reátegui *et al.* 2014) and for dense storage of information for efficient search and retrieval (Miliotis *et al.* 2008); (d) Noise-resistant versions of GLS maps are useful for signal multiplexing in the presence of noise (Nagaraj and Vaidya 2009) and error detection (Nagaraj 2019); (e) GLS maps are shown to be useful in a number of cryptographic protocols (Nagaraj 2012) - for joint source coding and encryption (Nagaraj *et al.* 2009; Wong *et al.* 2010) and also for generating pseudo-random numbers (Palacios-Luengas *et al.* 2019; Addabbo *et al.* 2006); (f) Skew-tent maps have been employed in chaos based communications (Hasler and Schimming 2000). The unique properties and rich features of the Tent Map and its skewed cousins (GLS maps) that enable these wide variety of engineering applications will be discussed.

This paper is organized as follows. In the next section we introduce the Tent map, Binary map and Generalized Lüroth Series (GLS). The unique properties of GLS are enumerated. What makes these GLS maps so attractive to a host of engineering applications? In the following section, the unreasonable effectiveness of the chaotic Tent map/GLS maps are discussed. We conclude with some open issues and suggest a few pointers for exciting future research.

TENT MAP, BINARY MAP AND GENERALIZED LÜROTH SERIES (GLS)

In this section, we define the Tent map and other closely related maps. We shall also describe the properties of these maps.

The Tent map (Figure 1(a)) is defined as $T : [0, 1) \rightarrow [0, 1)$:

$$T(x) = \begin{cases} 2x, & 0 \leq x < 0.5, \\ 2 - 2x, & 0.5 \leq x < 1. \end{cases} \tag{1}$$

The Skew-Tent map (Figure 1(b)) is a generalization of the Tent map and is defined as $T_b : [0, 1) \rightarrow [0, 1)$:

$$T_b(x) = \begin{cases} \frac{x}{b}, & 0 \leq x < b, \\ \frac{(1-x)}{(1-b)}, & b \leq x < 1, \end{cases} \tag{2}$$

where $0 < b < 1$ is the skew parameter. Setting $b = 0.5$ in Eq. 2 gives us the Tent map $T(x)$.

The Binary map (also known as Bernoulli Shift map, Figure 1(e)) is defined as $T : [0, 1) \rightarrow [0, 1)$:

$$T_{binary}(x) = \begin{cases} 2x, & 0 \leq x < 0.5, \\ 2x - 1, & 0.5 \leq x < 1. \end{cases} \tag{3}$$

A similar extension to Skew Binary map is also possible. In these examples, the maps are piecewise linear onto $[0, 1)$ with either a positive slope or negative slope. Generalizing this to an arbitrary finite number of intervals yields the 1D Generalized Lüroth Series or GLS maps (Figure 1(f)). The GLS map is defined as T_{GLS} : $[0, 1) \rightarrow [0, 1)$:

$$T_{GLS}(x) = \begin{cases} \frac{x}{p_1}, & 0 \leq x < p_1, \\ \frac{x-p_1}{p_2}, & p_1 \leq x < p_1 + p_2. \\ \dots & \\ \frac{x-\sum_1^{N-1} p_i}{p_N}, & \sum_1^{N-1} p_i \leq x < 1, \end{cases} \tag{4}$$

where the set of intervals $\{a_1, a_2, \dots, a_N\}$ have lengths $\{p_1, p_2, \dots, p_N\}$ respectively (note: $\sum_1^N p_i = 1$). In each of the intervals a_i, we have a linear mapping with a positive slope, but we could have chosen a line with negative slope instead. Thus, there are 2^N different GLS maps (piecewise linear) having the exact same set of intervals with the same lengths. They only differ in the sign of the slope of the linear mapping in one or more of the intervals (without any intrinsic change in chaotic dynamics).

It is easy to see that the Tent map, Binary map and their skewed cousins are all special instances of this family of 1D Generalized Lüroth Series (GLS) maps which we have defined above. We have to appropriately choose the set of intervals $\{a_i\}$ and their lengths $\{p_i\}$. Note that the set of intervals forms a Generating Markov Partition (GMP) on $[0, 1)$ for the GLS map.

Properties of GLS

GLS maps exhibit several interesting properties. We list a few of them here:

1. Continuity: GLS maps are piecewise linear and could be either continuous or not. This depends on the transition of the linear mapping across adjacent intervals a_i, a_j - whether there is a corner or not. Even if a GLS is continuous, it is not differentiable at the corner points.

2. Lebesgue measure and invariant distribution: GLS maps preserve the Lebesgue measure (Dajani and Kraaikamp 2002; Boyarski and Gora 1998) and has the uniform distribution on $[0, 1)$ as the invariant distribution (Figure 1(d)).

3. Generating Markov Partition: the set of intervals $\{a_1, a_2, \dots, a_N\}$ with lengths $\{p_1, p_2, \dots, p_N\}$ forms a Generating Markov Partition (GMP).

4. Symbolic dynamics on GLS: given the GMP on the GLS, we can associate symbols from the alphabet $\{'0', '1', \dots, 'N-1'\}$ to the N intervals $\{a_1, a_2, \dots, a_N\}$ respectively. Every initial value on the GLS yields a trajectory which can be associated with a *unique symbolic sequence* consisting of symbols from this alphabet (Dajani and Kraaikamp 2002; Nagaraj 2008).

5. Relationship between Lyapunov Exponent and Shannon Entropy: the lyapunov exponent λ of GLS map with the GMP defined in Figure 1(f) is given by:

$$\lambda_{GLS} = -\sum_{i=1}^{N} p_i \log(p_i). \tag{5}$$

The Shannon Entropy H of the symbolic sequence on the GLS is given by:

$$H_{GLS} = -\sum_{i=1}^{N} p_i \log_2(p_i) \;\; bits/symbol. \tag{6}$$

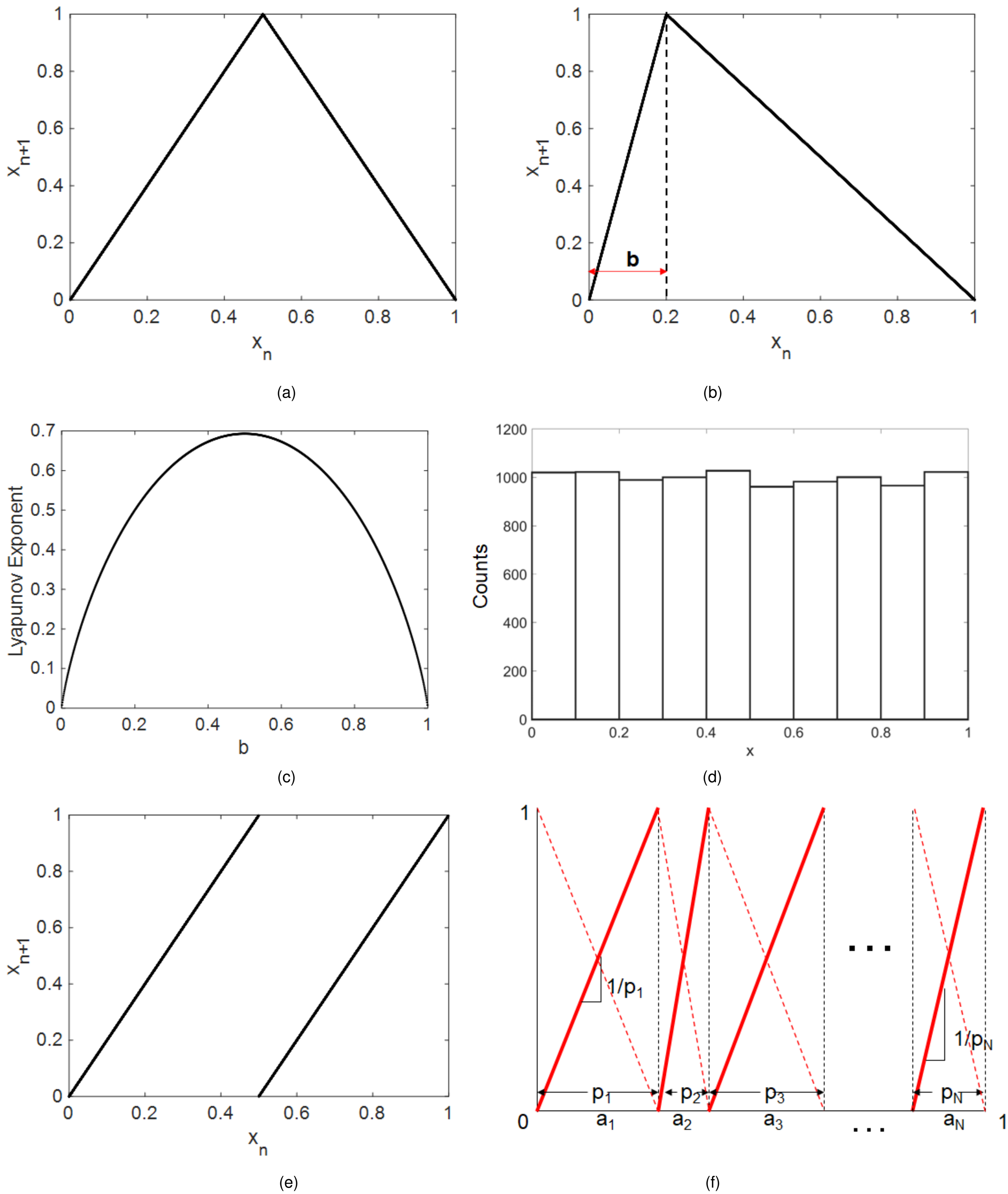

Figure 1 (a) Tent map, (b) Skew Tent Map, (c) Lyapunov exponents of the skew tent maps for different values of skew parameter b, (d) Histogram of a trajectory on the Tent map for a randomly chosen initial value, (e) Binary map, (f) Generalized Lüroth Series (GLS).

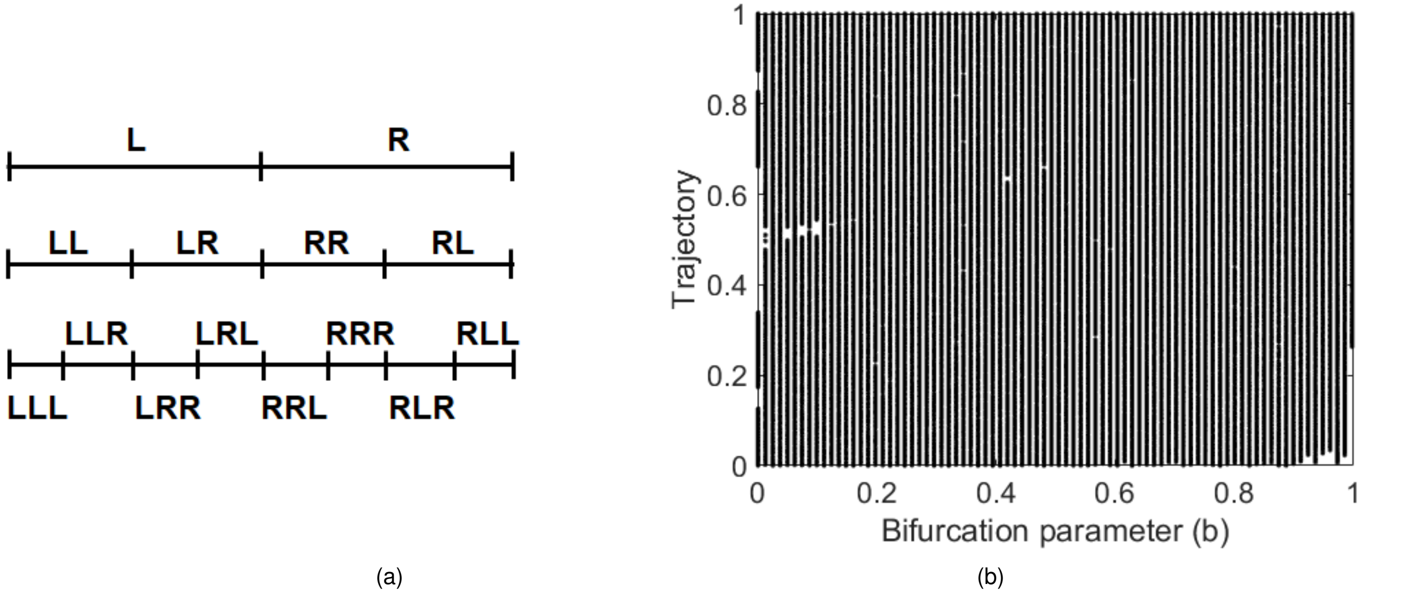

Figure 2 (a) Symbolic sequences of different orders on the Tent map. These form a *Gray Code*, (b) Bifurcation diagram for the Skew Tent Map. Absence of *windows* or attracting periodic orbits indicating *Robust Chaos*.

If the base of the logarithm is chosen as 2 in computation of the lyapunov exponent in Eq. 5, then we have $\lambda_{GLS} = H_{GLS}$ (Nagaraj 2008). We can interpret the lyapunov exponent as the number of bits of information of the initial value that is revealed at every iteration of the map. This equality plays a very significant role in lossless data compression of i.i.d sources (Nagaraj *et al.* 2009).

6. Periodic, quasi-periodic and chaotic behaviour: it is well known that the Tent map, Binary map, their skewed versions - all of these maps exhibit chaotic behaviour (Devaney 2018). This is indicated both by a positive lyapunov exponent (Figure 1(c)) for every value of $0 < b < 1$ and by the bifurcation diagram that reveals the fact that the entire family of GLS maps exhibit *Robust Chaos* (Banerjee *et al.* 1998; Glendinning 2017) - characterized by complete absence of attracting periodic orbits (Figure 2(b)), also known as *windows*. This is a very desirable property for a number of applications such as pseudorandom number generators (Nagaraj *et al.* 2008), chaos based cryptographic protocols and joint compression and encryption algorithms. As per Devaney's definition of chaos (Devaney 2018), GLS maps also have countably infinite number of periodic and quasi-periodic orbits that are dense, uncountably infinity of non-periodic trajectories and also exhibit sensitive dependence on initial values (the *Butterfly Effect*). The topological entropy of GLS is positive.

7. Ergodicity, mixing properties - as already noted, GLS maps preserve the Lebesque measure. They are also known to be ergodic and exhibit mixing. Variations of GLS show different degrees of mixing - weak and strong mixing. For more details, the reader is referred to Dajani and Kraaikamp (2002).

8. Topological transitivity and Universal orbits: GLS maps exhibit topological transitivity property defined as follows.

   Topological transitivity: for every pair of non-empty open sets $A, B \in [0, 1)$, there exists a non-negative integer m such that $T^m(A) \cap B \neq \emptyset$. Equivalently, there exists at least one initial value in A which when iterated a finite number of times ($m \geq 0$ iterations) reaches B.

   Universal orbits (also known as *dense orbits*) are special non-periodic trajectories which visit every possible non-empty neighbourhoods of $[0, 1)$. Equivalently, there exists an initial value $x_0 \in [0, 1)$ such that the set $\{x_0, T(x_0), T^2(x_0), \ldots, T^k(x_0), \ldots\}$ is dense in $[0, 1)$. This property is of utmost importance in Neurochaos Learning (NL) (Balakrishnan *et al.* 2019).

9. Number theoretic properties - Dajani and Kraaikamp (2002) discuss number theoretic properties of GLS and other variations of GLS (β expansions).

10. Implementation in software and hardware: given the piecewise linear nature of Tent map (and GLS), it enjoys a very low computational complexity for software implementation. There have been a number of hardware/electronic circuit realizations of the Tent map (Valtierra *et al.* 2017; Kumar *et al.* 2018; Hernandez *et al.* 2003; Campos-Cantón *et al.* 2009).

11. Nonlinear GLS: Nagaraj *et al.* (2009) propose a non-linear extension to GLS which preserves the Lebesgue measure that finds applications in joint compression and encryption techniques.

Table 1 Details of research works that have employed Tent/GLS maps (or their variations) in applications pertaining to electronics, communication (coding, error correction/detection, encryption) and computer science and engineering. This is not an exhaustive list.

Reference	Properties of GLS used	Applications
Hasler et al., 2000	Chaotic synchronization	Chaos shift keying using iterations of the skew tent map, chaotic communication systems.
Dajani et al., 2002	Ergodicity, mixing	Number theory.
Miliotis et al., 2008	"Be-headed" Tent map	Efficient and flexible storage, very fast search, amenable for parallel implementation.
Nagaraj et al., 2009	$\lambda_{GLS} = H_{GLS}$	GLS-coding: Shannon optimal lossless compression for i.i.d sources. Generalization of Arithmetic Coding that is used in JPEG2000.
Nagaraj et al., 2009	Noise-resistant GLS maps, symbolic sequence invariance	Multiplexing and de-multiplexing chaotic signals in the presence of noise.
Campos-Cantón et al., 2013	Chaos and ergodicity of the Tent map	Reconfigurable logical cell using evolutionary computation.
Wong et al., 2010	GLS maps with key-based switching	Simultaneous arithmetic coding or GLS-coding and encryption.
Nagaraj, 2012	Ergodicity/mixing	Joint compression and encryption. One-Time Pads that achieve unbreakable encryption (Perfect Secrecy) are nothing but switched GLS-coding.
Nagaraj, 2019	Cantor sets on GLS maps with a forbidden symbol	Error detection, joint compression and error control coding.
Palacios-Luengas et al., 2019	Ergodcity/mixing properties of Skew-Tent map	Psuedo Random Number Generators.
Balakrishnan et al., 2019	GLS used as a neuron. Topological transitivity, universal orbits, chaotic features	Brain-inspired machine learning (Neurochaos Learning or NL) for classification. State-of-the-art performance in low training sample regime.
Balakrishnan et al., 2021	Stochastic resonance at a GLS neuron	NL for classification tasks.
Balakrishnan et al., 2022	Topological transitivity, universal orbits, chaotic features of trajectories on GLS	Efficient classification of SARS-CoV-2 viral genome sequences using NL.
Balakrishnan et al., 2022	Causality preservation property of network of GLS neurons	Causality and machine Learning using NL. Deep learning fails to preserve causality.
Ajai et al., 2022	Heterogeneous network of GLS and Logistic map neurons	Classification tasks, further boost in performance of heterogeneous NL architecture

UNREASONABLE EFFECTIVENESS OF TENT/GLS MAPS

Table 1 is an attempt to succinctly summarize some of the published past research works that employ the Tent/GLS maps (or their variations) for applications in electronics and communications, computer science and engineering. It is by no means an exhaustive list of such published research. The specific maps used and the properties of these maps that enable these applications are also mentioned.

Why not Logistic map?

One may be wondering why we have not discussed the logistic map which is also an equally popular 1D chaotic map. In fact, logistic map is continuous and differentiable (unlike GLS maps which can only be continuous at best). There are a number of published research papers on properties and applications of the logistic map as well.

One of the most important reasons why the Tent map and GLS maps are preferable (over the Logistic map) in engineering applications is due to the piecewise linear nature of these maps. This enables an easy implementation of these maps in hardware and software. Furthermore, one of the important issues in the implementation of any dynamical system is finite numerical precision. Given an arbitrarily long symbolic sequence from a GMP of a GLS, it is possible to find the initial value to arbitrary precision. This is made possible because of the connection between GLS and Arithmetic Coding. Using ideas of finite precision implementation (such as scaling and re-normalization) of Arithmetic Coding, we can determine the initial value of a given arbitrarily long symbolic sequence on the GLS. This is used in GLS-coding as well as in multiplexing and de-multiplexing of chaotic signals in the presence of noise. Please see Nagaraj and Vaidya (2009) (Appendix) which describes this algorithm in detail.

This is also the reason that other 1D/2D maps such as the Standard Map, Sine Map, Circle Map, Hénon Map etc. are not preferred in practical engineering applications where finite precision effects can lead to problems.

Symbolic dynamics on GLS

Figure 2(a) depicts the symbolic sequences on the Tent map upto order 3. These sequences produces a *Binary Gray code*. A Gray code has the unique property that successive codes differ only in one location. A binary Gray code would have the property that consecutive codewords differ by exactly one hamming distance. Gray codes are widely employed in electromechanical switches to prevent spurious outputs and also in digital communications to enable error correction. This property can be extended to N-ary Gray codes using GLS with N intervals. The requirement for Gray codes is that the GLS should be continuous on the entire set $[0, 1)$.

Nagaraj and Vaidya (2009) construct noise-resistant versions of the Tent map (and Binary map) to enable efficient multiplexing and de-multiplexing of chaotic signals in the presence of noise. They employ the symbolic sequence invariance property and provide a finite precision implementation of finding the initial condition of an arbitrarily long symbolic sequence on the Tent/Binary map. Their scheme is able to multiplex/de-multiplex up to 20 chaotic signals in the presence of additive noise.

Compression, Coding and Cryptography Applications

GLS maps find applications in lossless data compression, joint compression and error detection and in several cryptographic schemes. The reason for the success of GLS maps in these kind of applications is due to the unique property that $\lambda_{GLS} = H_{GLS}$ (along with the property of chaos and ergodcity/mixing). To the best of one's knowledge, such a property is not true with any other map. This allows a very efficient handshake between dynamical properties with infotheoretic properties. The well known Kraft–McMillan inequality and its converse for prefix-free codes and the celebrated Huffman Coding are both related to symbolic dynamics on GLS maps (Nagaraj 2009, 2011).

The Tent map (and GLS maps) preserves the Lebesgue measure, has an uniform distribution as invariant, positive lyapunov exponent for all values of the bifurcation parameter, positive topological entropy and is ergodic. This is highly desirable for cryptographic algorithms, methods and protocols. Block ciphers and stream ciphers are required to have the properties of *confusion* and *diffusion*. These can be translated to strong mixing/ergodicity of the underlying chaotic map (Alvarez and Li 2006). The fact that Skew Tent map with the skew parameter b exhibits *Robust Chaos* for all values of b is very desirable for hardware implementation of cryptographic methods. Since there are no *windows* or attracting periodic orbits for any value of b, this means that perturbations to the parameter b due to noise in hardware implementations do not result in low periodicity. This is the problem with most maps that exhibit *fragile chaos*, i.e., the presence of windows or attracting periodic orbits. The logistic family of maps with the bifurcation parameter a: $x_{n+1} = ax_n(1 - x_n)$ exhibits fragile chaos which is problematic in cryptography applications.

Researchers have also employed GLS maps for simultaneous compression and encryption (Wong *et al.* 2010; Nagaraj *et al.* 2009; Nagaraj 2008, 2019).

Chaos based computing applications

The power of chaotic maps is their ability to generate a wide variety of patterns. This feature is available in even simple 1D maps that exhibit chaos such as the Logistic map and Tent map. Sinha and Ditto (1998) proposed, for the very first time, thresholded logistic map to emulate logic gates, encode numbers and perform simple arithmetic operations such as addition, multiplication and least common multiplier of a given sequence of integers. Since the publication of this pioneering work by Sinha and Ditto, several other researchers have contributed to this rich field of chaos-based computing. Ditto *et al.* (2010) proposes the *Chaogate* – a dynamical universal computing device which can be rapidly morphed to serve as any desired logic gate. Experimental realization of the same using a chaotic circuit has also been accomplished (Murali *et al.* 2005).

While the above research works focused on the Logistic map, it is easily translatable to the Tent map since there exists a topological conjugacy between these two maps (Alligood *et al.* 2000). There have been attempts also to use the Tent map for designing a reconfigurable logical cell (Campos-Cantón *et al.* 2013). Miliotis *et al.* (2008) employs the "be-headed" Tent map (or thresholded Tent map) ingeniously to efficiently and flexibly store information. They demonstrate how a single element can store M items (M could potentially be very large) and a very fast search by means of a single global shift operation is possible. Such a scheme is amenable for parallel implementation of chaos-based computing architectures.

Machine Learning: Neurochaos Learning using GLS neurons

One of the recent applications of GLS maps is in the design and construction of a novel neural network composed of GLS neurons as the input layer. This learning architecture is dubbed *Neurochaos Learning* or NL (Balakrishnan *et al.* 2019; Harikrishnan and Nagaraj 2021; Harikrishnan *et al.* 2022b). NL draws inspiration from the empirical fact that chaos is ubiquitous in the brain and found to manifest at several spatiotemporal scales - at the level of single neurons, coupled neurons and network of neurons (Korn and Faure 2003). The performance of NL on publicly available datasets for classification tasks in the domains of medical diagnosis, banknote fraud detection, environmental applications and spoken-digit classification is impressive and comparable to state-of-the-art Machine Learning (ML) and Deep Learning (DL) algorithms. Sethi *et al.* (2022) propose a hybrid learning architecture composed of chaos-based features from GLS neurons (NL) fed to classical ML algorithms such as Support Vector Machines, Logistic Regression, AdaBoost, Decision Trees, Random Forest, $k-$Nearest Neighbours and Naive Bayes classifiers. Such an approach provides a significant boost to the performance of standalone ML algorithms thereby indicating the efficacy of chaos-based features extracted from GLS neurons.

GLS neurons are shown to satisfy a version of the *Universal Approximation Theorem* (UAT) (Harikrishnan *et al.* 2022b) which is very desirable for learning algorithms since it allows for approximating complicated decision boundaries. The property of topological transitivity combined with presence of universal/dense orbits and ergodic/mixing properties of GLS makes it effective for machine learning applications. Another surprising property of GLS neurons is that an intermediate amount of noise added to the input is beneficial for classification performance in NL architecture (Harikrishnan and Nagaraj 2021). This is the well known *Stochastic Resonance* or noise-enhanced signal processing property found in certain non-linear systems. Ajai *et al.* (Sep. 2022) provide a heterogeneous Neurochaos Learning architecture using both GLS neurons and logistic map neurons to further enhance classification performance. GLS neurons also help preserve causality unlike Deep Learning architectures (Harikrishnan *et al.* 2022a).

CONCLUSION AND FUTURE WORKS

In this paper, we have explored the unique properties of the chaotic Tent map and more generally of Generalized Lüroth Series (GLS) maps. The Tent map, Binary map, Skew Tent maps are all examples of 1D GLS maps. We have discussed which specific properties of these maps contribute to their effectiveness in various engineering applications. To conclude, we shall list some pointers and directions for further research:

1. Harikrishnan *et al.* (2022b) have proven a version of the Universal Approximation Theorem (UAT) using GLS maps as neurons in the input layer of a novel learning architecture (NL). Further explorations on various versions of UAT and connections to standard Artificial Neural Networks (ANNs) and NL is a research direction worth investigating.

2. Coupled GLS maps for Neurochaos Learning: currently NL architecture consists of an input layer of 1D GLS maps which are independent of each other. Going forward, it would be interesting to explore addition of hidden layers to the network that consists of GLS maps which are coupled to the previous layers. Coupling between GLS neurons within each layer also needs to be explored. Such a *Deep* Neurochaos Learning architecture could further boost classification performance.

3. GLS maps have already proved their effectiveness in lossless data compression, joint compression and error detection and joint compression and encryption. However, incorporating an efficient error correction property along with compression and/or encryption has been elusive. This is an open problem.

4. GLS maps and other 1D chaotic maps (such as Logistic map) have shown promise in chaos based computing schemes. Have these matured to a stage where they can give serious competition to classical computer architectures?

5. GLS and their variations, especially β expansions have rich number theoretic properties. Is it possible to use these in practical engineering applications?

6. It is well understood that digital implementation of dynamical systems results in degradation of chaotic and ergodic properties (Li *et al.* 2005). Have we realized the full implications of these in practical applications involving GLS maps? What corrections are necessitated to combat the dynamical degradation of digital piecewise chaotic maps?

7. Another open problem is an efficient software implementation of determining the initial value for an arbitrarily long symbolic sequence on other 1D maps such as the Logistic map (without running into numerical precision issues). As mentioned earlier, such a method exists for the Tent map (GLS maps) but no such efficient method is known for other maps which are not piecewise linear.

To conclude, we foresee exciting novel applications of GLS maps in new domains. The (un)reasonable effectiveness of GLS maps (Tent map included) in engineering applications owe to a unique set of properties which other maps do not enjoy.

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Availability of data and material

Not applicable.

Conflicts of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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