Cumhuriyet Üniversitesi Fen Fakültesi
Fen Bilimleri Dergisi (CFD), Cilt 38, No. 2 (2017)
ISSN: 1300-1949

Cumhuriyet University Faculty of Science
Science Journal (CSJ), Vol. 38, No. 2 (2017)
ISSN: 1300-1949

http://dx.doi.org/10.17776/cumuscij.290566

Notes on Prime Near-Rings with Multiplicative Derivation

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Received: 07.02.2017; Accepted: 21.04.2017

Abstract: Let N be a left near ring. A map $d: N \to N$ is called a nonzero multiplicative derivation if $d(xy) = xd(y) + d(x)y$ holds for all $x, y \in N.$ In the present paper, we shall extend some well known results concerning commutativity of prime rings for nonzero multiplicative derivations of a left prime near-ring $N.$

Keywords: Prime ring, near-ring, derivation, multiplicative derivation

Çarpımsal Türevli Asal Yakın Halkalar Üzerine Notlar

Özet: N bir sol yakın halka olsun. $d: N \to N$ dönüşümü her $x, y \in N$ için $d(xy) = xd(y) + d(x)y$ koşulunu sağlıyorsa d ye bir çarpımsal türev denir. Bu makalede, asal halkalarda iyi bilinen bazı komütatiflik koşulları, çarpımsal türevli sol asal yakın halkalar için genelleştirilecektir.

Anahtar Kelimeler: Asal halka, yakın halka, türev, çarpımsal türev

1. INTRODUCTION

An additively written group $(N,+)$ equipped with a binary operation $. : N \to N, (x, y) \to xy,$ such that $x(yz) = (xy)z$ and $x(y+z) = xy + xz$ for all $x, y, z \in N$ is called a left near-ring. A near-ring N is called zero symmetric if $0x = 0$ for all $x \in N$ (recall that left distributive yields $x0 = 0$). A near-ring N is said to be 3-prime if $xNy = \{0\}$ implies $x = 0$ or $y = 0.$ For any $x, y \in N,$ as usual $[x, y] = xy - yx$ and $xoy = xy + yx$ will denote the well-known Lie and Jordan products respectively. The set $Z = \{x \in N \mid yx = xy$ for all $y \in N\}$ is called multiplicative center of $N.$ A mapping $d: N \to N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$. N is said to be $2-$torsion free if $x \in N$ and $x + x = 0$ implies $x = 0.$

Since Posner published his paper [11] in 1957, many authors have investigated properties of derivations of prime and semiprime rings. The study of derivations of near-rings was initiated by Bell and Mason in 1987 [1]. There has been a great deal of work concerning commutativity of prime and semiprime rings and near-rings with derivations satisfying with certain differential identities. (see references for a partial bibliography).

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In [7], Herstein has proved that if R is a prime ring of characteristic different from 2 and if d is a nonzero derivation of R such that $d(R) \subseteq Z,$ then R is commutative. In [3], Bell and Kappe have proved that d is a derivation of R which is either a homomorphism or an anti-homomorphism in semiprime ring R or a nonzero right ideal of R then $d = 0.$ In [5], Daif and Bell proved that if R is semiprime ring, U is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) = \pm[x, y],$ for all $x, y \in U,$ then $U \subseteq Z.$ All of these results were extended to near rings.

In [4], the notion of multiplicative derivation was introduced by Daif motivated by Martindale in [8]. $d : R \to R$ is called a multiplicative derivation if $d(xy) = xd(y) + d(x)y$ holds for all $x, y \in R.$ These maps are not additive. In [6], Goldman and Semrl gave the complete description of these maps. We have $R = C[0,1],$ the ring of all continuous (real or complex valued) functions and define a map $d : R \to R$ such as

$$d(f)(x) = \begin{cases} f(x)\log|f(x)|, & f(x) \neq 0 \\ 0, & \text{otherwise} \end{cases}.$$

It is clear that d is multiplicative derivation, but d is not additive.

Recently, some results concerning commutativity of prime rings with derivations were proved for multiplicative derivations. It is natural to look for comparable results with multiplicative derivations of near-rings. In the present paper, we shall extend above mentioned results for multiplicative derivations of 3-prime near-ring $N.$ Also, we will prove some commutativity conditions.

Chapter 1:

Lemma 1 *[2, Lemma 1.2]Let N be a $3-$prime near-ring.*

(i) *If $z \in Z \setminus \{0\}$, then z is not a zero divisor.*

(ii) *If Z contains a nonzero element z for which $z + z \in Z$, then $(N,+)$ is abelian.*

(iii) *If $z \in Z \setminus \{0\}$ and $x \in N$* such that $xz \in Z$ or $zx \in Z,$ *then $x \in Z.$*

Lemma 2 *[2, Lemma 1.5] Let N be a $3-$prime near ring. If Z contains a nonzero semigroup ideal of N, then N is commutative ring.*

Lemma 3 *[9, Lemma 2.1]A near-ring N admits a multiplicative derivation if and only if it is zero symmetric.*

Lemma 4 *Let N be a near-ring and $d : N \to N$ multiplicative derivation of $N.$ Then*

$(xd(y) + d(x)y)z = xd(y)z + d(x)yz,$ for all $x, y, z \in N.$

Proof: By calculating $d(xyz)$ in two different ways, we see that

$$d((xy)z) = xyd(z) + d(xy)z$$

and

$$d(x(yz)) = xd(yz) + d(x)yz$$
$$= xyd(z) + xd(y)z + d(x)yz.$$

Hence we have

$$d(xy)z = xyd(z) + xd(y)z$$

and so

$$(xd(y) + d(x)y)z = xd(y)z + d(x)yz, \text{for all } x, y, z \in N.$$

Lemma 5 *Let N be a 3-prime near-ring and $a \in N$. If N admits a nonzero multiplicative derivation d such that $d(N)a = 0$ (or $ad(N) = 0$), then $a = 0$.*

Proof. By the hypothesis, we get

$$d(xy)a = 0, \text{for all } x, y \in N.$$

Expanding this equation with Lemma 4 and using the hypothesis, we have

$$d(x)Na = (0), \text{for all } x \in N.$$

Since N is 3-prime near-ring and $d \neq 0$, we obtain that $a = 0$.

$ad(N) = 0$ can be proved by applying the same techniques.

Theorem 1 *Let N be a 3-prime near-ring. If N admits a nonzero multiplicative derivation d such that $d(N) \subseteq Z$, then N is a commutative ring.*

Proof. For any $x, y \in N$, we get $d(xy) \in Z$, and so

$$d(xy)y = yd(xy).$$

That is

$$(xd(y) + d(x)y)y = y(xd(y) + d(x)y).$$

Using Lemma 4, we get

$$xd(y)y + d(x)yy = yxd(y) + yd(x)y.$$

Since $d(N) \subseteq Z,$ we arrive at

$$d(y)xy + d(x)yy = d(y)yx + d(x)yy$$

and so

$$d(y)[x, y] = 0.$$

Using Lemma 1 (i), we have for each fixed $y \in N$ either $d(y) = 0$ or $y \in Z.$

Now, we assume $d(y) = 0.$ For any $x \in N,$ we have $d(xy) \in Z$ by the hypothesis. Since $d(y) = 0,$ we get $d(xy) = d(x)y \in Z,$ for all $x \in N.$ By Lemma 1 (iii), we get $d(x) = 0,$ for all $x \in N$ or $y \in Z.$ Since $d \neq 0,$ we must have $y \in Z.$ Hence we arrive at $y \in Z$ for any cases. That is $N \subseteq Z,$ and so N is commutative near-ring by Lemma 2.

Theorem 2 *Let* N *be a 3-prime near-ring and* d *a multiplicative derivation of* N *such that* $d(xy) = d(x)d(y),$ *for all* $x, y \in N,$ *then* $d = 0.$

Proof. In view of our hypothesis, we have

$$xd(y) + d(x)y = d(x)d(y), \text{for all } x, y \in N. \tag{2.1}$$

Replacing y by yz in (2.1), we get

$$xd(yz) + d(x)yz = d(x)d(yz).$$

By our hypothesis, we have

$$xd(y)d(z) + d(x)yz = d(x)d(y)d(z)$$

and so

$$xd(y)d(z) + d(x)yz = d(xy)d(z).$$

Since d is multiplicative derivation of $N,$ we arrive at

$$xd(y)d(z) + d(x)yz = (xd(y) + d(x)y)d(z).$$

By Lemma 4, we get

$$xd(y)d(z) + d(x)yz = xd(y)d(z) + d(x)yd(z), \text{for all } x, y, z \in N.$$

That is

$$d(x)yz = d(x)yd(z), \text{for all } x, y, z \in N.$$

Since N is left near-ring, we have

$$d(x)N(d(z) - z) = (0), \text{for all } x, z \in N.$$

By the 3-primeness of N, we arrive at

$$d = 0 \text{ or } d(z) = z, \text{for all } z \in N.$$

If $d(z) = z$, for all $z \in N$, then

$$d(xy) = xd(y) + d(x)y$$

$$xy = xy + xy$$

$$xy = 0, \text{for all } x, y \in N.$$

This yields that $N = (0)$, a contradiction. So, we must have $d = 0$. This completes the proof of our theorem.

Theorem 3 *Let* N *be a 3-prime near-ring and* d *a multiplicative derivation of* N *such that* $d(xy) = d(y)d(x)$, *for all* $x, y \in N$, *then* $d = 0$.

Proof. By our hypothesis, we have

$$xd(y) + d(x)y = d(y)d(x), \text{for all } x, y \in N. \tag{2.2}$$

Replacing y by xy in (2.2), we get

$$xd(xy) + d(x)xy = d(xy)d(x).$$

In view of our hypothesis, we have

$$xd(y)d(x) + d(x)xy = d(xy)d(x).$$

Using d is multiplicative derivation of N, we arrive at

$$xd(y)d(x) + d(x)xy = (xd(y) + d(x)y)d(x).$$

By Lemma 4, we get

$$xd(y)d(x) + d(x)xy = xd(y)d(x) + d(x)yd(x)$$

and so

$$d(x)xy = d(x)yd(x), \text{for all } x, y \in N. \tag{2.3}$$

Taking yz instead of y in (2.3) and using (2.3), we obtain that

$$d(x)N[z, d(x)] = 0, \text{for all } x, z \in N.$$

By the 3-primeness of $N,$ we get

$$d(x) = 0 \text{ or } d(x) \in Z.$$

Now, $d(x) = 0$ implies that $d(x) \in Z.$ So, we have $d(N) \subseteq Z$ for any cases. By Theorem 1, we obtain that N is commutative ring or $d = 0.$ If N is commutative ring, then $d(xy) = d(y)d(x) = d(x)d(y),$ for all $x, y \in N.$ Hence, we get $d = 0$ by Theorem 2. This completes the proof.

Theorem 4 *Let N be a 3-prime near-ring and d a nonzero multiplicative derivation of N such that $d([x, y]) = [d(x), y],$ for all $x, y \in N,$ then N is commutative ring.*

Proof. Replacing xy instead of y in the hypothesis, we get

$$d(x[x, y]) = [d(x), xy].$$

Expanding this equation and using the hypothesis, we have

$$xd([x, y]) + d(x)[x, y] = [d(x), xy]$$

$$x[d(x), y] + d(x)[x, y] = [d(x), xy]$$

$$xd(x)y - xyd(x) + d(x)[x, y] = d(x)xy - xyd(x).$$

On the other hand, replacing $y = 0$ in the hypothesis, we arrive at $d(0) = 0.$ Again replacing x instead of y in the hypothesis, we get

$$[d(x), x] = 0$$

and so

$$d(x)x = xd(x), \text{for all } x \in N.$$

Now, using this in the above equation, we find that

$$d(x)xy - xyd(x) + d(x)[x, y] = d(x)xy - xyd(x)$$

$$d(x)[x, y] = 0$$

and so

$$d(x)xy = d(x)yx, \text{for all } x, y \in N.$$

Replacing y by yz in this equation and using this, we have

$$d(x)N[x, z] = (0), \text{for all } x, z \in N.$$

This yields that

$$d(x) = 0 \text{ or } x \in Z.$$

If $d(x) = 0,$ then $d(x) \in Z.$ On the otherwise, if $x \in Z$ then $[d(x), y] = 0,$ for all $y \in N$ by the hypothesis. Hence we have $d(x) \in Z.$ Thus we arrive at $d(x) \in Z,$ for both cases. That is $d(N) \subseteq Z,$ and so, we obtain that N is commutative ring by Theorem 1.

Theorem 5 *Let N be a 3-prime near-ring and d a nonzero multiplicative derivation of N such that $[d(x), y] = [d(x), d(y)],$ for all $x, y \in N,$ then N is commutative ring.*

Proof. If $d(x) \in Z,$ then there is nothing to prove. So we assume that $d(x) \notin Z,$ for any $x \in N.$ In the view of the hypothesis, we get

$$[d(x), y] = [d(x), d(y)], \text{for all } x, y \in N.$$

Writing $d(x)y$ instead of y in this equation, we get

$$[d(x), d(d(x)y)] = [d(x), d(x)y]$$

$$d(x)d(d(x)y) - d(d(x)y)d(x) = d(x)[d(x), y].$$

Using d is multiplicative derivation of N and Lemma 4, we arrive at

$$d(x)d(x)d(y) + d(x)d^2(x)y - (d(x)d(y)d(x) + d^2(x)yd(x)) = d(x)[d(x), y].$$

By the hypothesis, we have

$$d(x)d(x)d(y) + d(x)d^2(x)y - (d(x)d(y)d(x) + d^2(x)yd(x)) = d(x)[d(x), d(y)].$$

Expanding this term and using $-(a+b)=-b-a,$ we arrive at

$$d(x)d(x)d(y)+d(x)d^2(x)y-d^2(x)yd(x)-d(x)d(y)d(x)=d(x)d(x)d(y)-d(x)d(y)d(x)$$

and so

$$d(x)d^2(x)y=d^2(x)yd(x), \text{for all } x,y\in N.$$

Replacing yz instead of y in the last equation, we find that

$$d^2(x)N[d(x),z]=(0), \text{for all } x,z\in N.$$

By the 3-primeness of $N,$ we get for each $x\in N$

$$d^2(x)=0 \text{ or } d(x)\in Z.$$

Since $d(x)\notin Z,$ we must have $d^2(x)=0,$ for all $x\in N.$ Writing $d(y)$ instead of y in the hypothesis and using $d^2(y)=0,$ we arrive at $[d(x),d(y)]=0.$ Again using this in the hypothesis, we have $[d(x),y]=0,$ and so $d(x)\in Z,$ a contradiction. Hence, we must have $d(N)\subseteq Z,$ and so, N is commutative ring by Theorem1. This completes the proof.

Theorem 6 *Let N be a 3-prime near-ring, d a multiplicative derivation of N. If $[d(x),y]\in Z,$ for all $x,y\in N,$ then N is a commutative ring.*

Proof. Replacing y by $d(x)y$ in the hypothesis yields that

$$d(x)[d(x),y]\in Z, \text{for all } x,y\in N.$$

By Lemma 1 (iii), we get

$$d(x)\in Z \text{ or } [d(x),y]=0, \text{for all } x,y\in N.$$

For any cases, we obtain that $d(N)\subseteq Z.$ By Theorem 1, we obtain that N is a commutative ring.

Acknowledgments: This work is supported by the Scientific Research Project Fund of Cumhuriyet University under the project number F-496.

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