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Nonconvex integro-differential sweeping processes involving maximal monotone operators

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Abstract

This paper is devoted to the study of a perturbed differential inclusion governed by a nonconvex sweeping process in a Hilbert space. The sweeping process is perturbed by a sum of an integral forcing term which the integrand depends on two time-variables and a maximal monotone operator. By using a semi-regularization method combined with a Gronwall-like inequality we prove solvability of the initial value problem.

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1. Introduction

After their introduction by Stampacchia in the 1960s (see [10,19]), variational and quasi-variational inequalities have been a rich field of research for the mathematical community, with many applications to physics, mechanics, and economics, among others. As the generalization of variational inequalities, the theory of hemivariational inequalities was first introduced and studied by Panagiotopoulos in [17]. The mathematical theory of hemivariational inequalities has been of great interest recently, which is due to the intensive development of applications of hemivariational inequalities in contact mechanics, control theory, games and so forth. Some comprehensive references are [11–14]. One of the typical formulations of the evolution variational inequality problem found in the literature is the sweeping process which was introduced and largely treated by J. J. Moreau in a series of papers, in particular in [15,16]. It was shown in [15] that such processes play a fundamental role in mechanics, especially in elasto-plasticity, quasi-statics, dynamics. The mathematical model of the sweeping process (see [15,16]) corresponds to a point which is swept by a moving closed convex set $C(t)$ in a Hilbert space H according to the differential inclusion

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) \qquad a.e.\ \ t \in [T_0, T] \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

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where $T_0, T \in \mathbb{R}$ with $0 \leq T_0 < T$ and $N_{C(t)}(\cdot)$ denotes the normal cone of $C(t)$ (in the standard sense). The analysis of systems with external forces led to consider and analyze the following perturbed variant

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f(t, x(t)) & a.e.\ t \in [T_0, T] \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where $f : [T_0, T] \times H \to H$ is a Carathéodory mapping, i.e., $f(t, \cdot)$ is continuous and $f(\cdot, x)$ is Bochner measurable for $[T_0, T]$ endowed with the Borel σ-field $\mathcal{B}([T_0, T])$. By Bochner measurable mapping we mean here any limit of uniformly convergent sequence of simple mappings from $[T_0, T]$ into H with $[T_0, T]$ endowed with its Borel σ-field.

In this paper, we are interested in a variant of *integro-differential sweeping process of Volterra type* associated with maximal monotone operators of following form,

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + Ax(t) + \int\limits_0^t f(t, s, x(s)) ds & \text{a.e. } t \in [0, T] \\ x(0) = x_0 \in C(0), \end{cases} \tag{$P_{A,f}$}$$

where $A : H \rightrightarrows H$ is a set-valued maximal monotone operator, $C(t)$ is a prox-regular moving set of the Hilbert space H and $N_{C(t)}(\cdot)$ is its proximal normal cone.

The well-posedness of the sweeping process with a maximal monotone perturbation $(P_{A,0})$, i.e., $f \equiv 0$, has been studied in [2]. Sweeping process involving integral perturbation, i.e., $P_{0,f}$ (with $A \equiv 0$) was considered recently in [4, 5].

In the present paper, we obtain results on the existence and uniqueness of a solution to the Volterra sweeping process $(P_{A,f})$ in a Hilbert space. This is done with the help of a Gronwall-like inequality and of a semi-regularization corresponding to the existence and uniqueness of absolutely continuous solutions for the integro-differential sweeping processes

$$\begin{cases} -\dot{x}_\lambda(t) \in N_{C(t)}(x_\lambda(t)) + A_\lambda x_\lambda(t) + \int\limits_0^t f(t, s, x_\lambda(s))\, ds \quad \text{a.e.} \quad t \in [0, T], \\ x_\lambda(0) = x_0 \in C(0). \end{cases}$$

where $\lambda > 0$.

The paper is organized as follow. In Section 2, we recall some preliminary results that we use throughout. In Section 3, we present our main existence and uniqueness result.

2. Notations and preliminaries

In all the paper H is a real Hilbert space whose scalar product will be denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\|\cdot\|$. For any $x \in H$ and $r \geq 0$, the closed (respectively open) ball centered at x with radius r will be denoted by $B[x, r]$ (respectively $B(x, r)$). For $x = 0$ and $r = 1$, we will put $\mathbb{B}_H$ or $\mathbb{B}$ in place of $B[0, 1]$. Further, if C is a subset of H, we denote by $\delta_C(\cdot)$ or $\delta(\cdot, C)$ the indicator function of C, that is, $\delta(x, C) = 0$ if $x \in C$ and $+\infty$ otherwise. For a set $J \subset \mathbb{R}$, the notation $\mathbf{1}_J(\cdot)$ stands for the characteristic function in the sense of measure theory, i.e., for all $x \in \mathbb{R}, \mathbf{1}_J(x) = 1$ if $x \in J$ and $\mathbf{1}_J(x) = 0$ otherwise.

We first give some background material on variational analysis. We state only the definitions and results which will be needed in the development of the paper. Throughout this section, we will denote by S a nonempty closed subset of H. A vector $v \in H$ is said to be a proximal normal to S at $x \in S$ if there exists a real $\rho > 0$ such that $x \in \mathrm{Proj}_S(x + \rho v)$, where $\mathrm{Proj}(\cdot) : H \rightrightarrows H$ denotes the metric projection on S defined by

$$\mathrm{Proj}_S(z) := \{u \in H : d_S(z) = \|z - u\|\}.$$

Here $d_S(z) := \inf_{y\in S}\|z-y\|$ is the distance function from S; sometimes it will be convenient to put $d(\cdot, S)$ instead of $d_S(\cdot)$. If the $\mathrm{Proj}_S(z)$ is a singleton set, we say that the metric projection of z on S is well defined and we recall the convention to denote by $\mathrm{proj}_S(z)$ its unique point. The set of all proximal normal vectors to S at $x \in S$ will be denoted by $N(S;x)$ or $N_S(x)$ and called the proximal normal cone to S at x (see e.g., Clarke et al. [7]). If $x \notin S$, we put by convention $N_S(x) = \emptyset$.

Definition 2.1. Let $r \in]0, +\infty]$. The nonempty closed set S is said to be r-prox-regular (or uniformly prox-regular with constant r) if each point x in the open r-enlargement of S

$$U_r(S) := \{u \in H : d(u, S) < r\},$$

has a unique nearest point $\mathrm{proj}_S(x)$ and the mapping $\mathrm{proj}_S(\cdot)$ is continuous over $U_r(S)$. It is clear that the r-prox-regularity of S with $r = +\infty$ corresponds to its convexity. This class of sets was initialy introduced by Federer [9] in the finite-dimensional framework under the name "positively reached sets".

The following proposition provides some useful characterizations and properties of uniform prox-regular sets for which we refer to [8, 18, 20].

Theorem 2.2. *Let S be a closed set in H and $r > 0$. The followings are equivalent.*

(a) *S is r-prox-regular.*

(b) *For all $x \in S$ and $\xi \in N(S;x)$, we have*

$$\langle \xi, y - x\rangle \leq \frac{\|\xi\|}{2r}\|y-x\|^2 \quad \forall y \in S.$$

(c) *For all $x, x' \in S$, for all $\xi \in N(S;x) \cap \mathbb{B}$, and for all $\xi' \in N(S;x') \cap \mathbb{B}$, we have*

$$\langle \xi - \xi', x - x'\rangle \geq -\frac{1}{r}\|x-x'\|^2.$$

The property (c) of the latter theorem means that the multimapping $N(S;\cdot) \cap \mathbb{B}$ is Hypomonotone. In the following, we summarize some known definitions and results concerning maximal monotone operators. The domain and graph of a set-valued operator $A : H \rightrightarrows H$ are defined, respectively, by

$$\mathrm{Dom}(A) := \{x \in H : A(x) \neq \emptyset\}, \quad \mathrm{gph}(A) := \{(x,y) : x \in H, y \in A(x)\}.$$

The operator A is called monotone if for all $x, y \in H$, $x^* \in A(x), y^* \in A(y)$, we have $\langle x^* - y^*, x - y\rangle \geq 0$. In addition, if there is no monotone mapping B such that $\mathrm{gph}A$ is contained strictly in $\mathrm{gph}B$, then A is called maximal monotone. When A is a maximal monotone operator, we denote by $A^\circ(x)$ the element of the closed convex set $A(x)$ of minimal norm, that is,

$$\|A^\circ(x)\| = \min\{\|\xi\|, \xi \in A(x)\}.$$

Let $\lambda > 0$, the Yosida approximation of index λ of A is the operator $A_\lambda : H \to H$ defined on H by

$$A_\lambda(x) := \frac{1}{\lambda}(x - J_\lambda(x)),$$

where $J_\lambda : H \to H$ is the resolvent map of A defined by $J_\lambda(x) := (I + \lambda A)^{-1}(x)$.

We now present the basic properties of the Yosida approximation of a maximal mononotone operator and its resolvent map. For the proofs of these results we refer the reader to [3, 6].

Proposition 2.3. *Let $A : H \rightrightarrows H$ be maximal monotone operator and let $\lambda > 0$. Then*

(i) *J_λ is a non-expansive single-valued map from H to H, that is*

$$\|J_\lambda(x) - J_\lambda(y)\| \leq \|x - y\| \text{ for all } x, y \in H.$$

(ii) A_λ *is* $1/\lambda$*-Lipschitz continuous, maximal monotone and for all* $x \in H, A_\lambda(x) \in A(J_\lambda(x))$. *Further, if* $x \in \operatorname{Dom}(A)$ *then* $\|A_\lambda(x)\| \leq \|A^\circ(x)\|$.

(iii) *If* $x_\lambda \longrightarrow x$ *as* $\lambda \downarrow 0$ *and* $(A_\lambda x_\lambda)_\lambda$ *is bounded then* $x \in \operatorname{Dom}(A)$.

The next proposition addresses some closeness properties of the graph of a maximal monotone operator as well as its extension to L^2.

Proposition 2.4 ([6])**.**
Let $A : H \rightrightarrows H$ *be maximal monotone. Then*

(a) *A is sequentially weak-strong and strong-weak closed.*

(b) *A is locally bounded in* $\operatorname{int}\operatorname{Dom}(A)$, *where* $\operatorname{int} S$ *denotes the interior of a subset* S *of* H.

(c) *Let* $T > 0$. *Then, the extension of* A *to* $L^2([0,T];H)$ *noted by* $\mathcal{A} : L^2([0,T];H) \rightrightarrows L^2([0,T];H)$ *and defined by*

$$v(\cdot) \in \mathcal{A}u(\cdot) \Leftrightarrow v(t) \in A(u(t)) \text{ a.e } t \in [0,T],$$

is maximal monotone.

3. Main result

In this section, we study the existence of solutions of $(P_{A,f})$ by using a semi-regularization approach and some properties of a classical class of integral perturbed sweeping processes. Before going on, we start firstly with the following auxiliary lemma.

Lemma 3.1 (Gronwall-like differential inequality [5])**.** *Let* $\rho : [T_0,T] \to \mathbb{R}_+$ *be a non-negative absolutely continuous function and let* $K_1, K_2, \varepsilon : [T_0,T] \to \mathbb{R}_+$ *be non-negative Lebesgue integrable functions. Suppose for some* $\epsilon > 0$

$$\dot{\rho}(t) \leq \varepsilon(t) + \epsilon + K_1(t)\rho(t) + K_2(t)\sqrt{\rho(t)} \int_{T_0}^{t} \sqrt{\rho(s)}\, ds \ \text{ a.e } t \in [T_0,T].$$

Then for all $t \in [T_0,T]$, *one has*

$$\begin{aligned} \rho(t) \leq & \sqrt{\rho(T_0)+\epsilon}\exp\left(\int_{T_0}^{t}(K(s)+1)\,ds\right) + \frac{\sqrt{\epsilon}}{2}\int_{T_0}^{t}\exp\left(\int_{s}^{t}(K(\tau)+1)\,d\tau\right)ds \\ & + 2\left(\sqrt{\int_{T_0}^{t}\varepsilon(s)\,ds+\epsilon} - \sqrt{\epsilon}\exp\left(\int_{T_0}^{t}(K(\tau)+1)\,d\tau\right)\right) \\ & + 2\int_{T_0}^{t}(K(s)+1)\exp\left(\int_{s}^{t}(K(\tau)+1)\,d\tau\right)\sqrt{\int_{T_0}^{s}\varepsilon(\tau)\,d\tau+\epsilon}\; ds, \end{aligned}$$

where $K(t) := \max\left\{\dfrac{K_1(t)}{2}, \dfrac{K_2(t)}{2}\right\}$, *a.e* $t \in [T_0,T]$.

For the sake of readability, we collect the hypotheses used along with the paper.
Assumption 1: For each $t \in [0,T]$, $C(t)$ is a nonempty closed subset of H which is r-prox-regular for some constant $r \in]0,+\infty]$, and has an absolutely continuous variation, in the sense that there is some absolutely continuous function $v : [0,T] \to \mathbb{R}$ such that

$$C(t) \subset C(s) + |v(t) - v(s)|\mathbb{B}, \quad \forall\, t,s \in [0,T].$$

Assumption 2: The set-valued mapping $A : H \rightrightarrows H$ is a maximal monotone operator.
Assumption 3: $f : Q_\Delta \times H \longrightarrow H$ is a measurable mapping such that, there exists a

non-negative function $\beta(\cdot,\cdot) \in L^1(Q_\Delta, \mathbb{R}_+)$ such that

$$\|f(t,s,x)\| \leq \beta(t,s)(1+\|x\|), \text{ for all } (t,s) \in Q_\Delta \text{ and for any } x \in \operatorname{rge}(C),$$

where $\operatorname{rge}(C) := \bigcup_{t\in[0,T]} C(t)$ denotes the range of the set-valued mapping $C(\cdot)$ and

$$Q_\Delta := \{(t,s) \in [0,T] \times [0,T] : s \leq t\}.$$

Assumption 4: for each real $\eta > 0$ there exists a non-negative function $L^\eta(\cdot) \in L^1([0,T], \mathbb{R}_+)$ such that for all $(t,s) \in Q_\Delta$ and for any $(x,y) \in B[0,\eta] \times B[0,\eta]$,

$$\|f(t,s,x) - f(t,s,y)\| \leq L^\eta(t)\|x-y\|.$$

We come now to our main result in this work which gives the existence result of $(P_{A,f})$.

Theorem 3.2. *Assume that the assumptions 1-4 above hold. Under the following additional conditions*

(1) $\operatorname{rge}(C) \subset \operatorname{Dom}(A)$.
(2) *there exists a non-negative functions* $\alpha(\cdot), \delta(\cdot) \in L^2([0,T], \mathbb{R}_+)$ *with* $\alpha(\cdot)$ *is a continuous function, such that*

$$\|A^\circ x\| \leq \alpha(t)\|x\| + \delta(t), \text{ for a.e. } t \in [0,T] \text{ and } x \in C(t),$$

(3) *there exists two functions* $\gamma_1(\cdot), \gamma_2(\cdot) \in L^1([0,T], \mathbb{R}_+)$ *such that*

$$\beta(t,s) \leq \gamma_1(t).\gamma_2(s) \text{ for a.e. } t \in [0,T],$$

for each $x_0 \in C(0)$, *the differential inclusion* $(P_{A,f})$ *admits, at least, an absolutely continuous solution* $x(\cdot) : [0,T] \to H$.

Proof. The idea of the proof is to study a family of sweeping process involving an integral perturbation and the Yosida approximation of the maximal monotone operator A, and then pass to the limit on the approximate solutions. This semi-regularization process is possible thanks to nice properties of A_λ.

Step 1. A family of approximate solutions.
Fix any $\lambda > 0$ and consider the following approximate problem

$$\begin{cases} -\dot{x}_\lambda(t) \in N_{C(t)}(x_\lambda(t)) + A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds \quad \text{a.e.} \quad t \in [0,T], \\ x_\lambda(0) = x_0 \in C(0). \end{cases} \tag{3.1}$$

From Theorem 4.1 in [5], it results that for any $\lambda > 0$, the differential inclusion with integral perturbation (3.1) has a unique absolutely continuous solution $x_\lambda(\cdot)$ on $[0,T]$. Moreover, for almost every $t \in [0,T]$

$$\|\dot{x}_\lambda(t) + A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds\| \leq \|A_\lambda x_\lambda(t)\| + \int_0^t \|f(t,s,x_\lambda(s))\|\,ds + |\dot{v}(t)|. \tag{3.2}$$

Step 2. Let us establish an upper bound of the norm of the approximate solutions $x_\lambda(\cdot)$.
From the assumptions above, one has $\|A_\lambda(x_\lambda(t))\| \leq \|A^\circ(x_\lambda(t))\| \leq \alpha(t)\|x_\lambda(t)\| + \delta(t)$ and

$$\begin{aligned} \int_0^t \|f(t,s,x_\lambda(s))\|\,ds &\leq \int_0^t \beta(t,s)(1+\|x_\lambda(s)\|)\,ds \leq \int_0^t \beta(t,s)\,ds + \int_0^t \beta(t,s)\|x_\lambda(s)\|\,ds \\ &\leq \int_0^t \beta(t,s)\,ds + \gamma_1(t)\int_0^t \gamma_2(s)\|x_\lambda(s)\|\,ds, \text{ a.e } t \in [0,T]. \end{aligned}$$

Hence, from (3.2)

$$\|\dot{x}_\lambda(t)\| \le 2\|A_\lambda x_\lambda(t)\| + 2\int_0^t \|f(t,s,x_\lambda(s))\|\,ds + |\dot{v}(t)|$$
$$\le 2\alpha(t)\|x_\lambda(t)\| + 2\,\gamma_1(t)\int_0^t \gamma_2(s)\|x_\lambda(s)\|\,ds + 2\,\delta(t) + 2\int_0^t \beta(t,s)\,ds + |\dot{v}(t)|.$$

On the other hand, one has

$$\|x_\lambda(t)\| = \|x_0 + \int_0^t \dot{x}_\lambda(t)\,ds\| \le \|x_0\| + \int_0^t \|\dot{x}_\lambda(t)\|\,ds$$
$$\le \|x_0\| + \int_0^t \Big(2\alpha(\tau)\|x_\lambda(\tau)\| + 2\gamma_1(\tau)\int_0^\tau \gamma_2(s)\|x_\lambda(s)\|\,ds + 2\delta(\tau)$$
$$+ 2\int_0^\tau \beta(\tau,s)\,ds + |\dot{v}(\tau)|\Big)d\tau$$
$$= \|x_0\| + \Xi(t) + 2\int_0^t \alpha(\tau)\|x_\lambda(\tau)\|d\tau + 2\int_0^t \gamma_1(\tau)\Big(\int_0^\tau \gamma_2(s)\|x_\lambda(s)\|\,ds\Big)d\tau,$$

where

$$\Xi(t) := 2\int_0^t \delta(\tau)d\tau + 2\int_0^t\int_0^\tau \beta(\tau,s)\,ds\,d\tau + \int_0^t |\dot{v}(\tau)|d\tau,$$

using the fact that

$$\int_0^t \gamma_1(\tau)\Big(\int_0^\tau \gamma_2(s)\|x_\lambda(s)\|\,ds\Big)d\tau \le \|\gamma_1\|_{L^1([0,T],\mathbb{R}_+)}\int_0^t \gamma_2(s)\|x_\lambda(s)\|\,ds,$$

we obtain

$$\|x_\lambda(t)\| \le \|x_0\| + \Xi(t) + \int_0^t \Upsilon(\tau)\|x_\lambda(\tau)\|\,d\tau,$$

where

$$\Upsilon(t) := 2\alpha(t) + 2\|\gamma_1\|_{L^1([0,T],\mathbb{R}_+)}\gamma_2(t).$$

Hence, by the classical Gronwall's inequality, it follows that

$$\|x_\lambda(t)\| \le (\|x_0\| + \Xi(t))\exp\Big(\int_0^t \Upsilon(\tau)d\tau\Big) \le (\|x_0\| + \Xi(T))\exp\Big(\int_0^T \Upsilon(\tau)d\tau\Big) =: M,$$

which translates the boundness property of $x_\lambda(\cdot)$ independently of λ on $[0,T]$.

Step 3. We show that $(x_\lambda)_\lambda$ is a Cauchy sequence in $C([0,T];H)$.

Let us set $\xi(t) := M\alpha(t) + \delta(t) + (M+1)\int_0^t \beta(t,s)\,ds$, then $\xi \in L^2([0,T],\mathbb{R}_+)$ and for a.e.

$t \in [0, T]$ we have

$$\begin{aligned}\Big\|\dot{x}_\lambda(t) + A_\lambda x_\lambda(t) + \int_0^t f(t, s, x_\lambda(s))\, ds\Big\| &\leq \|A_\lambda x_\lambda(t)\| + \int_0^t \|f(t, s, x_\lambda(s))\|\, ds + |\dot{v}(t)| \\ &\leq \|A^\circ(x_\lambda(t))\| + \int_0^t \|(1 + \|x_\lambda(s)\|)\beta(t, s)\|\, ds + |\dot{v}(t)| \\ &\leq \xi(t) + |\dot{v}(t)|,\end{aligned}$$

and

$$\Big\|A_\lambda x_\lambda(t) + \int_0^t f(t, s, x_\lambda(s))\, ds\Big\| \leq \xi(t),$$

which implies that

$$-\frac{1}{\xi(t) + |\dot{v}(t)|}\Big(\dot{x_\lambda}(t) + A_\lambda x_\lambda(t) + \int_0^t f(t, s, x_\lambda(s))\, ds\Big) \in N_{C(t)}(x_\lambda(t)) \cap \mathbb{B}. \tag{3.3}$$

Let now $\lambda, \mu > 0$. Since the sets $C(t)$ are r-prox-regular then, by using the hypomonotonicity property given in (c) of Theorem 2.2 and the inclusion (3.3), one has for a.e. $t \in [0, T]$

$$\begin{aligned}&\langle\, \dot{x}_\lambda(t) + A_\lambda x_\lambda(t) + \int_0^t f(t, s, x_\lambda(s))\, ds - \dot{x}_\mu(t) - A_\mu x_\mu(t) \\ &\quad - \int_0^t f(t, s, x_\mu(s))\, ds, x_\lambda(t) - x_\mu(t)\,\rangle \leq \frac{\xi(t) + |\dot{v}(t)|}{r}\|x_\lambda(t) - x_\mu(t)\|^2.\end{aligned} \tag{3.4}$$

It is clear that $x_\lambda(t) = J_\lambda(x_\lambda(t)) + \lambda A_\lambda(x_\lambda(t))$ further, the inclusion $A_\lambda x_\lambda(t) \in A(J_\lambda x_\lambda(t))$ holds true thanks to the Proposition 2.3. Recalling that the operator A is monotone, so

$$\langle\, A_\lambda x_\lambda(t) - A_\mu x_\mu(t), J_\lambda x_\lambda(t) - J_\mu x_\mu(t)\,\rangle \geq 0.$$

It results that

$$\begin{aligned}\langle\, A_\lambda x_\lambda(t) - A_\mu x_\mu(t), &x_\lambda(t) - x_\mu(t)\,\rangle = \\ &\langle\, A_\lambda x_\lambda(t) - A_\mu x_\mu(t), J_\lambda x_\lambda(t) + \lambda A_\lambda x_\lambda(t) - J_\mu x_\mu(t) - \mu A_\mu x_\mu(t)\,\rangle \\ &\geq \langle\, A_\lambda x_\lambda(t) - A_\mu x_\mu(t), \lambda A x_\lambda(t) - \mu A_\mu x_\mu(t)\,\rangle \\ &\geq \lambda\| A_\lambda x_\lambda(t)\|^2 + \mu\|A_\mu x_\mu(t)\|^2 - \lambda\|A_\lambda x_\lambda(t)\|\|A_\mu x_\mu(t)\| \\ &- \mu\|A_\mu x_\mu(t)\|\|A_\lambda x_\lambda(t)\|\,,\end{aligned}$$

but

$$\begin{aligned}0 &\leq \Big(\sqrt{\lambda}\|A_\lambda x_\lambda(t)\| - \frac{\sqrt{\lambda}}{2}\|A_\mu x_\mu(t)\|\Big)^2 \\ &= \lambda\|A_\lambda x_\lambda(t)\|^2 + \frac{\lambda}{4}\|A_\mu x_\mu(t)\|^2 - \lambda\|A_\lambda x_\lambda(t)\|\|A_\mu x_\mu(t)\|,\end{aligned}$$

which means that

$$-\lambda\|A_\lambda x_\lambda(t)\|\|A_\mu x_\mu(t)\| \geq -\lambda\|A_\lambda x_\lambda(t)\|^2 - \frac{\lambda}{4}\|A_\mu x_\mu(t)\|^2,$$

hence

$$\langle A_\lambda x_\lambda(t) - A_\mu x_\mu(t), x_\lambda(t) - x_\mu(t) \rangle \geq -\frac{1}{4}(\lambda\|A_\mu x_\mu(t)\|^2 + \mu\|A_\lambda x_\lambda(t)\|^2)$$
$$\geq -\frac{1}{4}(\lambda+\mu)\Big(M\alpha(t)+\delta(t)\Big)^2,$$

this entails

$$\langle A_\lambda x_\lambda(t) - A_\mu x_\mu(t), x_\lambda(t) - x_\mu(t) \rangle \geq -\frac{1}{4}(\lambda+\mu)\xi^2(t). \tag{3.5}$$

On the other hand, according to assumption 4 one has

$$\Big\langle \int_0^t f(t,s,x_\lambda(s))ds - \int_0^t f(t,s,x_\mu(s))ds, x_\lambda(t) - x_\mu(t) \Big\rangle$$
$$\geq -L^\eta(t)\|x_\lambda(t) - x_\mu(t)\| \int_0^t \|x_\lambda(s) - x_\mu(s)\|\, ds.$$

Combining this last inequality with (3.4) and (3.5), we find

$$\frac{d}{dt}\|x_\lambda(t) - x_\mu(t)\|^2 \leq \frac{1}{2}(\lambda+\mu)\xi^2(t) + 2\frac{\xi(t)+|\dot{v}(t)|}{r}\|x_\lambda(t) - x_\mu(t)\|^2$$
$$+ 2L^\eta(t)\|x_\lambda(t) - x_\mu(t)\| \int_0^t \|x_\lambda(s) - x_\mu(s)\|\, ds\,.$$

Applying Lemma 3.1 with $\rho(t) = \|x_\lambda(t) - x_\mu(t)\|^2$, $K_1(t) = 2\frac{\xi(t)+|\dot{v}(t)|}{r}$, $K_2(t) = 2L^\eta(t)$, $\varepsilon(t) = \frac{1}{2}(\lambda+\mu)\xi^2(t), \epsilon > 0$ and taking into account the equality $x_\lambda(0) = x_\mu(0) = x_0$, we obtain that for all $t \in [0,T]$

$$\|x_\lambda(t) - x_\mu(t)\| \leq \sqrt{\epsilon}\exp\left(\int_0^t (K(s)+1)\, ds\right) + \frac{\sqrt{\epsilon}}{2}\int_0^t \exp\left(\int_s^t (K(\tau)+1)\, d\tau\right) ds$$
$$+ 2\left(\sqrt{\int_0^t \varepsilon(s)\, ds + \epsilon} - \sqrt{\epsilon}\exp\left(\int_0^t (K(\tau)+1)\, d\tau\right)\right)$$
$$+ 2\int_0^t (K(s)+1)\exp\left(\int_s^t (K(\tau)+1)\, d\tau\right)\sqrt{\int_0^s \varepsilon(\tau)\, d\tau + \epsilon}\, ds,$$

where $K(t) := \max\left\{\frac{K_1(t)}{2}, \frac{K_2(t)}{2}\right\}$, a.e $t \in [0,T]$.

By taking $\epsilon \to 0$, we deduce that $(x_\lambda(\cdot))_{\lambda>0}$ is a Cauchy sequence in $C([0,T];H)$ and, therefore, it converges uniformly to some function $x(\cdot) \in C([0,T];H)$ as $\lambda \downarrow 0$. Furthermore, $x(t) \in C(t)$ because $x_\lambda(t) \in C(t)$ for all $\lambda > 0$ and $C(t)$ is closed. Also, one has $x(t) \in \mathrm{Dom}(A)$ thanks to property (iii) of the proposition 2.3 and the inequality $\|A_\lambda(x_\lambda(t))\| \leq M\alpha(t) + \delta(t)$.

Step 4. We prove that $x(\cdot)$ is a solution of $(P_{A,f})$.
Since $\|\dot{x}_\lambda(t)\| \leq 2\xi(t) + |\dot{v}(t)|$ for almost all $t \in [0,T]$, there exists a subsequence $(\dot{x}_{\lambda_n})_n$ converges weakly to some $g(\cdot) \in L^1([0,T];H)$. That is

$$\int_0^T \langle \dot{x}_{\lambda_n}(s), h(s)\rangle\, ds \longrightarrow \int_0^T \langle g(s), h(s)\rangle\, ds, \quad \forall h \in L^\infty([0,T];H).$$

In particular, for any $z \in H$, taking $h(t) := z \cdot \mathbf{1}_{[0,t]}(t), t \in [0,T]$ then

$$\int_0^T \Big\langle \dot{x}_{\lambda_n}(s), z \cdot \mathbf{1}_{[0,t]}(s) \Big\rangle \, ds = \int_0^t \langle \dot{x}_{\lambda_n}(s), z \rangle \, ds = \Big\langle \int_0^t \dot{x}_{\lambda_n}(s) ds, z \Big\rangle,$$

also

$$\int_0^T \Big\langle g(s), z \cdot \mathbf{1}_{[0,t]}(s) \Big\rangle \, ds = \int_0^t \langle g(s), z \rangle \; ds = \langle \int_0^t g(s) ds, z \rangle.$$

It results that

$$\int_0^t \dot{x}_{\lambda_n}(s) \, ds \text{ converges weakly in in } H \text{ to } \int_0^t g(s) \, ds,$$

and so

$$x_{\lambda_n}(t) = x_{\lambda_n}(0) + \int_0^t \dot{x}_{\lambda_n}(s) \, ds \text{ converges weakly in in } H \text{ to } x(0) + \int_0^t g(s) \, ds,$$

taking into account that $x_n(\cdot)$ converge uniformly to $x(\cdot)$, we deduce

$$x(t) = x_0 + \int_0^t g(s) ds,$$

which translates the absolute continuous property of $x(\cdot)$, moreover $\dot{x}(\cdot) = g(\cdot)$ a.e. $t \in [0,T]$, and by the way

$$\|x(t)\| \leq M_1 := \|x_0\| + \int_0^T g(s) ds.$$

Through the continuity property of $x \mapsto f(t,s,x)$ and the uniform convergence of $x_{\lambda_n}(\cdot)$ to $x(\cdot)$ we get

$$\lim_{n \to +\infty} f(t,s,x_{\lambda_n}(s)) = f(t,s,x(s)). \tag{3.6}$$

Let us set for each $t \in [0,T]$

$$\phi_n(t) := \int_0^t f(t,s,x_{\lambda_n}(s)) \, ds \quad \text{and} \quad \phi(t) := \int_0^t f(t,s,x(s)) \, ds,$$

and let $\eta_0 := \max\{M, M_1\}$ then

$$(x(t), x_\lambda(t)) \in B[0,\eta_0] \times B[0,\eta_0], \text{ for all } t \in [0,T].$$

Therefore, by **assumption 4** there exists $L^{\eta_0}(\cdot) \in L^1([0,T];\mathbb{R}_+)$ such that

$$\int_0^T \|\phi_n(t) - \phi(t)\| dt \leq \int_0^T L^{\eta_0}(t) \int_0^t \|x_{\lambda_n}(s) - x(s)\| ds \, dt. \tag{3.7}$$

Note that for every $(t,s) \in Q_\Delta$

$$L^{\eta_0}(t) \int_0^t \|x_{\lambda_n}(s) - x(s)\| ds \leq 2\eta_0 T L^{\eta_0}(t), \tag{3.8}$$

then, putting (3.6), (3.7) and (3.8) together and applying the Lebesgue dominated convergence theorem to obtain

$$\phi_n(\cdot) \text{ converges strongly to } \phi(\cdot) \text{ in } L^1([0,T];H).$$

On the other hand, we have

$$\|A_{\lambda_n} x_{\lambda_n}(t)\| \leq M\alpha(t) + \delta(t) \leq \xi(t) \text{ for a.e. } t \in [0,T],$$

hence, there exist a subsequence, still denoted by $(A_{\lambda_n} x_{\lambda_n}(\cdot))_n$ and $\vartheta \in L^1([0,T];H)$ such that $A_{\lambda_n} x_{\lambda_n}$ converges to ϑ weakly in $L^1([0,T];H)$. Then we obtain

$$\Psi_n(\cdot) := \dot{x}_{\lambda_n} + A_{\lambda_n} x_{\lambda_n} + \phi_{\lambda_n} \text{ converges weakly to } \Psi(\cdot) := \dot{x} + \vartheta + \phi \text{ in } L^1([0,T];H).$$

By virtue of Mazur's lemma, for each $n \in \mathbb{N}$ there exists some sequence of convex combinations of the form

$$\Big(\sum_{k=n}^{T(n)} S_{k,n}\Psi_k\Big)_n \quad \text{with} \quad S_{k,n} \geq 0 \quad \text{and} \quad \sum_{k=n}^{T(n)} S_{k,n} = 1,$$

converges strongly to Ψ in $L^1([0,T];H)$. Extracting a subsequence, we may suppose that there exists some negligible set $\mathcal{N} \subset [0,T]$ such that $\Big(\sum_{k=n}^{T(n)} S_{k,n}\Psi_k(t)\Big)_n$ converges in H to $\Psi(t)$ as $n \to +\infty$ for all $t \in [0,T]\backslash\mathcal{N}$ and such that for all $n \in \mathbb{N}$

$$-\Psi_n(t) \in N_{C(t)}(x_{\lambda_n}(t)) \text{ a.e. } \; t \in [0,T]\,.$$

Fix any $t \in [0,T] \setminus \mathcal{N}$, from the prox-regularity of $C(t)$, one has

$$\langle \Psi_k(t), y - x_{\lambda_k}(t)\rangle \geq -\frac{\xi(t) + |\dot{v}(t)|}{2r}\|y - x_{\lambda_k}(t)\|^2, \; \forall y \in C(t).$$

Hence, for all $y \in C(t)$

$$\sum_{k=n}^{T(n)} S_{k,n} \langle \Psi_k(t), y - x_{\lambda_k}(t)\rangle \geq -\frac{\xi(t) + |\dot{v}(t)|}{2r} \sum_{k=n}^{T(n)} S_{k,n}\|y - x_{\lambda_k}(t)\|^2. \tag{3.9}$$

Note that

$$\left|\sum_{k=n}^{T(n)} S_{k,n} \langle \Psi_k(t), x(t) - x_{\lambda_k}(t)\rangle\right| \leq (\xi(t) + |\dot{v}(t)|) \sum_{k=n}^{T(n)} S_{k,n}\|x(t) - x_{\lambda_k}(t)\|,$$

and hence

$$\sum_{k=n}^{T(n)} S_{k,n} \langle \Psi_k(t), x(t) - x_{\lambda_k}(t)\rangle \underset{n\to\infty}{\longrightarrow} 0, \tag{3.10}$$

since it is easily seen that $\sum_{k=n}^{T(n)} S_{k,n}\|x(t) - x_{\lambda_k}(t)\| \underset{n\to\infty}{\longrightarrow} 0$ because $\|x(t) - x_{\lambda_n}(t)\| \underset{n\to\infty}{\longrightarrow} 0$ and $\sum_{k=n}^{T(n)} S_{k,n} = 1$. The convergence (3.10) and the equality

$$\sum_{k=n}^{T(n)} S_{k,n} \langle \Psi_k(t), y - x_{\lambda_k}(t)\rangle = \left\langle \sum_{k=n}^{T(n)} S_{k,n}\Psi_k(t), y - x(t)\right\rangle + \sum_{k=n}^{T(n)} S_{k,n} \langle \Psi_k(t), x(t) - x_{\lambda_k}(t)\rangle\,,$$

give us

$$\sum_{k=n}^{T(n)} S_{k,n} \langle \Psi_k(t), y - x_{\lambda_k}(t)\rangle \underset{n\to\infty}{\longrightarrow} \langle \Psi(t), y - x(t)\rangle\,. \tag{3.11}$$

On the other hand

$$\sum_{k=n}^{T(n)} S_{k,n}\|y - x_{\lambda_k}(t)\|^2 \longrightarrow \|y - x(t)\|^2, \tag{3.12}$$

since $x_{\lambda_n}(t) \underset{n\to\infty}{\longrightarrow} x(t)$. Passing to the limit on n in the inequality (3.9), we obtain through (3.11) and (3.12)

$$\langle \Psi(t), y - x(t)\rangle \geq -\frac{\xi(t) + |\dot{v}(t)|}{2r}\|y - x(t)\|^2, \; \forall y \in C(t),$$

which means that

$$\Psi(t) = \dot{x}(t) + \vartheta(t) + \int_0^t f(t,s,x(s))ds \in N_{C(t)}(x(t)). \tag{3.13}$$

Now, in order to complete the proof, let us demonstrate that $\vartheta(t) \in Ax(t)$ for a.e. $t \in [0,T]$. To this end, let us recall that

$$A_{\lambda_n} x_{\lambda_n}(t) \in A(J_{\lambda_n} x_{\lambda_n}(t)), \text{ for a.e. } t \in [0,T],$$

and $(A_{\lambda_n} x_{\lambda_n})_n$ converges weakly to ϑ in $L^2([0,T];H)$. In addition $(J_{\lambda_n} x_{\lambda_n})_n$ converges strongly to x in $L^2([0,T];H)$. Indeed

$$\begin{aligned}\|J_{\lambda_n} x_{\lambda_n}(t) - x(t)\| &\leq \|J_{\lambda_n} x_{\lambda_n}(t) - x_{\lambda_n}(t)\| + \|x_{\lambda_n}(t) - x(t)\| \\ &\leq \lambda_n \| A_{\lambda_n} x_{\lambda_n}(t)\| + \|x_{\lambda_n}(t) - x(t)\| \\ &\leq \lambda_n \xi(t) + \|x_{\lambda_n}(t) - x(t)\| \underset{n\to+\infty}{\longrightarrow} 0.\end{aligned}$$

Consequently, we have

$$\begin{cases} A_{\lambda_n} x_{\lambda_n}(\cdot) \in \mathcal{A}(J_{\lambda_n} x_{\lambda_n}(\cdot)) \text{ in } L^2([0,T];H) \\ A_{\lambda_n} x_{\lambda_n}(\cdot) \overset{w}{\longrightarrow} \vartheta(\cdot) \text{ in } L^2([0,T];H) \\ J_{\lambda_n} x_{\lambda_n} \overset{\|\cdot\|}{\longrightarrow} x(\cdot) \text{ in } L^2([0,T];H), \end{cases}$$

where $\mathcal{A}$ denotes the extension of A given in Proposition 2.4. Combining this last three properties with the strong-weak closeness of $\mathcal{A}$ in $L^2([0,T];H)$ (see Proposition 2.4, (a)) we deduce that

$$\vartheta(\cdot) \in \mathcal{A}(x(\cdot)) \text{ in } L^2([0,T];H) \Leftrightarrow \vartheta(t) \in Ax(t) \text{ for a.e. } t \in [0,T]. \tag{3.14}$$

Putting (3.13) and (3.14) together, we conclude that

$$\dot{x}(t) \in -N_{C(t)}(x(t)) - Ax(t) - \int_0^t f(t,s,x(s))\,ds \quad \text{for a.e. } t \in [0,T],$$

which completes the proof of Theorem. □

Under some additional assumptions, we can prove a uniqueness result. The following theorem is in this sense.

Theorem 3.3. *Let the* **Assumptions 1-4** *be satisfied. Suppose that*

$$\operatorname{rge}(C) \subset \operatorname{intDom}(A). \tag{3.15}$$

Then, for any $x_0 \in C(0)$, the problem $(P_{A,f})$ has at most one solution.

Proof. Let $x_1(\cdot)$ and $x_2(\cdot)$ be two solutions of $(P_{A,f})$ such that $x_1(0) = x_2(0) = x_0 \in C(0) \subset \operatorname{rge}(C)$. Since A is maximal monotone then, A is locally bounded in $\operatorname{intDom}(A)$ according to the Proposition 2.4. That is, there exists $R, \rho > 0$ such that $B(x_0,\rho) \subset \operatorname{intDom}(A)$

$$\|\omega\| \leq R, \text{ for all } w \in A(y) \text{ and all } y \in B(x_0,\rho). \tag{3.16}$$

The continuity of $x_i(\cdot), i = 1,2$ on $[0,T]$ implies that for any $\varepsilon > 0$, there exists $0 < T' < T$ such that

$$\|x_i(t) - x_i(0)\| < \varepsilon, \text{ for all } t \in [0,T'],$$

in particular, for $\varepsilon = \rho$, one obtains $x_i([0,T']) \subset B(x_0,\rho), i = 1,2$. It results from (3.16) that

$$\|\omega\| \leq R, \text{ for all } w \in A(x_i(t)) \text{ and all } t \in [0,T'], i = 1,2. \tag{3.17}$$

Let $g_i(\cdot) \in A(x_i(\cdot))$ such that for a.e. $t \in [0,T']$

$$-\dot{x}_i(t) \in N_{C(t)}(x_i(t)) + g_i(t) + \int_0^t f(t,s,x_i(s))\,ds \;\; i = 1,2.$$

Then, according to Theorem 4.1 in [5], one has

$$\left\| \dot{x}_i(t) + g_i(t) + \int_0^t f(t,s,x_i(s))\,ds \right\| \le |\dot{v}(t)| + \|g_i(t)\| + \int_0^t \|f(t,s,x_i(s))\|\,ds$$
$$\le |\dot{v}(t)| + R + \kappa(t),$$

where $\kappa(t) := (1+\rho+\|x_0\|)\int_0^t \beta(t,s)\,ds$. The r-prox-regularity of $C(t)$, the monotonicity of A and the hypomonotonicity of the proximal normal cone give

$$\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t) \rangle \le L^{\eta_1}(t)\|x_1(t)-x_2(t)\| \int_0^t \|x_1(s)-x_2(s)\|\,ds$$
$$+ \frac{1}{r}\Big(\dot{v}(t) + R + \kappa(t)\Big)\|x_1(t)-x_2(t)\|^2,$$

where $\eta_1 := \rho + \|x_0\|$. Hence

$$\frac{d}{dt}\|x_1(t)-x_2(t)\|^2 \le 2L^{\eta_1}(t)\|x_1(t)-x_2(t)\| \int_0^t \|x_1(s)-x_2(s)\|\,ds$$
$$+ \frac{2}{r}\Big(\dot{v}(t) + R + \kappa(t)\Big)\|x_1(t)-x_2(t)\|^2.$$

Applying Lemma 3.1, we deduce that

$$x_1(\cdot) = x_2(\cdot) \text{ on } [0,T']. \tag{3.18}$$

Let $E_\tau := \{t \in [0,\tau] : x_1(t) \ne x_2(t)\}$ where $\tau \in [0,T]$ is such that $x_1(\tau) \ne x_2(\tau)$. It is clear that $E_\tau \subset]0,\tau]$. Further, from (3.18) and $T' > 0$ one has $\varrho := \inf E_\tau \in]0,\tau]$. Then $x_1(t) = x_2(t)$ for all $t \in [0,\varrho[$. Letting t tending to ϱ we get $x_1(\varrho) = x_2(\varrho)$ thanks to the continuity of $x_i(\cdot), i = 1,2$. Hence $0 < \varrho < \tau$ because $x_1(\tau) \ne x_2(\tau)$. With the same argument as above, there exists some $T' > 0$ such that $x_1(\cdot) = x_2(\cdot)$ on $[0,\varrho+T']$. This constitutes a contradiction with the definition of $\varrho := \inf E_\tau$. Thus $x_1(\cdot) = x_2(\cdot)$ on $[0,T]$. □

Let us provide an example in parabolic variational inequalities with Volterra type operators. Our example completes that in [2].

Example 3.4. Let Ω be a bounded subset of $\mathbb{R}^n$ and $H = L^2(\Omega)$, $U = H^2(\Omega) \cap H_0^1(\Omega)$. Let be given $\psi \in L^2(0,T;U)$, $M \in L^2(0,T;\mathbb{R}_+)$ such that $\psi(\cdot)$ is k-Lipschitz continuous with respect to the supremum norm. For each $t \in [0,T]$, we define

$$C(t) := C_1(t) \cup C_2(t),$$

where

$$C_1(t) = \{v \in U : v \ge \psi(t) \text{ a.e., on } \Omega \text{ and } \|\Delta\, v\| \le M(t)\},$$

and

$$C_2(t) = \{v \in U : v \le \psi(t) - 1 \text{ a.e., on } \Omega \text{ and } \|\Delta\, v\| \le M(t)\}.$$

It is easy to see that for each t, the set $C(t)$ is closed, prox-regular (but non-convex) since $C_1(t)$, $C_2(t)$ are two disjoint closed, convex sets and $\|v_1 - v_2\| \ge \sqrt{m(\Omega)}$ for all $v_1 \in C_1(t)$, $v_2 \in C_2(t)$, where $m(\Omega)$ is the volume of Ω. Furthermore, $C(\cdot)$ is k-Lipschitz continuous since $\psi(\cdot)$ is k-Lipschitz continuous. Let be given x_0. We consider the following parabolic variational inequalities with a moving obstacle : find $x(t) \in C(t)$ such that $x(0) = x_0 \in C(0)$ and for a.e. $t \in [0,T]$, there exists $\delta_t > 0$ satisfying

$$\int_\Omega \dot{x}(t)(v(t)-x(t))\,dy + \int_\Omega \nabla x(t)(\nabla v(t) - \nabla x(t))\,dy \\ + \int_\Omega \left(\int_0^t B(t-s)x(s)\,ds\right)(v(t)-x(t))\,dy \geq -\delta_t \|v(t)-x(t)\|^2, \forall\, v(t) \in C(t). \tag{3.19}$$

Here, $B : [0,T] \to L^\infty(\Omega)$, is a prescribed mapping. Let us define the operator $A : H \to H$ as $A = -\Delta$, where Δ is the Laplace operator. Then A is a self-adjoint maximal monotone operator with $\mathrm{Dom}(A) = U$ and $\overline{\mathrm{Dom}(A)} = H$ (see, e.g. [1,6]). It is easy to see that

$$\int_\Omega Ax(t)(v(t)-x(t))\,dy = \int_\Omega \nabla x(t)(\nabla v(t) - \nabla x(t))\,dy, \quad \forall\, x, v \in U.$$

Then the problem (3.19) can be rewritten as follows

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + Ax(t) + \int_0^t B(t-s)x(s)\,ds \quad \text{a.e.} \quad t \in [0,T]\,. \tag{3.20}$$

Assume that the operator B satisfies the following condition $B \in C([0,T], L^\infty(\Omega))$, so the function

$$f(t,s,v) := B(t-s)v \quad \text{ for all } (t,s) \in Q_\Delta \text{ and } v \in H,$$

satisfies the assumptions 3-4 with

$$\beta(t,s) = \|B(t-s)\|_{L^\infty(\Omega)} \text{ and } L(t) = \sup_{t\in[0,T]} \|B(t)\|_{L^\infty(\Omega)} \text{ for all } (t,s) \in Q_\Delta.$$

Further, all the assumptions of Theorem 3.2 are satisfied. Thus, for $x_0 \in C(0)$, there exists an absolutely continuous solution $x(\cdot)$ of (3.20), or equivalently, of (3.19). In addition, if $M(\cdot)$ is a constant function, by using Remark 3.5, one deduces the uniqueness of solutions.

Remark 3.5. In Theorem 3.3, we can also relax the condition $\mathrm{rge}(C) \subset \mathrm{int}\,\mathrm{Dom}(A)$ by the assumption that A is locally bounded on $\mathrm{rge}(C)$, i.e., for all $u \in \mathrm{rge}(C)$, there exists $k > 0$, $\rho > 0$ such that A is bounded by K in $\mathbb{B}(u,\rho) \cap \mathrm{rge}(C)$.

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