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Research Article

On the boundedness and compactness of extended Cesàro composition operators between weighted Bloch-type spaces

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Abstract

Let $\psi \in H(\mathbb{B}_n)$, the space of all holomorphic functions on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$, $\varphi = (\varphi_1, \ldots, \varphi_n) \in S(\mathbb{B}_n)$ the set of holomorphic self-maps of $\mathbb{B}_n$. Let $C_{\psi,\varphi} : \mathcal{B}_\nu$ (and $\mathcal{B}_{\nu,0}$) $\to \mathcal{B}_\mu$ (and $\mathcal{B}_{\mu,0}$) be weighted extended Cesàro operators induced by products of the extended Cesàro operator C_φ and integral operator T_ψ. In this paper, we characterize the boundedness and compactness of $C_{\psi,\varphi}$ via the estimates for either $|\varphi|$ or $|\varphi_k|$ *for some* $k \in \{1, \ldots, n\}$. At the same time, we also give the asymptotic estimates of the norms of these operators.

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1. Introduction

For $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, we define

$$\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n,$$

where $\overline{w}_k$ is the complex conjugate of w_k. We also write

$$\|z\| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The space $\mathbb{C}^n$ becomes an n-dimensional Hilbert space when endowed with the inner product above. The standard basis for $\mathbb{C}^n$ consists of the following vectors:

$$e_1 = (1, 0, \ldots, 0),\ e_2 = (0, 1, 0, \ldots, 0), \ldots,\ e_n = (0, \ldots, 0, 1).$$

Let $\mathbb{B}_n$ be the unit ball in the n-dimensional complex space $\mathbb{C}^n$. Let $H(\mathbb{B}_n)$ be the class of all holomorphic functions on $\mathbb{B}_n$ and $S(\mathbb{B}_n)$ the collection of all the holomorphic self-mappings of $\mathbb{B}_n$.

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Let E be space of holomorphic functions on $\mathbb{B}_n$. For a holomorphic self-map $\varphi \in S(\mathbb{B}_n)$ and a holomorphic function $\psi \in H(\mathbb{B}_n)$, the composition operator C_φ and the extended Cesàro operato T_ψ are defined by

$$(C_\varphi f)(z) := (f \circ \varphi)(z), \quad (T_\psi f)(z) := \int_0^1 f(tz) R\psi(tz) \frac{dt}{t} \quad \forall f \in E, \ \forall z \in \mathbb{B}_n.$$

where

$$R\psi(z) := \sum_{j=1}^n z_j \frac{\partial \psi}{\partial z_j}(z)$$

be the radial derivative of ψ.

The study of composition operators consists in the comparison of the properties of the C_φ with that of the function φ itself, which is called the symbol of C_φ. One can characterize boundedness and compactness of C_φ and many other properties.

The problem of studying of composition operators on various Banach spaces of holomorphic functions on the unit disk or the unit ball, such as Hardy and Bergman spaces, the space H^∞ of all bounded holomorphic functions, the disk algebra and weighted Banach spaces with sup-norm, etc. received a special attention of many authors during the past several decades. The weighted composition operators on these spaces appeared in some works with different applications. There is a great number of topics on operators of such a type: boundedness and compactness, compact differences, topological structure, dynamical and ergodic properties.

The extended Cesàro operator T_ψ is a natural extension of the Cesàro operator acting $f \in H(\mathbb{B}_1)$

$$C[f](z) := \sum_{j=0}^\infty \left(\frac{1}{j+1} \sum_{k=0}^j a_k \right)$$

with $f(z) = \sum_{j=0}^\infty a_j z^j$, the Taylor expansion of f.

It is well know that $C[\cdot]$ acts as a bounded linear operator on various spaces of holomorphic functions, including the Hardy and Bergman spaces. But it is not bounded on the Bloch space (see [16]). A little calculation shows

$$C[f](z) = \frac{1}{z} \int_0^z f(t) \Big(\log \frac{1}{1-t} \Big)' dt.$$

Hence, on most holomorphic function spaces, $C[\cdot]$ is bounded if and only if the integral operator $f \mapsto \frac{1}{z} \int_0^z f(t) \big(\log \frac{1}{1-t} \big)' dt$ is bounded. From this point of view it is natural to consider the extended Cesàro operator with holomorphic symbol ψ,

$$T_\psi f(z) := \int_0^z f(t) \psi'(t) dt.$$

The boundedness and compactness of this operator on Hardy spaces, Bergman spaces, Bloch- type spaces and Lipschitz spaces have been studied in [1, 2, 15]. The extended Cesàro operator is a generalization of this operator. It has been well studied in many papers, see, for example, [3, 5, 7] as well as the related references therein.

It is natural to discuss the product of extended Cesàro operator and composition operator. For $\varphi \in S(\mathbb{B}_n)$ and $\psi \in H(\mathbb{B}_n)$ the product can be expressed as

$$C_{\psi,\varphi}(f) := T_\psi C_\varphi f(z) = \int_0^1 f(\varphi(tz)) R(\psi(tz)) \frac{dt}{t} \quad f \in H(\mathbb{B}_n), z \in \mathbb{B}_n. \tag{1.1}$$

This operator is called *extended Cesàro composition operator*. It is interesting to characterize the boundedness and compactness of the product operator on all kinds of function spaces. Even on the disk of $\mathbb{C}$, some properties are not easily managed; see some recent papers in [8–10, 13, 17].

Building on those foundations, the present paper continues this line of research and discusses the operator in infinite dimension. The study of extended Cesàro composition operators $C_{\psi,\varphi}$ on vector-valued function spaces involves some important basic principles which hold for large classes of function spaces.

It can be said that, in most of necessary as well as sufficient conditions that have been given before for the boundedness (or compactness) for compositon operators, the evaluations depend on the module $|\varphi|$ of holomorphic self-maps φ but not only on $|\varphi_k|$ of some component function φ_k of φ. The question arises as to whether, under a certain condition, the evaluations for bounded as well as compact characterizarion of $W_{\psi,\varphi}$ depend only some component function φ_k of φ?

In this paper, we consider the weighted extended Cesàro operators $C_{\psi,\varphi}$ between weighted (little) Bloch-type spaces of holomorphic functions in $\mathbb{B}_n$, where ν, μ are normal weights on $\mathbb{B}_n$. Besides investigating the estimates of the boundedness and the compactness of $W_{\psi,\varphi}$ via the module $|\varphi|$, we are interested in finding the answer of the question mentioned above. To do this we shall concentrate our attention on the case where the range $\varphi(\mathbb{B}_n)$ of the holomorphic self-map φ contains all of the unit disks $\mathbb{D}_j = \{\lambda e_j : |\lambda| < 1\}$, $j = 1, \dots, n$, of coordinate planes.

The paper is organized as follows. We set up review in Section 2 some notations of holomorphic functions of several variables, the weighted (little) Bloch-type spaces $\mathcal{B}^{\Diamond}_{\mu}$, $\mathcal{B}^{\Diamond}_{\mu,0}$ of holomorphic functions on $\mathbb{B}_n$ and some estimates concerning the radial derivative, the holomorphic gradient of holomorphic functions pertaining to our work. Several helpful test functions concerning our computations will be introduced in Section 3. In Section 4 (resp. Section 5), we characterize the boundedness (resp. compactness) of the operators $W_{\psi,\varphi}$ between the weighted Bloch-type spaces of introduced in Section 2, as well as the equivalent relationships between them.

Throughout this paper, we use the notions $X \lesssim Y$ and $X \asymp Y$ for non negative quantities X and Y to mean $X \le CY$ and, respectively, $Y/C \le X \le CY$ for some inessential constant $C > 0$.

2. Preliminaries and auxiliary results

2.1. Holomorphic functions

A function $f : \mathbb{B}_n \to \mathbb{C}$ is said to be holomorphic if it is Fréchet differentiable at every $z \in \mathbb{B}_n$, or equivalently, if

$$f(z) = \sum_m a_m z^m, \quad z \in \mathbb{B}_n.$$

Here the summation is over all multi-indexes $m = (m_1, \dots, m_n)$, where each m_k is a nonnegative integer and $z^m = z_1^{m_1} \dots z_n^{m_n}$.

The series above is called the Taylor series of f at the origin; it converges absolutely and uniformly on each of the sets $r\mathbb{B}_n = \{z \in \mathbb{C}^n : \|z\| \le r\}$, $0 < r < 1$.

If we let

$$f_k(z) = \sum_{|m|=k} a_m z^m$$

for each $k \ge 0$, where $|m| = m_1 + \dots + m_n$ then the Taylor series of f can be rewritten as

$$f(z) = \sum_{k=0}^{\infty} f_k(z).$$

This is called the homogeneous expansion of f; each f_k is a homogeneous polynomial of degree k. Both the Taylor series and the homogeneous expansion of f are uniquely determined by f. The space of all holomorphicfunctions in $\mathbb{B}_n$ will be denoted by $H(\mathbb{B}_n)$.

2.2. The radial derivative and the holomorphic gradient

A very important concept of differentiation on the unit ball is that of the *radial derivative*, which is based on the usual partial derivatives of a holomorphic function. Thus for a holomorphic function f in $\mathbb{B}_n$ we write

$$Rf(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z).$$

A simple verification shows that if

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \sum_{|m|=k} a_m z^m \tag{2.1}$$

is the homogeneous expansion of f, then

$$Rf(z) = \sum_{k=0}^{\infty} k f_k(z) = \sum_{k=1}^{\infty} k f_k(z).$$

This is called the radial derivative of f because

$$Rf(z) = \lim_{r\to 0} \frac{f(z+rz) - f(z)}{r}$$

whenever f is holomorphic. Here r is a real parameter so that $z + rz$ is a radial variation of the point z. For every holomorphic function f in $\mathbb{B}_n$, it is easy to see that

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} dt \tag{2.2}$$

for all $z \in \mathbb{B}_n$. This formula will come in handy when we need to recover a holomorphic function from its radial derivative.

We also need the following lemma:

Lemma 2.1. *For every $f, \psi \in H(\mathbb{B}_n)$, and $\varphi \in S(\mathbb{B}_n)$ we have*

$$R(C_{\psi,\varphi} f)(z) = f(\varphi(z)) R\psi(z) \quad \forall z \in \mathbb{B}_n. \tag{2.3}$$

Indeed, assume that $\sum_{n=1}^{\infty} P_n(z)$ is the Taylor series of $(f \circ \varphi)(z)R\psi(z)$, where P_n is a homogeneous polynomial of degree n. Then we have

$$R(C_{\psi,\varphi} f)(z) = R\int_0^1 \sum_{n=1}^{\infty} P_n(z) t^n \frac{dt}{t} = R\sum_{n=1}^{\infty} \frac{P_n(z)}{n} = \sum_{n=1}^{\infty} P_n(z) = f(\varphi(z)) R\psi(z).$$

2.3. The weighted spaces of Bloch type

In this subsection we introduce the weighted spaces of Bloch type and some relationships between them.

Definition 2.2. A positive continuous function ω on the interval $[0, 1)$ is called normal if there are three constants $0 \le \delta < 1$ and $0 < a < b < \infty$ such that

$$\frac{\omega(t)}{(1-t)^a} \text{ is decreasing on } [\delta, 1), \quad \lim_{t\to 1} \frac{\omega(t)}{(1-t)^a} = 0, \tag{W_1}$$

$$\frac{\omega(t)}{(1-t)^b} \text{ is increasing on } [\delta, 1), \quad \lim_{t\to 1} \frac{\omega(t)}{(1-t)^b} = \infty. \tag{W_2}$$

If we say that a function $\omega : \mathbb{B}_n \to [0, \infty)$ is normal, we also assume that it is radial, that is, $\omega(z) = \omega(\|z\|)$ for every $z \in \mathbb{B}_n$.

Strictly positive continuous functions on $\mathbb{B}_n$ are called weights.

Remark 2.3. It follows from (W_1) that the weight ω is decreasing on $[\delta, 1)$. Indeed, since $[(1-t)^a]^{-1}$ is increasing on $[0, 1)$ and (W_1), the weight ω must be decreasing on $[\delta, 1)$.

Since ω is positive, continuous, we also obtain from (W_1) and (W_2) that

$$m_{\omega,\delta} := \min_{t\in[0,\delta]} \omega(t) > 0, \quad M_\omega := \max_{t\in[0,1)} \omega(t) < \infty.$$

Now, because of the decrease of ω on $(\delta, 1)$ it is easy to check that

$$\omega(z)\int_0^{\|z\|} \frac{dt}{\omega(t)} < R_\omega := \delta\frac{M_\omega}{m_{\omega,\delta}} + 1 - \delta < \infty \quad \text{for every } z \in \mathbb{B}_n. \tag{2.1}$$

We need the following lemma whose proof parallels that of Lemma 13 in [11] and will be omitted.

Lemma 2.4. *Let ω be a non-increasing normal weight on $\mathbb{B}_n$. Then there exists $C_\omega > 0$ such that*

$$C_\omega \le \frac{\omega(r)}{\omega(r^2)} \le 1 \quad \forall r \in [0, 1).$$

We put

$$\mathcal{B}_\omega = \Big\{f \in H(\mathbb{B}_n) : \sup_{z\in\mathbb{B}_n} \omega(z)|Rf(z)| < \infty\Big\};$$
$$\mathcal{B}_{\omega,0} = \Big\{f \in \mathcal{B}_\omega : \lim_{\|z\|\to 1} \omega(z)|Rf(z)| = 0\Big\}.$$

It is easy to check that the mapping

$$f \mapsto \|f\|_{s\mathcal{B}_\omega} = \sup_{z\in\mathbb{B}_n} \omega(z)|Rf(z)|$$

is a seminorm on $\mathcal{B}_\omega$ which can be endowed with Banach space structure via the norm

$$\|f\|_{\mathcal{B}_\omega} = |f(0)| + \|f\|_{s\mathcal{B}_\omega}.$$

We denote

$$\mathcal{B}_\omega := (\mathcal{B}_\omega, \|\cdot\|_{\mathcal{B}_\omega}).$$

It is easy to see that the little Bloch-type space $\mathcal{B}_{\omega,0}$ endowed with the norm induced by $\|\cdot\|_{\mathcal{B}_\omega}$ is also a Banach space.

We recall the following from [13, Theorem 2.1]

Lemma 2.5 ([13])**.** *Let ω be a normal weight on $\mathbb{B}_n$. There exists $C > 0$ such that for every $f \in \mathcal{B}_\omega$ and for every $z \in \mathbb{B}_n$ we have*

$$|f(z)| \le C\left(1 + \int_0^{\|z\|} \frac{dt}{\omega(t)}\right)\|f\|_{\mathcal{B}_\omega}.$$

Remark 2.6. In fact, the estimate in Lemma 2.5 can be written as follows

$$|f(z)| \le |f(0)| + \int_0^{\|z\|} \frac{dt}{\omega(t)}\|f\|_{s\mathcal{B}_\omega}.$$

3. The test functions

In this section we consider ν is a normal weight on $\mathbb{B}_n$ and $\varphi = (\varphi_1, \dots, \varphi_n) \in S(\mathbb{B}_n)$.

We begin this section by constructing test functions that are usefull for the proofs of our main results.

First we consider the holomorphic function

$$g(z) := 1 + \sum_{k>k_0} 2^k z^{n_k} \quad \forall z \in \mathbb{B}_1 \tag{3.1}$$

where $k_0 = \left[\log_2 \frac{1}{\nu(\delta)}\right]$, $n_k = \left[\frac{1}{1-r_k}\right]$ with $r_k = \nu^{-1}(1/2^k)$ for every $k \geq 1$. Here the symbol $[x]$ means the greatest integer not more than x. By [6, Theorem 2.3], $g(t)$ is increasing on $[0, 1)$ and

$$|g(z)| \leq g(\|z\|) \in \mathbb{R} \quad \forall z \in \mathbb{B}_1, \tag{3.2}$$

$$0 < C_1 := \inf_{t\in[0,1)} \nu(t)g(t) \leq \sup_{t\in[0,1)} \nu(t)g(t) \leq \sup_{z\in\mathbb{B}_1} \nu(z)|g(z)| =: C_2 < \infty. \tag{3.3}$$

Moreover, Hadama [4, Lemma 2.1(ii)] proved that there exists a positive constant C_3 such that the inequality

$$\int_0^r g(t)dt \leq C_3 \int_0^{r^2} g(t)dt \tag{3.4}$$

holds for all $r \in [r_1, 1)$, where $r_1 \in (0, 1)$ is a constant such that $\int_0^{r_1} g(t)dt = 1$.

Now, for $w \in \mathbb{B}_n$ and $j \in \{1, \ldots, n\}$ fixed put the test function

$$\beta_{w,j}(z) := \int_0^{\langle z, w_j e_j\rangle} g(t)dt, \quad z \in \mathbb{B}_n; \tag{3.5}$$

and in addition assume that $|w| > r$ for some $r > 0$ put

$$\gamma_w(z) := \frac{1}{\int_0^{\|w\|^2} g(t)dt} \left(\int_0^{\langle z,w\rangle} g(t)dt \right)^2, \quad z \in \mathbb{B}_n. \tag{3.6}$$

Proposition 3.1. *We have $\beta_{w,j}, \gamma_w \in \mathcal{B}_{\nu,0}$ and*

$$\|\beta_{w,j}\|_{\mathcal{B}_\nu(\mathbb{B}_n)} \leq C_2, \ \|\gamma_w\|_{\mathcal{B}_\nu(\mathbb{B}_n)} \leq 2C_2.$$

Proof. It suffices to prove for the function $\beta_{w,j}$ because the proofs for other ones are very similar.

(i) Fix $j \in \{1, \ldots, n\}$. Since $\nu(z) \to 0$ and from (3.3) we have

$$\begin{aligned}\nu(z)|R\beta_{w,j}(z)| &= |\langle z, w_j e_j\rangle| \nu(z)||g(\langle z, w_j e_j\rangle)| \\ &\leq |\langle z, w_j e_j\rangle| \nu(z)|g(|\langle z, w_j e_j\rangle|) \leq \nu(z)g(w_j e_j) \to 0 \quad \text{as } \|z\| \to 1.\end{aligned}$$

This implies that $\beta_{w,j} \in \mathcal{B}_{\nu,0}$.

(ii) Obviously, $\beta_{w,j}(0) = 0$. On the other hand, since g is increasing on $[0; 1)$ we obtain

$$\begin{aligned}\|\beta_{w,j}\|_{\mathcal{B}_\nu(\mathbb{B}_n)} = \sup_{z\in\mathbb{B}_n} \nu(z)|R\beta_{w,j}(z)| &\leq \sup_{z\in\mathbb{B}_n} \nu(z)|\langle z, w_j e_j\rangle||g(\langle z, w_j e_j\rangle)| \\ &\leq \sup_{z\in\mathbb{B}_n} \nu(z)g(z) \leq C_2 < \infty.\end{aligned}$$

□

Let $r > 0$ and $\{w^m\}_{m\geq1}$ be a sequence in $\mathbb{B}_n$ such that $\|w^m\| \geq r$ for every $m \geq 1$ and $\lim_{m\to\infty} w^m = w^0$ with $\|w^0\| = 1$.

Proposition 3.2. *We have*

(1) *The sequences $\{\beta_{w^m,j}\}_{m\geq1}$, $\{\gamma_{w^m}\}_{m\geq1}$ are bounded in $\mathcal{B}_\nu$;*
(2) *$\gamma_{w^m} \to 0$ uniformly on any compact subset of $\mathbb{B}_n$ if $\int_0^1 \frac{dt}{\nu(t)} = \infty$.*

Proof. (1) It follows from Proposition 3.1.

(2) By (3.3) and the assumption $\int_0^1 \frac{1}{\nu(t)} = \infty$, we have $\int_0^1 g(t)dt = \infty$. Then it is easy to check that $\gamma_{w^m,j} \to 0$ uniformly on any compact subset of $\mathbb{B}_n$. And we are done. □

4. The boundedness of operators

Let $S^*(\mathbb{B}_n)$ be the set of holomorphic self-maps $\varphi \in S(\mathbb{B}_n)$ which satisfies the following condition $(*)$:

$$\varphi(\mathbb{B}_n) \supseteq \bigcup_{j=1}^{n} \{\lambda e_j : \ \lambda \in \mathbb{C}, |\lambda| < 1\}. \tag{$*$}$$

Obviously, if $\varphi \in S(\mathbb{B}_n)$ is surjective then $\varphi \in S^*(\mathbb{B}_n)$.

Next we give an example of non-surjective holomorphic self-map which satisfies $(*)$.

Let n be integer number, $n \geq 2$.

Consider the function $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ given by $\varphi(z_1, z_2, \dots, z_n) = (z_1, z_2, \dots, z_{n-1}, z_n^2)$ for every $(z_1, z_2, \dots, z_n) \in \mathbb{B}_n$. Assume the contrary, then there exists $(z_1^0, z_2^0, \dots, z_n^0)$ such that $\varphi(z_1^0, z_2^0, \dots, z_n^0) = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}, \frac{1}{n}) \in \mathbb{B}_n$. However, $(z_1^0, z_2^0, \dots, z_n^0) = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}, \pm\frac{1}{\sqrt{n}})$ does not belong to $\mathbb{B}_n$.

Lemma 4.1. *Let $\varphi = (\varphi_1, \dots, \varphi_n) \in S^*(\mathbb{B}_n)$, $\varphi(0) = 0$ and ν be a normal weight and h be a bounded function defining on $\mathbb{B}_n$ satisfying $\lim_{\|z\|\to 1} h(z) > 0$. Assume that there exists $k \in \{1, \dots, n\}$ such that*

$$M_k := \sup_{w\in\mathbb{B}_n} h(w) \int_0^{|\varphi_k(w)|} \frac{dt}{\nu(t)} < \infty.$$

Then there exists constant $C_4 > 0$ such that for every $z \in \mathbb{B}_n$ we have

$$h(z) \int_0^{\|\varphi(z)\|} \frac{dt}{\nu(t)} \leq C_4(1 + M_k). \tag{4.1}$$

Proof. Obviously, (4.1) holds when $\|\varphi(z)\| = 0$. It suffices to consider the case $|\varphi(z)| > 0$.

Since $\lim_{\|z\|\to 1} h(z) > 0$ we can find δ_0 (we may assume that $\delta_0 \geq \delta$) such that $\inf_{\|z\|\geq\delta_0} h(z) > 0$. Then, by the boundedness of h we have $C^+_{\delta_0} := \frac{\sup_{\|z\|\geq\delta_0} h(z)}{\inf_{\|z\|\geq\delta_0} h(z)} < \infty$. Denote $C^-_{\delta_0} := \frac{\delta_0}{m_{\nu,\delta_0}} \sup_{w\in\mathbb{B}_n} h(w)$ where $m_{\nu,\delta_0} = \inf_{\|z\|\leq\delta_0} \nu(z) > 0$.

Assume that $M_k < \infty$ and fix $z \in \mathbb{B}_n$.

In the case $0 < \|\varphi(z)\| \leq \delta_0$ we have

$$\begin{aligned} h(z) \int_0^{\|\varphi(z)\|} \frac{dt}{\nu(t)} &\leq h(z) \int_0^{\delta_0} \frac{dt}{\nu(t)} \\ &\leq \sup_{w\in\mathbb{B}_n} h(w) \int_0^{\delta_0} \frac{dt}{m_{\nu,\delta_0}} \\ &\leq \frac{\delta_0}{m_{\nu,\delta_0}} \sup_{w\in\mathbb{B}_n} h(w) + M_k = C^-_{\delta_0} + M_k. \end{aligned}$$

Now we consider the case $\|\varphi(z)\| \geq \delta_0$. By $\varphi \in S^*(\mathbb{B}_n)$, there exists $z' \in \mathbb{B}_n$ such that $|\varphi_k(z')| = \|\varphi(z)\|$. Since $\varphi(0) = 0$, we have $\|z'\| \geq \|\varphi(z')\| \geq |\varphi_k(z')| = \|\varphi(z)\| > \delta_0$. Therefore, $h(z') \geq \inf_{\|z\|\geq\ \delta_0} h(z)$. Then, since ν is decreasing on $[\delta, 1)$ we get the following estimate

$$\begin{aligned}
h(z)\int_0^{\|\varphi(z)\|}\frac{dt}{\nu(t)} &= h(z)\int_0^1 \frac{\|\varphi(z)\|dt}{\nu(t\varphi(z))}\\
&= h(z)\int_0^{\delta_0/\|\varphi(z)\|}\frac{\|\varphi(z)\|dt}{\nu(t\varphi(z))} + h(z)\int_{\delta_0/\|\varphi(z)\|}^1 \frac{\|\varphi(z)\|dt}{\nu(t\varphi(z))}\\
&\le \sup_{w\in\mathbb{B}_n} h(w)\int_0^{\delta_0}\frac{dt}{m_{\nu,\delta_0}} + \frac{h(z)}{h(z')}h(z')\int_{\delta_0/|\varphi_k(z')|}^1 \frac{|\varphi_k(z')|dt}{\nu(t\varphi_k(z'))}\\
&\le C^-_{\delta_0} + C^+_{\delta_0}h(z')\int_{\delta_0}^{|\varphi_k(z')|}\frac{dt}{\nu(t)}\\
&\le C^-_{\delta_0} + C^+_{\delta_0}\sup_{w\in\mathbb{B}_n} h(w)\int_0^{|\varphi_k(w)|}\frac{dt}{\nu(t)}\\
&= C_4(1+M_k) < \infty
\end{aligned}$$

where $C_4 = \max\{C^-_{\delta_0}, C^+_{\delta_0}\}$. □

Lemma 4.2. *Let ν be a normal weight on $\mathbb{B}_n$, $\alpha\in\mathbb{B}_n\setminus\{0\}$, $\gamma=(\gamma_1,\dots,\gamma_n)\in Aut(\mathbb{B}_n)$, the class of automorphisms consists of symmetries of $\mathbb{B}_n$, defined by*

$$\gamma(z) = \frac{\alpha - P_\alpha(z) - s_\alpha Q_\alpha(z)}{1-\langle z,\alpha\rangle}, \quad z\in\mathbb{B}_n \tag{4.2}$$

where $s_\alpha=\sqrt{1-\|\alpha\|^2}$, $P_\alpha(z)=\frac{\langle z,\alpha\rangle}{\|\alpha\|^2}\alpha$, $Q_\alpha(z)=z-\frac{\langle z,\alpha\rangle}{\|\alpha\|^2}\alpha$. Then, for every $k\in\{1,\dots,n\}$ there exists $C(\alpha,k)>0$ such that

$$\int_0^{|\gamma_k(z)|}\frac{dt}{\nu(t)} \le C(\alpha,k) + \int_0^{|z_k|}\frac{dt}{\nu(t)} \quad \textit{for all} \quad |z_k|\ge\|\alpha\|.$$

Proof. First, recall [18] that P_α is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[\alpha]$ generated by α and Q_α is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n\ominus[\alpha]$. Then

$$\gamma_k(z) = \frac{\alpha_k - P_{\alpha,k}(z) - s_\alpha Q_{\alpha,k}(z)}{1-\langle z,\alpha\rangle}, \quad z\in\mathbb{B}_n$$

where $P_{\alpha,k}(z) := \frac{\langle z,\alpha\rangle}{\|\alpha\|^2}\alpha_k$, and $Q_{\alpha,k}(z) = z_k - \frac{\langle z,\alpha\rangle}{\|\alpha\|^2}\alpha_k$ for every $z\in\mathbb{B}_n$. Obviously that $\gamma(\alpha)=0$ and $(\gamma\circ\gamma)(z)=z$ for every $z\in\mathbb{B}_n$. Since $\alpha-P_\alpha(z)$ and $Q_\alpha(z)$ are perpendicular in $\mathbb{C}^n$ it is easy to see that $\alpha_k-P_{\alpha,k}(z)$ and $Q_{\alpha,k}(z)$ so are. Therefore, for every $|z_k|\ge\|\alpha\|$ we have

$$\begin{aligned}
&|\alpha_k - P_{\alpha,k}(z) - s_\alpha Q_{\alpha,k}(z)|^2\\
&\quad= |\alpha_k - P_{\alpha,k}(z)|^2 + (1-\|\alpha\|^2)||z_k|^2 - |Q_{\alpha,k}(z)|^2|\\
&\quad= |\alpha_k|^2\Big(1 - 2\mathrm{Re}\frac{\langle z,\alpha\rangle}{\|\alpha\|^2} + \frac{|\langle z,\alpha\rangle|^2}{\|\alpha\|^4}\Big) - \frac{|\langle z,\alpha\rangle|^2}{\|\alpha\|^4}|\alpha_k|^2 + \frac{|\langle z,\alpha\rangle|^2}{\|\alpha\|^2}|\alpha_k|^2 + (1-\|\alpha\|^2)|z_k|^2\\
&\quad= |\alpha_k|^2\Big(1 - 2\mathrm{Re}\frac{\langle z,\alpha\rangle}{\|\alpha\|^2} + \frac{|\langle z,\alpha\rangle|^2}{\|\alpha\|^2}\Big) + (1-\|\alpha\|^2)|z_k|^2\\
&\quad= \frac{|\alpha_k|^2}{\|\alpha\|^2}\Big(\|\alpha\|^2 - 2\mathrm{Re}\langle z,\alpha\rangle + |\langle z,\alpha\rangle|^2\Big) + (1-\|\alpha\|^2)|z_k|^2\\
&\quad= \frac{|\alpha_k|^2}{\|\alpha\|^2}\Big(\|\alpha\|^2 - 1 + 1 - 2\mathrm{Re}\langle z,\alpha\rangle + |\langle z,\alpha\rangle|^2\Big) + (1-\|\alpha\|^2)|z_k|^2\\
&\quad= \frac{|\alpha_k|^2}{\|\alpha\|^2}(\|\alpha\|^2-1) + \frac{|\alpha_k|^2}{\|\alpha\|^2}|1-\langle z,\alpha\rangle|^2 + (1-\|\alpha\|^2)|z_k|^2
\end{aligned}$$

This yields that, for $|z_k| \geq \|\alpha\|$, hence $\|z\| \geq \|\alpha\|$, we have

$$\begin{aligned}
|\gamma_k(z)|^2 &= \frac{|\alpha_k|^2}{\|\alpha\|^2} - \frac{(1-\|\alpha\|^2)\Big(\frac{|\alpha_k|^2}{\|\alpha\|^2} - |z_k|^2\Big)}{|1-\langle z,\alpha\rangle|^2} \\
&\leq \frac{|\alpha_k|^2}{\|\alpha\|^2} - \frac{(1-\|\alpha\|^2)\Big(\frac{|\alpha_k|^2}{\|\alpha\|^2} - |z_k|^2\Big)}{(1+|\langle z,\alpha\rangle|)^2} \\
&\leq \frac{|\alpha_k|^2}{\|\alpha\|^2} - \frac{(1-\|\alpha\|^2)\Big(\frac{|\alpha_k|^2}{\|\alpha\|^2} - |z_k|^2\Big)}{(1+\|\alpha\|)^2} \\
&= \frac{|\alpha_k|^2}{\|\alpha\|^2} - \frac{(1-\|\alpha\|)\Big(\frac{|\alpha_k|^2}{\|\alpha\|^2} - |z_k|^2\Big)}{1+\|\alpha\|} \\
&= \frac{|\alpha_k|^2}{\|\alpha\|(1+\|\alpha\|)} + \frac{1-\|\alpha\|}{1+\|\alpha\|}|z_k|^2 \\
&= A_k^2 + A^2|z_k|^2 \\
&< (A_k + A|z_k|)^2
\end{aligned}$$

where $A_k^2 := \frac{|\alpha_k|^2}{\|\alpha\|(1+\|\alpha\|)}$ and $A^2 := \frac{1-\|\alpha\|}{1+\|\alpha\|}$. It is easy to check that $A_k + A|z_k| < A_k + A < 1$. Thus we have

$$|\gamma_k(z)| \leq A_k + A|z_k| < 1 \quad \text{if } |z_k| \geq \|\alpha\|.$$

Then, we obtain the following the estimate

$$\begin{aligned}
\int_0^{|\gamma_k(z)|} \frac{dt}{\nu(t)} &\leq \int_0^{A|z_k|} \frac{dt}{\nu(t)} + \int_{A|z_k|}^{A_k+A|z_k|} \frac{dt}{\nu(t)} \\
&\leq \int_0^{|z_k|} \frac{dt}{\nu(t)} + \int_{A\|\alpha\|}^{A_k+A} \frac{dt}{\nu(t)} = C_{\alpha,k} + \int_0^{|z_k|} \frac{dt}{\nu(t)}
\end{aligned}$$

where $C_{\alpha,k} = \int_{A\|\alpha\|}^{A_k+A} \frac{dt}{\nu(t)} < \infty$. □

Lemma 4.3. *Let ν be a normal weight on $\mathbb{B}_n$, $\alpha \in \mathbb{B}_n \setminus \{0\}$ and $\gamma \in Aut(\mathbb{B}_n)$ defined by (4.2). Then, the composition operator $C_\gamma : \mathcal{B}_\nu \to \mathcal{B}_\nu$, $f \mapsto f \circ \gamma$, is an homeomorphism.*

Proof. First, since $\|\alpha\| < 1$ and $\|z\| < 1$ it is clear that

$$\|\nabla\gamma(z)\| = \frac{1-\|\alpha\|^2}{|1-\langle z,\alpha\rangle|^2} \leq 1 + \|\alpha\| \tag{4.3}$$

for every $z \in \mathbb{B}_n$.

Next, we will show that

$$\sup_{z\in\mathbb{B}_n} \frac{\nu(z)}{\nu(\gamma(z))} < \infty. \tag{4.4}$$

Indeed, by the continuity of γ the set $\{\gamma(z) : \ \|z\| \leq \|\alpha\|\}$ is relatively compact in $\mathbb{B}_n$. Since ν is positive and continuous $\inf_{\|z\|\leq\|\alpha\|} \nu(\gamma(z)) > 0$. It implies that

$$\sup_{\|z\|\leq\|\alpha\|} \frac{\nu(z)}{\nu(\gamma(z))} < \infty. \tag{4.5}$$

In the case $\|z\| \geq \|\alpha\|$, with a, b are in Definition 2.2 of the weight ν, by an estimate as in the previous lemma we have

$$\frac{(1-\|z\|)^a}{(1-\|\gamma(z)\|)^b} \leq \frac{(1-\|\alpha\|)^a}{(1-\frac{1}{1+\|\alpha\|} - \frac{1-\|\alpha\|}{1+\|a\|}|\|z\|)^b} \to \frac{(1-\|\alpha\|)^a}{(1-\frac{1}{1+\|\alpha\|} - \frac{1-\|\alpha\|}{1+\|a\|}|)^b} < \infty$$

as $\|z\| \to 1$. Then, by (W_1) and (W_2)

$$\lim_{\|z\|\to 1} \frac{\nu(z)}{\nu(\gamma(z))} = \lim_{\|z\|\to 1} \frac{\nu(z)}{(1-\|z\|)^a} \frac{(1-\|\gamma(z))\|)^b}{\nu(\gamma(z))} \frac{(1-\|z\|)^a}{(1-\|\gamma(z)\|)^b} = 0. \tag{4.6}$$

Then, we obtain (4.4) from (4.3), (4.5) and (4.5). Thus,

$$\begin{aligned}\|C_\gamma(f)\|_{\mathcal{B}_\nu} &= \sup_{z\in\mathbb{B}_n} \nu(z)\|\nabla(f\circ\gamma)(z)\| \\ &= \sup_{z\in\mathbb{B}_n} \frac{\nu(z)}{\nu(\gamma(z))}\nu(\gamma(z))\|\nabla(f(\gamma(z))\|\|\nabla\gamma(z)\| \\ &\le (1+\|\alpha\|) \sup_{z\in\mathbb{B}_n} \frac{\nu(z)}{\nu(\gamma(z))}\|f\|_{\mathcal{B}_\nu}.\end{aligned}$$

This means C_γ is bounded. Since $\gamma \in Aut(\mathbb{B}_n)$ it is easily seen that $C_{\gamma^{-1}} = C_\gamma^{-1}$ is also bounded. Hence, the lemma is proved. □

We use the following notation, which will be used in this work:

$$M_{R\psi,\varphi_*}(y) := \mu(y)|R\psi(y)| \int_0^{\|\varphi_*(y)\|} \frac{dt}{\nu(t)}$$

where φ_* denote either φ or φ_k, $k \in \{1,\dots,n\}$.

Theorem 4.4. *Let $\psi \in H(\mathbb{B}_n)$, $\varphi = (\varphi_1,\dots,\varphi_n) \in S(\mathbb{B}_n)$ and μ,ν be normal weights on $\mathbb{B}_n$. Then the following are equivalent:*

(1) *$C_{\psi,\varphi} : \mathcal{B}_\nu \to \mathcal{B}_\mu$ is bounded;*
(2) *$C^0_{\psi,\varphi} : \mathcal{B}_{\nu,0} \to \mathcal{B}_\mu$ is bounded;*
(3) *$\psi \in \mathcal{B}_\mu$ and $M_{R\psi,\varphi} := \sup_{y\in\mathbb{B}_n} M_{R\psi,\varphi}(y) < \infty$;*

In this case, the following asymptotic relation holds:

$$\|C_{\psi,\varphi}\| \asymp \|\psi\|_{\mathcal{B}_\mu} + 1 + M_{R\psi,\varphi}. \tag{4.7}$$

Moreover, under the additional conditions that $\psi \notin \mathcal{B}_{\mu,0}$, $\varphi \in S^*(\mathbb{B}_n)$ the assertions (1)-(3) and the following are equivalent:

(4) *$\psi \in \mathcal{B}_\mu$ and there exists $k \in \{1,2,\dots,n\}$ such that*

$$M_{R\psi,\varphi_k} := \sup_{y\in\mathbb{B}_n} M_{R\psi,\varphi_k}(y) < \infty.$$

Proof. It also suffices to prove for the case where $\varphi \in S^*(\mathbb{B}_n)$ because for $\varphi \in S(\mathbb{B}_n)$ the proof is similar but simpler.

It is clear that (3)⇒ (4) and since $\mathcal{B}_{\nu,0} \subset \mathcal{B}_\nu$ the implication (1) ⇒ (2) is obvious.

(4) ⇒ (1): Assume that there exists $k \in \{1,2,\dots,n\}$ such that $M_{R\psi,\varphi_k} < \infty$.

• First, we consider the case $\varphi(0) = 0$. Applying Lemma 4.1 with $h = \mu(\cdot)|R\psi(\cdot)|$ and Lemmas 2.1, 2.5, for every $f \in \mathcal{B}_\nu(\mathbb{B}_n)$, we have

$$\begin{aligned}&\|C_{\psi,\varphi}(f)\|_{s\mathcal{B}_\mu} = \sup_{z\in\mathbb{B}_n} \mu(z)|R(C_{\psi,\varphi}f(z))| \\ &= \sup_{z\in\mathbb{B}_n} \mu(z)|(f\circ\varphi)(z)||R\psi(z)| = \sup_{z\in\mathbb{B}_n} \mu(z)|f(\varphi(z))||R\psi(z)| \\ &\le \sup_{z\in\mathbb{B}_n} \mu(z)|R\psi(z)|C\Big(1 + \int_0^{\|\varphi(z)\|} \frac{dt}{\nu(t)}\Big)\|f\|_{\mathcal{B}_\nu} \\ &\le C\Big[\sup_{z\in\mathbb{B}_n} \mu(z)|R\psi(z)| + C_4\Big(1 + \sup_{z\in\mathbb{B}_n} \mu(z)|R\psi(z)| \int_0^{|\varphi_k(z)|} \frac{dt}{\nu(t)}\Big)\Big]\|f\|_{\mathcal{B}_\nu} \\ &\le C\Big[\|\psi\|_{\mathcal{B}_\mu} + C_4(1 + M_{R\psi,\varphi_k})\Big]\|f\|_{\mathcal{B}_\nu} < \infty.\end{aligned}$$

Consequently, by $C_{\psi,\varphi}(0) = 0$, we have

$$\|C_{\psi,\varphi}(f)\|_{\mathcal{B}_\mu} \le C_5\big[\|\psi\|_{\mathcal{B}_\mu} + 1 + M_{R\psi,\varphi_k}\big]\|f\|_{\mathcal{B}_\nu} \tag{4.8}$$

where $C_5 = C\max\{1, C_4\} > 0$. Thus, $C_{\psi,\varphi} : \mathcal{B}_\nu \to \mathcal{B}_\mu$ is bounded.

• Next, we consider the case $\varphi(0) = \alpha \in \mathbb{B}_n \setminus \{0\}$. Let $\gamma \in Aut(\mathbb{B}_n)$ given by (4.2). Then $\eta := \gamma \circ \varphi$ satisfies $\eta(0) = 0$.

By the continuity of γ the set $\{\eta(z) = \gamma(\varphi(z)) : \ \|\varphi(z)\| \le \delta_0\}$ is relatively compact in $\mathbb{B}_n$. Then, by $\psi \in \mathcal{B}_\mu$ we have

$$\sup_{|\varphi_k(z)|\le\delta_0} \mu(z)|R\psi(z)| \int_0^{|\eta_k(z)|} \frac{dt}{\nu(t)} \le \sup_{|\varphi_k(z)|\le\delta_0} \mu(z)|R\psi(z)| \int_0^{\|\eta(z)\|} \frac{dt}{\nu(t)} < \infty. \tag{4.9}$$

It remains to consider the case $|\varphi_k(z)| \ge \delta_0 \ge \|\alpha\|$.

By Lemma 4.2 there exists $C(\alpha, k) > 0$ such that

$$\int_0^{|\eta_k(z)|} \frac{dt}{\nu(t)} = \int_0^{|\gamma_k(\varphi(z))|} \frac{dt}{\nu(t)} \le C(\alpha, k) + \int_0^{|\varphi_k(z)|} \frac{dt}{\nu(t)}.$$

Thus

$$\begin{aligned} &\sup_{|\varphi_k(z)|\ge\delta_0} \mu(z)|R\psi(z)| \int_0^{|\eta_k(z)|} \frac{dt}{\nu(t)} \\ &\qquad \le C(\alpha, k)\|\psi\|_{\mathcal{B}_\mu} + \sup_{|\varphi_k(z)|\ge\delta_0} \mu(z)|R\psi(z)| \int_0^{|\varphi_k(z)|} \frac{dt}{\nu(t)} < \infty. \end{aligned} \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$\sup_{z\in\mathbb{B}_n} \mu(z)|R\psi(z)| \int_0^{|\eta_k(z)|} \frac{dt}{\nu(t)} < \infty.$$

Then, by the first case, $C_{\psi,\eta}$ is bounded.

On the other hand, it is easy to check that $C_{\psi,\eta} = C_{\psi,\varphi} \circ C_\gamma$. Then by Lemma 4.3, $C_{\psi,\varphi}$ is bounded, hence, (4) $\Rightarrow$ (1) is proved.

(2) $\Rightarrow$ (3): Suppose $C^0_{\psi,\varphi} : \mathcal{B}_{\nu,0} \to \mathcal{B}_\mu$ is bounded.

First, it is obvious that $\psi \in \mathcal{B}_\mu$ because $1 \in \mathcal{B}_{\mu,0}$ and

$$\psi(z) = \psi(0) + \int_0^1 R\psi(tz)\frac{dt}{t} = \psi(0) + (C_{\psi,\varphi}1)(z). \tag{4.11}$$

Now, fix $z \in \mathbb{B}_n$, put $w = \varphi(z)$ and for each $k \in \{1, \dots, n\}$ consider the test function $\beta_{w,k} \in \mathcal{B}_{\nu,0}$ defined by (3.5). Then,

$$\begin{aligned} \|C_{\psi,\varphi}\beta_{w,k}\|_{\mathcal{B}_\mu} &= \sup_{z\in\mathbb{B}_n} \mu(z)|R(\beta_{w,k}(\varphi(z))| = \sup_{z\in\mathbb{B}_n} \mu(z)|\beta_{w,k}(\varphi(z))|R\psi(z)| \\ &= \sup_{z\in\mathbb{B}_n} \mu(z)|R\psi(z)| \int_0^{|\varphi_k(z)|^2} g(t)dt \le \|\beta_{w,k}\|_{\mathcal{B}_\nu}\|C^0_{\psi,\varphi}\| \\ &\le C_2\|C^0_{\psi,\varphi}\| < \infty. \end{aligned} \tag{4.12}$$

Let $r_1 = \nu^{-1}(1/2)$.

If $|w_k| := |\varphi_k(z)| \ge r_1$ by (3.3) and (4.12) we have

$$\begin{aligned} \mu(z)|R\psi(z)| \int_0^{|w_k|} \frac{dt}{\nu(t)} &\le \mu(z)|R\psi(z)| \int_0^{|w_k|} \frac{g(t)}{C_1}dt \\ &\le \frac{C_3}{C_1}\mu(z)|R\psi(z)| \int_0^{|w_k|^2} g(t)dt \\ &\le \frac{C_3C_2}{C_1}\|C^0_{\psi,\varphi}\| < \infty. \end{aligned}$$

If $|w_k| < r_1$ by (3.3) again we obtain

$$\begin{aligned}\mu(z)|R\psi(z)|\int_0^{|w_k|}\frac{dt}{\nu(t)} &\leq \mu(z)|R\psi(z)|\int_0^{|w_k|}\frac{g(t)}{C_1}dt\\ &\leq \frac{1}{C_1}\mu(z)|R\psi(z)| = \frac{1}{C_1}\|C^0_{\psi,\varphi}1\|_{\mathcal{B}_\mu}\\ &\leq \frac{1}{C_1}\|C^0_{\psi,\varphi}\| < \infty.\end{aligned}$$

Then, combining with $\psi \in \mathcal{B}_\mu$ we get $M_{R\psi,\varphi_k} < \infty$ for every $k \in \{1,\dots,n\}$, hence, $M_{R\psi,\varphi} < \infty$.

Finally, combining the above estimates we get (4.7). The proof of Theorem 4.4 is completed. □

Next, we will touch the characterizations for the boundedness of the operators from $\mathcal{B}_{\nu,0}(\mathbb{B}_n)$ to $\mathcal{B}_{\mu,0}(\mathbb{B}_n)$.

Theorem 4.5. *Let $\psi \in H(\mathbb{B}_n)$ and $\varphi \in S(\mathbb{B}_n)$. Let μ,ν be normal weights on $\mathbb{B}_n$. Then the following are equivalent:*

(1) $C^{0,0}_{\psi,\varphi} : \mathcal{B}_{\nu,0} \to \mathcal{B}_{\mu,0}$ *is bounded;*

(2) $\psi \in \mathcal{B}_{\mu,0}$ *and* $M_{R\psi,\varphi} < \infty$.

In this case, as in Theorem 4.4, the following asymptotic relation holds

$$\|C^{0,0}_{\psi,\varphi}\| \asymp \|\psi\|_{\mathcal{B}_\mu} + 1 + M_{R\psi,\varphi}. \tag{4.13}$$

Moreover, under the additional condition that $\varphi \in S^*(\mathbb{B}_n)$ the assertions (1)-(3) and the following are equivalent:

(3) *There exists $k \in \{1,2,\dots,n\}$ such that $\psi \in \mathcal{B}_{\mu,0}$ and $M_{R\psi,\varphi_k} < \infty$.*

Proof. As in the previous one, it suffices to prove for the case $\varphi \in S^*(\mathbb{B}_n)$, $\varphi(0) = 0$ and (3) ⇒ (1) and (1) ⇒ (2).

(3) ⇒ (1): Assume that $\psi \in \mathcal{B}_{\mu,0}$ and $M_{R\psi,\varphi_k} < \infty$ for some $k \in \{1,\dots,n\}$.

First, we show that $C^{0,0}_{\psi,\varphi}f \in \mathcal{B}_{\mu,0}$ for every $f \in \mathcal{B}_{\nu,0}$.

Let $f \in \mathcal{B}_{\nu,0}$ be arbitrarily fixed. Let $\varepsilon > 0$ be fixed. Then there exists $r_0 \in (1/2,1)$ such that

$$\nu(z)|Rf(z)| < \frac{\varepsilon}{4M_{R\psi,\varphi_k}}, \quad \|z\| \geq r_0. \tag{4.14}$$

By the continuity of f we have

$$K := \sup_{\|w\|\leq r_0}|f(w)| < \infty. \tag{4.15}$$

Since $\psi \in \mathcal{B}_{\mu,0}$ we can find $\theta \in (0,1)$ such that

$$\mu(z)|R\psi(z)| < \frac{\varepsilon}{2K} \quad \text{whenever } \theta < \|z\| < 1. \tag{4.16}$$

For $\theta < \|z\| < 1$ we consider two cases:

- The case $\|w\| := \|\varphi(z)\| > r_0$: Let $\widehat{w} = r_0 \frac{w}{\|w\|}$. We have

$$
\begin{aligned}
|f(\varphi(z)) - f(\widehat{w})| = |f(w) - f(\widehat{w})| &\le \int_{r_0/\|w\|}^{1} \frac{|Rf(tw)|}{t} dt \\
&\le \frac{\|w\|}{r_0} \int_{r_0/\|w\|}^{1} |Rf(tw)| dt \\
&\le \frac{\varepsilon \|w\|}{4M_{R\psi,\varphi_k} r_0} \int_{r_0/\|w\|}^{1} \frac{1}{\nu(t\|w\|)} dt \\
&\le \frac{\varepsilon}{2M_{R\psi,\varphi_k}} \int_{r_0}^{\|w\|} \frac{1}{\nu(t)} dt.
\end{aligned}
$$

Combining (4.14) and (4.16), for $\theta < \|z\| < 1$, by (4.1) we obtain

$$
\begin{aligned}
\mu(z)\|RC_{\psi,\varphi}^{0,0} f(z)\| &= \mu(z)|f(\varphi(z))||R\psi(z)| \\
&\le \mu(z)|R\psi(z)||f(w) - f(\widehat{w})| + \mu(z)|R\psi(z)||f(\widehat{w})| \\
&\le \frac{\varepsilon}{2M_{R\psi,\varphi_k}} \mu(z)|R\psi(z)| \int_{r_0}^{\|w\|} \frac{1}{\nu(t)} dt + \mu(z)|R\psi(z)||f(\widehat{w})| \\
&\le \frac{\varepsilon}{2M_{R\psi,\varphi_k}} \sup_{y\in\mathbb{B}_n} \mu(y)|R\psi(y)| \int_{r_0}^{|\varphi_k(y)|} \frac{1}{\nu(t)} dt + \mu(z)|R\psi(z)||f(\widehat{w})| \\
&\le M_{R\psi,\varphi_k} \frac{\varepsilon}{2M_{R\psi,\varphi_k}} + K\frac{\varepsilon}{2K} = \varepsilon.
\end{aligned}
$$

- The case $\|w\| := \|\varphi(z)\| \le r_0$: We have

$$\mu(z)|R(C_{\psi,\varphi}^{0,0} f(z))| = \mu(z)|f(\varphi(z))||R\psi(z)| < K\frac{\varepsilon}{2K} < \varepsilon.$$

Thus, $C_{\psi,\varphi}^{0,0} f \in \mathcal{B}_{\mu,0}$.

Next, we prove that there exists $C > 0$ such that for every $z \in \mathbb{B}_n$

$$\mu(z)|R\psi(z)| \int_0^{\|\varphi(z)\|} \frac{dt}{\nu(t)} \le C(1 + M_{R\psi,\varphi_k}). \tag{4.17}$$

Then, by an argument similar to the proof (4) $\Rightarrow$ (1) for the case $\varphi(0) = 0$ of Theorem 4.4, we will get $C_{\psi,\varphi}^{0,0} : \mathcal{B}_{\nu,0} \to \mathcal{B}_{\mu,0}$ is bounded.

In the case $0 < \|z\| \le \delta$, where δ is in Definition 2.2 of the weight ν, we have $\{\varphi(z) : \|z\| \le \delta\}$ is relatively compact in $\mathbb{B}_n$. Then, it is obvious that

$$C_\delta := \sup_{\|z\|\le\delta} \mu(z)|R\psi(z)| \int_0^{\|\varphi(z)\|} \frac{dt}{\nu(t)} < \infty.$$

In the case $\delta \le \|z\| < 1$, by $\varphi \in S^*(\mathbb{B}_n)$, there exists $z' \in \mathbb{B}_n$ such that $|\varphi_k(z')| = \|\varphi(z)\|$. We consider the function $\beta_{\varphi_k(z')}$ given by

$$\beta_{\varphi_k(z')}(w) = \int_0^{\|w\||\varphi_k(z')|} g(t) dt$$

where g is defined by (3.2). As in Lemma 3.1, it is easy to check that $\beta_{\varphi_k(z')} \in \mathcal{B}_{\nu,0}$. Then

$$C_{\psi,\varphi}^{0,0} \beta_{\varphi_k(z')} \in \mathcal{B}_{\mu,0},$$

i.e.

$$\mu(w)|R\psi(w)| \int_0^{\|\varphi(w)\||\varphi_k(z')|} g(t) dt \to 0 \tag{4.18}$$

as $\|w\| \to 1$. This implies that for

$$\varepsilon_z := \mu(z')|R\psi(z')| \int_0^{|\varphi_k(z')|} \frac{dt}{\nu(t)}$$

there $\delta_z \geq \delta$ such that

$$\mu(w)|R\psi(w)| \int_0^{\|\varphi(w)\|\cdot|\varphi_k(z')|} g(t)dt < \varepsilon_z \tag{4.19}$$

for $\|w\| > \delta_z$. Put

$$\delta' = \inf_{\|z\|\geq\delta} \delta_z \geq \delta.$$

If $\|z\| \in [\delta, \delta']$ it is obvious that

$$\mu(z)|R\psi(z)| \int_0^{\|\varphi(z)\|} \frac{dt}{\nu(t)} \leq \sup_{\|w\|\in[\delta.\delta']} \mu(w)|R\psi(w)| \int_0^{\|\varphi(w)\|} \frac{dt}{\nu(t)} := C_{\delta,\delta'} < \infty. \tag{4.20}$$

If $\|z\| > \delta'$ we can find $w \in \mathbb{B}_n$ with $\|w\| > \delta$ such that $\delta' < \delta_w < \|z\|$. Then, by (4.19)

$$\mu(z)|R\psi(z)| \int_0^{\|\varphi(z)\|\cdot|\varphi_k(z')|} g(t)dt < \varepsilon_w = \mu(w)|R\psi(w)| \int_0^{|\varphi_k(w)|} \frac{dt}{\nu(t)}. \tag{4.21}$$

It follows from (3.3), (3.4), (4.20) and (4.21) that

$$\begin{aligned}
\mu(z)|R\psi(z)| \int_0^{\|\varphi(z)\|} \frac{dt}{\nu(t)} &\leq C_1^{-1}\mu(z)|R\psi(z)| \int_0^{\|\varphi(z)\|} g(t)dt \\
&\leq C_1^{-1}C_3\mu(z)|R\psi(z)| \int_0^{\|\varphi(z)\|^2} \frac{dt}{\nu(t)} \\
&= C_1^{-1}C_3\mu(z)|R\psi(z)| \int_0^{\|\varphi(z)\|\cdot|\varphi_k(z')|} \frac{dt}{\nu(t)} \\
&\leq C_{\delta,\delta'} + C_1^{-1}C_3 \sup_{\|w\|>\delta} \mu(w)|R\psi(w)| \int_0^{|\varphi_k(w)|} \frac{dt}{\nu(t)}.
\end{aligned}$$

And we obtain (4.17) with $C = \max\{C_\delta + C_{\delta,\delta'}, C_2C_3\}$, hence, the proof of (3) $\Rightarrow$ (1) is completed.

(1) $\Rightarrow$ (2): Assume that $C^{0,0}_{\psi,\varphi}$ is bounded. By (4.11), $\psi \in \mathcal{B}_{\mu,0}$. And then, we obtain $M_{R\psi,\varphi} < \infty$ by the proof of Theorem 4.4. □

5. The compactness of operators

In order to study the compactness of the operators $C_{\psi,\varphi}$, as in [14] we can prove the following.

Lemma 5.1. *Let E, F be two Banach spaces of holomorphic functions on $\mathbb{B}_n$. Suppose that*

(1) *The point evaluation functionals on E are continuous;*
(2) *The closed unit ball of E is a compact subset of E in the topology of uniform convergence on compact sets;*
(3) *$T : E \to F$ is continuous when E and F are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if given a bounded sequence $\{f_m\}$ in E such that $f_m \to 0$ uniformly on compact sets, then the sequence $\{Tf_m\}$ converges to zero in the norm of F.

We can now combine this result with Montel theorem and Lemma 2.5 to obtain the following proposition. The details of the proof are omitted here.

Proposition 5.2. *Let $\psi \in H(\mathbb{B}_n)$ and $\varphi \in S(\mathbb{B}_n)$. Then $C_{\psi,\varphi} : \mathcal{B}_\nu \to \mathcal{B}_\mu$ is compact if and only if $\|C_{\psi,\varphi}(f_m)\|_{\mathcal{B}_\mu} \to 0$ for any bounded sequence $\{f_m\}$ in $\mathcal{B}_\nu$ converging to 0 uniformly on compact sets in $\mathbb{B}_n$.*

Now we investigate the compactness of weighted composition operators $C_{\psi,\varphi}$.

Theorem 5.3. *Let $\psi \in H(\mathbb{B}_n)$ and $\varphi \in S(\mathbb{B}_n)$ and μ, ν be normal weights on $\mathbb{B}_n$ such that $\int_0^1 \frac{dt}{\nu(t)} = \infty$. Then the following are equivalent:*

(1) $C_{\psi,\varphi} : \mathcal{B}_\nu \to \mathcal{B}_\mu$ *is compact;*

(2) $C^0_{\psi,\varphi} : \mathcal{B}_{\nu,0} \to \mathcal{B}_\mu$ *is compact;*

(3) $\psi \in \mathcal{B}_\mu$ *and* $\lim\limits_{|\varphi(y)|\to 1} M_{R\psi,\varphi}(y) = 0$.

Moreover, under the additional conditions that $\psi \notin \mathcal{B}_{\mu,0}$, $\varphi \in S^*(\mathbb{B}_n)$ the assertions (1)-(3) and the following are equivalent:

(4) $\psi \in \mathcal{B}_\mu$ *and there exists $k \in \{1, 2, \dots, n\}$ such that* $\lim\limits_{|\varphi_k(y)|\to 1} M_{R\psi,\varphi_k}(y) = 0$.

Proof. As in Theorem 4.4, it suffices to prove for the case $\psi \notin \mathcal{B}_{\mu,0}$, $\varphi \in S^*(\mathbb{B}_n)$, $\varphi(0) = 0$ and (4) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3).

(4) $\Rightarrow$ (1): Since $\int_0^1 \frac{dt}{\nu(t)} = \infty$ and $\lim_{|\varphi_k(y)|\to 1} M_{R\psi,\varphi_k}(y) = 0$ for some $k \in \{1, \dots, n\}$, for any $\varepsilon > 0$, there exists $r_0 \in (1/2, 1)$ such that

$$\mu(z)|R\psi(z)| \int_{r_0}^{|\varphi_k(z)|} \frac{dt}{\nu(t)} < \frac{\varepsilon}{3C^+_{r_0}/C_2} \quad \text{for } r_0 < |w_k| := |\varphi_k(z)| < 1 \tag{5.1}$$

where $C^+_{r_0} := \frac{\sup_{\|z\|\ge r_0} \mu(z)|R\psi(z)|}{\inf_{\|z\|\ge r_0} \mu(z)|R\psi(z)|} < \infty$ because $\psi \notin \mathcal{B}_{\mu,0}$.

Let $\{f_m\}_{m\ge 1}$ be a bounded sequence in $\mathcal{B}_\nu$ converging to 0 uniformly on compact subsets of $\mathbb{B}_n$ and fix an $\varepsilon > 0$. We may assume that $\|f_m\|_{\mathcal{B}_\nu} \le 1$ for every $m \ge 1$. By the hypothesis on the sequence $\{f_m\}_{m\ge 1}$, there exists a positive integer m_0 such that

$$|f_m(w)| \le \frac{\varepsilon}{3\|\psi\|_{\mathcal{B}_\mu}}, \quad m \ge m_0, \ \|w\| \le r_0.$$

Therefore, for every $m \ge m_0$ and $\|w\| \le r_0$ we have

$$\mu(z)|R\psi(z)||f_m(w)| < \frac{\varepsilon}{3}. \tag{5.2}$$

Now, for every $m \ge m_0$ and $r_0 < \|w\| = \|\varphi(z)\| < 1$, with noting that $\|r_0 \frac{w}{\|w\|}\| = r_0$, by an argument in the proof of Lemma 4.1, (3.3), (5.1) and (5.2) we have

$$\begin{aligned}
&\mu(z)|R\psi(z)||f_m(w)| \\
&\quad \le \mu(z)|R\psi(z)|\Big|f_m(w) - f_m\Big(r_0\frac{w}{\|w\|}\Big)\Big| \\
&\qquad + \mu(z)|R\psi(z)|\Big|f_m\Big(r_0\frac{w}{\|w\|}\Big)\Big| \\
&\quad \le \mu(z)|R\psi(z)| \int_{r_0/\|w\|}^1 |Rf_m(tw)|\frac{dt}{t} + \frac{\varepsilon}{3} \\
&\quad \le \frac{1}{C_2}\mu(z)|R\psi(z)|\frac{\|w\|}{r_0} \int_{r_0/\|w\|}^1 \frac{1}{\nu(t\|w\|)}dt + \frac{\varepsilon}{3} \\
&\quad \le \frac{1}{C_2}\mu(z)|R\psi(z)| \int_{r_0}^{\|w\|} \frac{1}{\nu(t)}dt + \frac{\varepsilon}{3} \\
&\quad \le \frac{C^+_{r_0}}{C_2} \sup_{|\varphi_k(y)|>r_0} \mu(y)|R\psi(y)| \int_{r_0}^{|\varphi_k(y)|} \frac{1}{\nu(t)}dt + \frac{\varepsilon}{3} \\
&\quad \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.
\end{aligned} \tag{5.3}$$

Consequently, it follows from (5.2) and (5.2) that

$$\mu(z)|R\psi(z)||f_m(\varphi(z))| < \varepsilon$$

for every $z \in \mathbb{B}_n$ and every $m \geq m_0$. This means $\|C_{\psi,\varphi}(f_m)\|_{\mathcal{B}_\mu} \to 0$ as $m \to \infty$. Hence, by Proposition 5.2, $C_{\psi,\varphi}$ is compact.

(2) $\Rightarrow$ (3): Suppose $C^0_{\psi,\varphi}$ is compact. Then clearly, $C^0_{\psi,\varphi}$ is bounded.

Firstly, assume that $M_{R\psi,\varphi}(y) \not\to 0$ as $\|\varphi(y)\| \to 1$. Then we can take $\varepsilon_0 > 0$ and a sequence $\{z^m\}_{m\geq 1} \subset \mathbb{B}_n$ such that $|\varphi(z^m)| \to 1$ but

$$\mu(z^m)|R\psi(z^m)| \int_0^{\|\varphi(z^m)\|} \frac{dt}{\nu(t)} \geq \varepsilon_0 \quad \text{for every } m = 1, 2, \ldots \tag{5.4}$$

We may assume that $\|z^m\| > r_1 := \nu^{-1}(1/2)$. Denote $w^m = \varphi(z^m)$.

Consider the sequence $\{\gamma_{w^m}\}_{m\geq 1}$ defined by (3.6). By Proposition 3.2, this sequence is bounded in $\mathcal{B}_\nu$ and converges to 0 uniformly on compact subsets of $\mathbb{B}_n$. Then $\|C^0_{\psi,\varphi}\gamma_{w^m}\|_{\mathcal{B}_\mu} \to 0$ as $m \to \infty$ by Proposition 5.2.

On the other hand, by (3.3) and (5.4) we have

$$\begin{aligned}
\|C^0_{\psi,\varphi}\gamma_{w^m}\|_{\mathcal{B}_\mu} &= \sup_{z\in\mathbb{B}_n} \mu(z)|R\psi(z)||\gamma_{w^m}(\varphi(z))| \\
&\geq \mu(z^m)|R\psi(z^m)||\gamma_{w^m}(w^m)| \\
&= \mu(z^m)|R\psi(z^m)| \int_0^{\|w^m\|^2} g(t)dt \\
&\geq C_1C_3^{-1}\mu(z^m)|R\psi(z^m)| \int_0^{\|w^m\|} \frac{dt}{\nu(t)} \\
&\geq C_1C_3^{-1}\varepsilon_0.
\end{aligned}$$

Thus, we get a contradiction. And the proof is complete. □

Theorem 5.4. *Let $\psi \in H(\mathbb{B}_n)$, $\varphi \in S(\mathbb{B}_n)$ and μ, ν be normal weights on $\mathbb{B}_n$ such that $\int_0^1 \frac{dt}{\nu(t)} < \infty$. Then the following are equivalent:*

(1) *$C_{\psi,\varphi} : \mathcal{B}_\nu \to \mathcal{B}_\mu$ is compact;*
(2) *$C^0_{\psi,\varphi} : \mathcal{B}_{\nu,0} \to \mathcal{B}_\mu$ is compact;*
(3) *$\psi \in \mathcal{B}_\mu$.*

Proof. It suffices to prove (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1): Let $\{f_m\}_{m\geq 1}$ be a bounded sequence in $\mathcal{B}_\nu$ converging to 0 uniformly on compact subsets of $\mathbb{B}_n$. Since $\int_0^1 \frac{dt}{\nu(t)} < \infty$, by [13, Lemma 4.2], $\lim_{m\to\infty} \sup_{z\in\mathbb{B}_n} |f_m(z)| = 0$. Then, by $\psi \in \mathcal{B}_\mu$ we have

$$\lim_{m\to\infty} \sup_{z\in\mathbb{B}_n} \mu(z)\|\nabla\psi(z)\||f_m(\varphi(z))| = 0. \tag{5.5}$$

Thus, by Proposition 5.2, (3) $\Rightarrow$ (1) is proved.

(2) $\Rightarrow$ (3): Suppose $C^0_{\psi,\varphi} : \mathcal{B}_{\nu,0} \to \mathcal{B}_\mu$ is compact, it is bounded. As in the proof (2) $\Rightarrow$ (3) of Theorem 4.4 we get $\psi \in \mathcal{B}_\mu$.

This concludes the proof of the theorem. □

Next, we discus the compactness of the operator $C^{0,0}_{\psi,\varphi} : \mathcal{B}_{\nu,0} \to \mathcal{B}_{\mu,0}$.

Theorem 5.5. *Let $\psi \in H(\mathbb{B}_n)$ and $\varphi \in S(\mathbb{B}_n)$ and μ, ν be normal weights on $\mathbb{B}_n$ such that $\int_0^1 \frac{dt}{\nu(t)} = \infty$. Then the following are equivalent:*

(1) *$C^{0,0}_{\psi,\varphi} : \mathcal{B}_{\nu,0} \to \mathcal{B}_{\mu,0}$ is compact;*
(2) *$\psi \in \mathcal{B}_{\mu,0}$ and $\lim_{\|\varphi(y)\|\to 1} M_{R\psi,\varphi}(y) = 0$.*

Moreover, under the additional condition that $\varphi \in S^*(\mathbb{B}_n)$ the assertions (1)-(2) and the following are equivalent:

(3) *$\psi \in \mathcal{B}_{\mu,0}$ and there exists $k \in \{1,\dots,n\}$ such that* $\lim\limits_{|\varphi_k(y)|\to 1} M_{R\psi,\varphi_k}(y) = 0.$

Proof. As in the previous ones, it suffices to prove for the case $\varphi \in S^*(\mathbb{B}_n)$ and (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2).

(3) $\Rightarrow$ (1): By Theorems 4.5, 5.3 we obtain (1) from (3).

(1) $\Rightarrow$ (2): It follows from the hypothesis and Theorem 4.5 that $\psi \in \mathcal{B}_{\mu,0}$. In the same way as in the proof of (2) $\Rightarrow$ (3) of Theorem 5.3 we also obtain $\lim\limits_{\|\varphi(y)\|\to 1} M_{R\psi,\varphi}(y) = 0.$ □

By an argument similar to the proof of Theorem 5.4 we obtain the following, whose the proof will be omitted.

Theorem 5.6. *Let $\psi \in H(\mathbb{B}_n)$ and $\varphi \in S(\mathbb{B}_n)$ and μ, ν be normal weights on $\mathbb{B}_n$ such that $\int_0^1 \frac{dt}{\nu(t)} < \infty$. Then the following are equivalent:*

(1) *$C^{0,0}_{\psi,\varphi} : \mathcal{B}_{\nu,0} \to \mathcal{B}_{\mu,0}$ is compact;*
(2) *$\psi \in \mathcal{B}_{\mu,0}$.*

To finish this paper, we include the following remark:

Remark 5.7. Under the additional conditions that either $\varphi(0) = 0$ if $\varphi \in S(\mathbb{B}_n)$ or $\varphi_k(0) = 0$ for some $k \in \{1,\dots,n\}$ if $\varphi \in S^*(\mathbb{B}_n)$, the limits in Theorems 5.3, 5.5 are replaced by similar ones but with $\|z\| \to 1$ or $|z_k| \to 1$ respectively. Indeed, in a more general framework, it suffices to show $\|\varphi(z)\| \le \|z\|$ for every $z \in \mathbb{B}_n$. That means we have to give an n-dimensional version of Schwarz's lemma.

For each $z \in \mathbb{C}^n, z \neq 0$ and $w \in \overline{\mathbb{B}}_n$, applying classical Schwarz's lemma to the function $\phi_{z,w} : \mathbb{B}_1 \to \mathbb{B}_1$ given by

$$\phi_{z,w}(t) := \langle \varphi(tz/\|z\|), w\rangle \quad \forall t \in \mathbb{B}_1,$$

we have

$$|\phi_{z,w}(t)| \le |t|.$$

Then, choosing $t = \|z\|$ and $w = \frac{\overline{\varphi(z)}}{\|\varphi(z)\|}$ we get the desired inequality.

Remark 5.8. The additional condition $\varphi \in S^*(\mathbb{B}_n)$ in the main results can be replaced $\varphi \in S^*_k(\mathbb{B}_n) \supset S^*(\mathbb{B}_n)$ with

$$\varphi \in S^*_k(\mathbb{B}_n) := \Big\{\varphi \in S(\mathbb{B}_n) : \ \varphi(\mathbb{B}_n) \supseteq \{\lambda e_k : \ \lambda \in \mathbb{C}, |\lambda| < 1\}\Big\}.$$

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