



On the Leonardo quaternions sequence

P. D. Beites^{*1} , P. Catarino² 

¹*Department of Mathematics and CMA-UBI, University of Beira Interior, 6201 - 001 Covilhã, Portugal*

²*Department of Mathematics and CMAT-UTAD, University of Trás-os-Montes e Alto Douro, 5001 - 801 Vila Real, Portugal*

Abstract

In the present work, a new sequence of quaternions related to the Leonardo numbers – named the Leonardo quaternions sequence – is defined and studied. Binet’s formula and certain sum and binomial-sum identities, some of which derived from the mentioned formula, are established. Tagiuri-Vajda’s identity and, as consequences, Catalan’s identity, d’Ocagne’s identity and Cassini’s identity are presented. Furthermore, applying Catalan’s identity, and the connection between composition algebras and vector cross product algebras, Gelin-Cesàro’s identity is also stated and proved. Finally, the generating function, the exponential generating function and the Poisson generating function are deduced. In addition to the results on Leonardo quaternions, known results on Leonardo numbers and on Fibonacci quaternions are extended.

Mathematics Subject Classification (2020). 11R52, 05A10, 11B37, 05A15

Keywords. Leonardo numbers, Leonardo quaternions, Binet’s formula, (Tagiuri-Vajda’s, Sum and Binomial-sum) identities, generating function

1. Introduction

The problem of classifying all composition algebras with identity (that is, Hurwitz algebras, [6, 7]) became known as the Hurwitz’s problem. Briefly speaking, what are all pairs consisting of a not necessarily associative algebra A with identity and a non-degenerate quadratic form Q on A such that $Q(xy) = Q(x)Q(y)$? The problem was solved by Hurwitz, for the real and complex cases, and by Jacobson, for a field of characteristic different from 2. In fact, the complete answer was given by the generalized Hurwitz Theorem which asserts that, over a field of characteristic different from 2, A is isomorphic either to the base field, a separable quadratic extension of the base field, a generalized quaternion algebra or a generalized octonion algebra, respectively, of dimensions 1, 2, 4 and 8, [28]. With respect to quaternions, as can be seen by the number and the variety of publications produced over the years, the interest in this subject remains very much alive. For instance, among many others, some recent works on quaternions are related to: algebra, as [6] and [7]; analysis on manifolds and differential geometry, as [23]; differential and difference equations, as [8]; spatial analytic geometry, as [45]; numerical analysis,

*Corresponding Author.

Email addresses: pbeites@ubi.pt (P. D. Beites), pcatarin@utad.pt (P. Catarino)

Received: 01.11.2022; Accepted: 03.09.2023

as [27]; quaternionic analysis, as [38]; spectral theory and quaternion groups, as [11]; algebraic number theory, as [9]. The present work belongs to the latter area, concerning, more concretely, a new sequence, which is a quaternions sequence, named the Leonardo quaternions sequence. This sequence is related to the Leonardo numbers, whose associated sequence is part of the smoothsort algorithm due to Dijkstra, [25].

The sequence of Leonardo numbers was studied by Catarino and Borges in [15] and [16]. In the former reference, several properties and identities were established. An incomplete version of the Leonardo numbers was defined in the latter reference. In [2], Alves and Vieira studied the roots of the characteristic polynomial derived from the sequence of Leonardo, performing a root search through the Newton fractal. In [53], Vieira, Alves and Catarino discussed the bidimensional recurrent relations of the Leonardo numbers starting with its unidimensional model, establishing also the connection between this sequence and the sequence of Fibonacci numbers. In [54], Vieira, Mangueira, Alves and Catarino undertook research on generating matrices for the Leonardo numbers and, considering its n th powers, obtained new relations related to the sequence of Leonardo numbers. More recently, in [5], Alp and Koçer deduced several identities involving Leonardo numbers. In general, integer sequences have been the subject of several research studies. A vast bibliography covers several topics related to integer sequences. One can find in studies presented in [1, 13, 15–17, 19, 26, 34, 42, 44, 53, 54] some properties of these sequences, which are a very small proportion of articles devoted to the various number sequences. Sequences of polynomials have also been considered by several researchers. For instance, Cação, Malonek and Tomaz, in [12], work with shifted generalized Pascal matrices in the context of Appell sequences.

Earlier papers on distinct sequences involving quaternions are [3, 4, 9, 10, 14, 18, 20, 21, 29–32, 39, 42, 43, 47–51, 55], among others. Motivated by a rational sequence, quaternions whose components are Viëtoris' numbers are studied in [18]. In [3, 41, 51, 55], a split version of quaternions is considered. More concretely, in the latter references we find the study of split Fibonacci, Lucas, Jacobsthal, Jacobsthal-Lucas quaternions and their generalizations, as well as the split generalized Fibonacci quaternions. In [4], the non split version of generalized quaternions is introduced, while in [10] and [43] the authors study other generalizations of Fibonacci and Lucas quaternions. Another kind of Pell and Pell-Lucas quaternions are presented in [9], where some identities satisfied by these sequences of quaternions and their respective generating functions are stated. In [14], a new sequence of quaternions is defined, as well as in references [20, 21, 29–31, 47, 48, 50], where also some properties of these sequences are established. From a generalization for Fibonacci quaternions due to İpek in [33], (p, q) -Lucas quaternions are introduced in [39] and, afterwards, the authors derive some identities for both of these quaternions sequences. In reference [49], some matrix representations associated with the Horadam quaternions are presented. Furthermore, using the matrix technique, many identities related to the mentioned quaternions and some binomial-sum identities, as an application, are derived.

The structure of the present work, divided into two more sections, is as follows. In section 2, where some background is presented, known definitions and results related to the real division algebra of quaternions, to the Fibonacci numbers sequence, to the Fibonacci quaternions sequence and to the Leonardo numbers sequence are collected. In addition, known results on Leonardo numbers and on Fibonacci quaternions are extended. The connection between composition algebras and vector cross product algebras, used in a later section, is recalled in section 2 too. After this introductory section, section 3 is dedicated to the definition of a new sequence of quaternions related to the Leonardo numbers – named the Leonardo quaternions sequence – and to the study of its properties. In particular, beyond some basic properties, Binet's formula is deduced. Some sum and binomial-sum identities, some of which derived from Binet's formula, are also established.

Several identities, namely Tagiuri-Vajda’s identity and, as consequences, Catalan’s identity, d’Ocagne’s identity and Cassini’s identity, are presented. Moreover, applying Catalan’s identity, and the connection between composition algebras and vector cross product algebras, specifically for quaternions, Gelin-Cesàro’s identity is also stated and proved. Lastly, the generating function, the exponential generating function and the Poisson generating function are deduced. All the cited references may be found in the corresponding list at the end of the paper.

2. Some background

2.1. The real division algebra of quaternions

To start with, recall that the real algebra of quaternions can be defined as $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$ with multiplication determine by

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Recall also that \mathbb{H} is a non-commutative associative division algebra that is unital. Its identity denoted by 1, whenever clear from the context, is omitted.

This algebra splits as $\mathbb{H} = \mathbb{R}1 \oplus \text{Im}\mathbb{H}$, where $\text{Im}\mathbb{H} = \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$ is known as the subspace of purely imaginary quaternions. In this sense, every quaternion q can be written as $q = q_01 + \tilde{q}$ with $\tilde{q} \in \text{Im}\mathbb{H} \simeq \mathbb{R}^3$. The map $- : \mathbb{H} \rightarrow \mathbb{H}$ defined as

$$q = q_0 + \tilde{q} \mapsto \bar{q} = q_0 - \tilde{q} \tag{2.1}$$

is an involution, that is, for all $p, q \in \mathbb{H}$, $\bar{\bar{q}} = q$, $\overline{pq} = \bar{q}\bar{p}$, $\overline{p+q} = \bar{p} + \bar{q}$. The mentioned map is commonly called the standard involution.

The 4-dimensional algebra \mathbb{H} possesses the structure of composition algebra, provided by the non-degenerate quadratic form $N : \mathbb{H} \rightarrow \mathbb{R}$ defined as

$$q \mapsto N(q) = q\bar{q} = \bar{q}q, \tag{2.2}$$

which is mutliplicative, that is, for all $p, q \in \mathbb{H}$, $N(pq) = N(p)N(q)$. Usually, in the sense of composition algebras, $N(q)$ is called the norm of the quaternion q .

As can be seen in [8], [37] and references there in, there is a connection between composition algebras and vector cross product algebras. In particular, the multiplication of \mathbb{H} , as recalled in the following result, can be written in terms of the Euclidean inner product and the usual vector cross product in \mathbb{R}^3 , hereinafter denoted by $\langle \cdot, \cdot \rangle$ and \times , respectively.

Theorem 2.1 ([37]). *Let $p, q \in \mathbb{H}$. Then*

$$pq = p_0q_0 - \langle \tilde{p}, \tilde{q} \rangle + p_0\tilde{q} + q_0\tilde{p} + \tilde{p} \times \tilde{q}.$$

This theorem enhances the proof of part of a result – consisting of two equalities which involve quaternions – presented by Daşdemir in [24], where he mentions a proof by employing the multiplication table of \mathbb{H} . For completeness, making use of the relation among the multiplication of \mathbb{H} , $\langle \cdot, \cdot \rangle$ and \times expressed in Theorem 2.1, we present a different proof of one of the equalities.

Proposition 2.2 ([24]). *Let $p, q \in \mathbb{H}$. Then*

$$p^2 = 2p_0p - N(p), \tag{2.3}$$

$$pq = qp + 2\tilde{p} \times \tilde{q}. \tag{2.4}$$

Proof. Let $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$, $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$. As, by straightforward calculations,

$$\begin{aligned} p^2 &= p_0^2 - p_1^2 - p_2^2 - p_3^2 + 2(p_0p_1\mathbf{i} + p_0p_2\mathbf{j} + p_0p_3\mathbf{k}), \\ 2p_0p &= 2p_0^2 + 2(p_0p_1\mathbf{i} + p_0p_2\mathbf{j} + p_0p_3\mathbf{k}), \\ N(p) &= \bar{p}p = p_0^2 + p_1^2 + p_2^2 + p_3^2, \end{aligned}$$

then we arrive at (2.3). Concerning (2.4), Theorem 2.1 allows us to write

$$\begin{aligned} pq &= p_0q_0 - \langle \tilde{p}, \tilde{q} \rangle + p_0\tilde{q} + q_0\tilde{p} + \tilde{p} \times \tilde{q}, \\ qp &= q_0p_0 - \langle \tilde{q}, \tilde{p} \rangle + q_0\tilde{p} + p_0\tilde{q} + \tilde{q} \times \tilde{p}, \end{aligned}$$

and, subtracting member by member, to obtain (2.4). □

For other more aspects related to quaternions see, for instance, [22] – a general reference for the present section.

2.2. The Fibonacci numbers sequence

Although the Fibonacci sequence of numbers is one of the most well known sequences, for completeness, it is recalled.

Definition 2.3 ([36]). The sequence $\{F_n\}_{n=0}^\infty$ of Fibonacci numbers is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, n \in \mathbb{N}_0 \text{ such that } n \geq 2, \tag{2.5}$$

with initial conditions given by $F_0 = 0$ and $F_1 = 1$.

In the next theorem, Binet’s formula for Fibonacci numbers is also recalled.

Theorem 2.4 (Binet’s formula, [36]). For $n \in \mathbb{N}_0$,

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}, \tag{2.6}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Assuming that Binet’s formula (2.6) holds for negative subscripts, it is possible to extend Definition 2.3 to negative subscripts as follows.

Definition 2.5 ([36]). For $n \in \mathbb{N}$,

$$F_{-n} = (-1)^{n+1}F_n. \tag{2.7}$$

In the following proposition, several properties related to Fibonacci numbers, one also related to Lucas numbers, are collected.

Proposition 2.6. For $s, t \in \mathbb{Z}$, the following identities hold,

(a) [35]

$$F_{s+t} = F_tF_{s+1} + F_{t-1}F_s; \tag{2.8}$$

(b) [36]

$$F_{2s} = F_{s+1}^2 - F_{s-1}^2; \tag{2.9}$$

(c) [36]

$$F_{2s} = F_sL_s; \tag{2.10}$$

(d) [36]

$$(-1)^tF_{s-t} = F_sF_{t+1} - F_{s+1}F_t; \tag{2.11}$$

(e) [52]

$$(-1)^sF_tF_r = F_{s+t}F_{s+r} - F_sF_{s+t+r}. \tag{2.12}$$

2.3. The Fibonacci quaternions sequence

The Fibonacci quaternions were introduced by Horadam in [32], where he presented the definition and some properties.

Definition 2.7 ([32]). The s th element of the Fibonacci quaternions sequence $\{QF_n\}_{n=0}^\infty$ is given by

$$QF_s = F_s + F_{s+1}\mathbf{i} + F_{s+2}\mathbf{j} + F_{s+3}\mathbf{k}, \quad s \in \mathbb{N}_0. \tag{2.13}$$

Halici, in [29], continued Horadam’s study and presented the subsequent results that we recall, namely the following recurrence relation and Binet’s formula, respectively.

Proposition 2.8 ([29]). For $n \in \mathbb{N}_0$,

$$QF_{n+1} = QF_n + QF_{n-1}. \tag{2.14}$$

Theorem 2.9 (Binet’s formula, [29]). For $n \in \mathbb{N}_0$,

$$QF_n = \frac{\phi\phi^n - \psi\psi^n}{\phi - \psi}, \tag{2.15}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$, $\underline{\phi} = 1 + \phi\mathbf{i} + \phi^2\mathbf{j} + \phi^3\mathbf{k}$ and $\underline{\psi} = 1 + \psi\mathbf{i} + \psi^2\mathbf{j} + \psi^3\mathbf{k}$.

Assuming that Binet’s formula (2.15) holds for negative subscripts and taking into account Definition 2.5, it is possible to extend Definition 2.7 to negative subscripts as follows.

Definition 2.10. For $s \in \mathbb{N}$,

$$QF_{-s} = F_{-s} + F_{-s+1}\mathbf{i} + F_{-s+2}\mathbf{j} + F_{-s+3}\mathbf{k}. \tag{2.16}$$

The next lemma and theorem establish Tagiuri-Vajda’s identity for the Fibonacci quaternions.

Lemma 2.11. For $r \in \mathbb{Z}$,

$$\underline{\phi}\underline{\psi}\psi^r - \psi\phi\phi^r = (\phi - \psi)[-2QF_r + 3(2F_{r+1} - F_r)\mathbf{k}], \tag{2.17}$$

where $\phi = \frac{1 + \sqrt{5}}{2}$ and $\psi = \frac{1 - \sqrt{5}}{2}$.

Proof. Straightforward calculations lead to

$$\underline{\phi}\underline{\psi} = 2 + 2\psi\mathbf{i} + (2 + 2\psi)\mathbf{j} + (5 - 2\psi)\mathbf{k}$$

and

$$\underline{\psi}\underline{\phi} = 2 + 2\phi\mathbf{i} + (2 + 2\phi)\mathbf{j} + (5 - 2\phi)\mathbf{k}.$$

Invoking Binet’s formula (2.6) for the Fibonacci numbers in Theorem 2.4,

$$\begin{aligned} & \underline{\phi}\underline{\psi}\psi^r - \psi\phi\phi^r \\ &= 2(\psi^r - \phi^r) + 2(\psi^{r+1} - \phi^{r+1})\mathbf{i} + 2(\psi^r + \psi^{r+1} - \phi^r - \phi^{r+1})\mathbf{j} \\ & \quad + (5\psi^r - 2\psi^{r+1} - 5\phi^r + 2\phi^{r+1})\mathbf{k} \\ &= -2(\phi - \psi) \left[\frac{\phi^r - \psi^r}{\phi - \psi} + \frac{\phi^{r+1} - \psi^{r+1}}{\phi - \psi} \mathbf{i} + \left(\frac{\phi^r - \psi^r}{\phi - \psi} + \frac{\phi^{r+1} - \psi^{r+1}}{\phi - \psi} \right) \mathbf{j} \right] \\ & \quad + (\phi - \psi) \left(-5\frac{\phi^r - \psi^r}{\phi - \psi} + 2\frac{\phi^{r+1} - \psi^{r+1}}{\phi - \psi} \right) \mathbf{k} \\ &= -2(\phi - \psi)(F_r + F_{r+1}\mathbf{i} + F_r\mathbf{j} + F_{r+1}\mathbf{j}) + (\phi - \psi)(-5F_r + 2F_{r+1})\mathbf{k} \\ &= -2(\phi - \psi)(F_r + F_{r+1}\mathbf{i} + F_{r+2}\mathbf{j} + F_{r+3}\mathbf{k}) + 2(\phi - \psi)F_{r+3}\mathbf{k} \\ & \quad + (\phi - \psi)(-5F_r + 2F_{r+1})\mathbf{k} \\ &= (\phi - \psi)(-2QF_r + 2F_{r+3}\mathbf{k} - 5F_r\mathbf{k} + 2F_{r+1}\mathbf{k}) \\ &= (\phi - \psi)(-2QF_r + 4F_{r+1}\mathbf{k} + 2F_{r+2}\mathbf{k} - 5F_r\mathbf{k}) \\ &= (\phi - \psi)(-2QF_r + 6F_{r+1}\mathbf{k} - 3F_r\mathbf{k}), \end{aligned}$$

and the lemma follows. □

Theorem 2.12 (Tagiuri-Vajda’s identity). *For $n, k, r \in \mathbb{Z}$,*

$$QF_{n+k}QF_{n+r} - QF_nQF_{n+k+r} = (-1)^n F_k [2QF_r - 3(2F_{r+1} - F_r)\mathbf{k}]. \tag{2.18}$$

Proof. Applying Binet’s formula (2.15) for the Fibonacci quaternions in Theorem 2.9, we have, on the one hand,

$$\begin{aligned} & QF_{n+k}QF_{n+r} \\ &= \left(\frac{\phi\phi^{n+k} - \psi\psi^{n+k}}{\phi - \psi} \right) \left(\frac{\phi\phi^{n+r} - \psi\psi^{n+r}}{\phi - \psi} \right) \\ &= \frac{1}{(\phi - \psi)^2} \left(\phi^{2n+k+r} \phi^2 - \psi^{n+r} \phi^{n+k} \phi\psi - \psi^{n+k} \phi^{n+r} \psi\phi + \psi^{2n+k+r} \psi^2 \right), \end{aligned}$$

and, on the other hand,

$$\begin{aligned} & QF_nQF_{n+k+r} \\ &= \left(\frac{\phi\phi^n - \psi\psi^n}{\phi - \psi} \right) \left(\frac{\phi\phi^{n+k+r} - \psi\psi^{n+k+r}}{\phi - \psi} \right) \\ &= \frac{1}{(\phi - \psi)^2} \left(\phi^{2n+k+r} \phi^2 - \phi^n \psi^{n+k+r} \phi\psi - \psi^n \phi^{n+k+r} \psi\phi + \psi^{2n+k+r} \psi^2 \right). \end{aligned}$$

Hence, Binet’s formula (2.6) for the Fibonacci numbers in Theorem 2.4 and, once again, (2.15) lead to

$$\begin{aligned} & QF_{n+k}QF_{n+r} - QF_nQF_{n+k+r} \\ &= \frac{(\phi\psi)^n}{(\phi - \psi)^2} (-\psi^r \phi^k \phi\psi - \psi^k \phi^r \psi\phi + \psi^{k+r} \phi\psi + \phi^{k+r} \psi\phi) \\ &= \frac{(-1)^n}{(\phi - \psi)^2} (\phi^k - \psi^k) (\psi\phi\phi^r - \phi\psi\psi^r) \\ &= \frac{(-1)^n}{\phi - \psi} F_k (\psi\phi\phi^r - \phi\psi\psi^r) \\ &= \frac{(-1)^{n+1}}{\phi - \psi} F_k (\phi\psi\psi^r - \psi\phi\phi^r) \\ &= \frac{(-1)^{n+1}}{\phi - \psi} F_k (\phi - \psi) [-2QF_r + 3(2F_{r+1} - F_r)\mathbf{k}], \end{aligned}$$

which, after simplification, is the right hand side of (2.18). □

As consequences of Tagiuri-Vajda’s identity, the last results of this section establish, respectively, Catalan’s identity, d’Ocagne’s identity and Cassini’s identity for the Fibonacci quaternions. We point out that Halici presented Cassini’s identity for the Fibonacci quaternions in [29], which she obtained, for non-negative subscripts, in a different way – from the Fibonacci quaternion matrix. In addition, as a particular case of Catalan’s identity for k -Fibonacci quaternions due to Polath and Kesim, [40], Daşdemir presented Catalan’s identity, with positive subscripts, for the Fibonacci quaternions in [24].

Corollary 2.13 (Catalan’s identity). *For $n, r \in \mathbb{Z}$,*

$$QF_{n-r}QF_{n+r} - QF_n^2 = (-1)^{n-r+1} (2F_r QF_r - 3F_{2r}\mathbf{k}). \tag{2.19}$$

Proof. Taking $k = -r$ in (2.18) gives

$$\begin{aligned} & QF_{n-r}QF_{n+r} - QF_n^2 \\ &= (-1)^n F_{-r} [2QF_r - 3(2F_{r+1} - F_r)\mathbf{k}] \\ &= (-1)^{n+r+1} F_r [2QF_r - 3(2F_{r+1} - F_r)\mathbf{k}] \\ &= (-1)^{n-r+1} [2F_r QF_r - 3F_r (2F_{r+1} - F_r)\mathbf{k}] \\ &= (-1)^{n-r+1} [2F_r QF_r - 3(F_{r+1} - F_{r-1})(F_{r+1} + F_{r-1})\mathbf{k}], \end{aligned}$$

which, by (2.9) in Proposition 2.6, leads to (2.19). □

Corollary 2.14 (d’Ocagne’s identity). *For $n, m \in \mathbb{Z}$,*

$$QF_{n+1}QF_m - QF_nQF_{m+1} = (-1)^n[2QF_{m-n} - 3(2F_{m-n+1} - F_{m-n})\mathbf{k}]. \tag{2.20}$$

Proof. It suffices to consider $r = m - n$ and $k = 1$ in (2.18). □

Corollary 2.15 (Cassini’s identity). *For $n \in \mathbb{Z}$,*

$$QF_{n-1}QF_{n+1} - QF_n^2 = (-1)^n(2QF_1 - 3\mathbf{k}). \tag{2.21}$$

Proof. With $r = -k = 1$ in (2.18), we get (2.21). In fact,

$$\begin{aligned} QF_{n-1}QF_{n+1} - QF_n^2 &= (-1)^n F_{-1}[2QF_1 - 3(2F_2 - F_1)\mathbf{k}] \\ &= (-1)^n (-1)^2 F_1[2QF_1 - 3(2F_2 - F_1)\mathbf{k}]. \end{aligned} \tag{2.21}$$

□

2.4. The Leonardo numbers sequence

The sequence of Leonardo numbers is entry A001595 of the On-line Encyclopedia of Integers Sequences, [46], and was studied by Catarino and Borges in [15].

Definition 2.16 ([15]). The sequence $\{Le_n\}_{n=0}^\infty$ of Leonardo numbers is defined by the recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1, n \in \mathbb{N}_0 \text{ such that } n \geq 2, \tag{2.22}$$

with initial conditions given by $Le_0 = Le_1 = 1$.

As recalled in the following results, the sequence of Leonardo numbers can be related to the well-known sequence of Fibonacci numbers, as well as to the Lucas sequence $\{L_n\}_{n=0}^\infty$. The latter sequence of numbers has been presented in several research papers in the literature.

Proposition 2.17 ([15]). *For $n \in \mathbb{N}_0$,*

$$Le_n = 2F_{n+1} - 1. \tag{2.23}$$

Proposition 2.18 ([15]). *For $n \in \mathbb{N}_0$,*

$$Le_n = 2 \left(\frac{L_n + L_{n+2}}{5} \right) - 1, \tag{2.24}$$

$$Le_{n+3} = \frac{L_{n+1} + L_{n+7}}{5} - 1, \tag{2.25}$$

$$Le_n = L_{n+2} - F_{n+2} - 1. \tag{2.26}$$

In the next result, the Binet’s formula for Leonardo numbers is also recalled.

Proposition 2.19 (Binet’s formula, [15]). *For $n \in \mathbb{N}_0$,*

$$Le_n = 2 \frac{\phi^{n+1} - \psi^{n+1}}{\phi - \psi} - 1, \tag{2.27}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ e $\psi = \frac{1-\sqrt{5}}{2}$.

Assuming that Binet’s formula (2.27) holds for negative subscripts, it is possible to extend Definition 2.16 to negative subscripts. In fact,

$$\begin{aligned}
Le_{-n} &= 2 \frac{\phi^{-n+1} - \psi^{-n+1}}{\phi - \psi} - 1 \\
&= 2 \frac{(-\psi)^{n-1} - (-\phi)^{n-1}}{\phi - \psi} - 1 \\
&= 2(-1)^n \frac{\phi^{n-1} - \psi^{n-1}}{\phi - \psi} - 1 \\
&= 2(-1)^n F_{n-1} - 1,
\end{aligned}$$

which leads to the next definition. This definition appears also in [5], although introduced in a different way – making use of the definition of Fibonacci numbers with negative subscripts.

Definition 2.20. For $n \in \mathbb{N}$,

$$Le_{-n} = (-1)^n (Le_{n-2} + 1) - 1. \quad (2.28)$$

In the following theorem, Tagiuri-Vajda's identity for the Leonardo numbers is established. An alternative form of presenting this identity was given by Alp and Koçer in [5], where the authors expressed it, only for non-negative subscripts, in terms of Fibonacci numbers and Leonardo numbers.

Theorem 2.21 (Tagiuri-Vajda's identity). For $n, k, r \in \mathbb{Z}$,

$$\begin{aligned}
&Le_{n+k}Le_{n+r} - Le_nLe_{n+k+r} \\
&= (-1)^{n+1}(Le_{k-1} + 1) \left[\frac{1}{2}(-1)^{n+1}(Le_{n+r-1} - Le_{n-1}) + Le_{r-1} + 1 \right] \\
&\quad + \frac{1}{2}(Le_k - 1)(Le_{n+r} - Le_n). \quad (2.29)
\end{aligned}$$

Proof. From (2.23) in Proposition 2.17, (2.8) and (2.12) in Proposition 2.6,

$$\begin{aligned}
&Le_{n+k}Le_{n+r} - Le_nLe_{n+k+r} \\
&= (2F_{n+k+1} - 1)(2F_{n+r+1} - 1) - (2F_{n+1} - 1)(2F_{n+k+r+1} - 1) \\
&= 4(F_{n+1+k}F_{n+1+r} - F_{n+1}F_{n+1+k+r}) \\
&\quad + 2(F_{n+1} + F_{n+1+k+r} - F_{n+k+1} - F_{n+r+1}) \\
&= 4(-1)^{n+1}F_kF_r + 2(F_{n+1} + F_{k-1}F_{n+1+r} + F_kF_{n+r+2} - F_{n+k+1} - F_{n+r+1}) \\
&= 4(-1)^{n+1}F_kF_r \\
&\quad + 2[F_{n+1+r}(F_{k-1} - 1) + F_k(F_{n+r+1} + F_{n+r}) + F_{n+1} - F_{n+k+1}] \\
&= 4(-1)^{n+1}F_kF_r + 2[F_{n+1+r}(F_{k-1} + F_k - 1) + F_kF_{n+r} + F_{n+1} - F_{n+k+1}] \\
&= 4(-1)^{n+1}F_kF_r \\
&\quad + 2[F_{n+1+r}(F_{k+1} - 1) + F_kF_{n+r} + F_{n+1} - F_kF_n - F_{k+1}F_{n+1}] \\
&= 4(-1)^{n+1}F_kF_r + 2[F_{n+1+r}(F_{k+1} - 1) + F_k(F_{n+r} - F_n) + F_{n+1}(1 - F_{k+1})] \\
&= 4(-1)^{n+1}F_kF_r + 2(F_{k+1} - 1)(F_{n+1+r} - F_{n+1}) + 2F_k(F_{n+r} - F_n) \\
&= (-1)^{n+1}(Le_{k-1} + 1)(Le_{r-1} + 1) + \frac{1}{2}(Le_k - 1)(Le_{n+r} - Le_n) \\
&\quad + \frac{1}{2}(Le_{k-1} + 1)(Le_{n+r-1} - Le_{n-1}),
\end{aligned}$$

which easily leads to (2.29). \square

As consequences of Tagiuri-Vajda's identity, Catalan's identity, d'Ocagne's identity and Cassini's identity for Leonardo numbers are, respectively, obtained in the following results. The latter three identities, without using Tagiuri-Vajda's identity and only for non-negative subscripts, have been previously presented by Catarino and Borges in [15].

Corollary 2.22 (Catalan’s identity). For $n, r \in \mathbb{Z}$,

$$Le_{n-r}Le_{n+r} - Le_n^2 = -Le_{n-r} - Le_{n+r} + 2Le_n + (-1)^{n-r}(Le_{r-1} + 1)^2. \tag{2.30}$$

Proof. Taking $k = -r$ in (2.29) leads to

$$\begin{aligned} & (-1)^{n+1}(Le_{-r-1} + 1)[Le_{r-1} + 1 + \frac{1}{2}(-1)^{n+1}(Le_{n+r-1} - Le_{n-1})] \\ & \quad + \frac{1}{2}(Le_{-r} - 1)(Le_{n+r} - Le_n) \\ & = (-1)^{n+1}[(-1)^{r+1}(Le_{r-1} + 1)][Le_{r-1} + 1 + (-1)^{n+1}\frac{1}{2}(Le_{n+r-1} - Le_{n-1})] \\ & \quad + \frac{1}{2}[(-1)^r(Le_{r-2} + 1) - 2](Le_{n+r} - Le_n) \\ & = (-1)^{n+r}(Le_{r-1} + 1)[Le_{r-1} + 1 + (-1)^{n+1}\frac{1}{2}(Le_{n+r-1} - Le_{n-1})] \\ & \quad + \frac{1}{2}[(-1)^r(Le_{r-2} + 1) - 2](Le_{n+r} - Le_n) \\ & = (-1)^{n+r}(Le_{r-1} + 1)^2 + \frac{1}{2}(-1)^{r+1}(Le_{r-1} + 1)(Le_{n+r-1} - Le_{n-1}) \\ & \quad + \left[\frac{1}{2}(-1)^r(Le_{r-2} + 1) - 1\right](Le_{n+r} - Le_n) \\ & = (-1)^{n-r}(Le_{r-1} + 1)^2 + \frac{1}{2}(-1)^r(Le_{r-1} + 1)(Le_{n-1} - Le_{n+r-1}) \\ & \quad + \frac{1}{2}(-1)^r(Le_{r-2} + 1)(Le_{n+r} - Le_n) - Le_{n+r} + Le_n \end{aligned}$$

As, applying (2.11) in Proposition 2.6 and (2.23) in Proposition 2.17,

$$\begin{aligned} & \frac{1}{2}(-1)^r(Le_{r-1} + 1)(Le_{n-1} - Le_{n+r-1}) + \frac{1}{2}(-1)^r(Le_{r-2} + 1)(Le_{n+r} - Le_n) \\ & = \frac{1}{2}(-1)^r[2F_r(2F_n - 2F_{n+r}) + 2F_{r-1}(2F_{n+r+1} - 2F_{n+1})] \\ & = 2(-1)^r(F_nF_r - F_{n+1}F_{r-1} + F_{r-1}F_{n+r+1} - F_rF_{n+r}) \\ & = 2(-1)^r[(-1)^{r-1}F_{n-r+1} + (-1)^{n+r}F_{-n-1}] \\ & = 2(-1)^r[(-1)^{r-1}F_{n-r+1} + (-1)^{n+r}(-1)^{n+2}F_{n+1}] \\ & = 2(F_{n+1} - F_{n-r+1}) \\ & = Le_n - Le_{n-r}, \end{aligned}$$

then (2.30) holds. □

Corollary 2.23 (d’Ocagne’s identity). For $m, n \in \mathbb{Z}$,

$$Le_{n+1}Le_m - Le_nLe_{m+1} = 2(-1)^{n+1}(Le_{m-n-1} + 1) + Le_{m-1} - Le_{n-1}. \tag{2.31}$$

Proof. Taking $r = m - n$ and $k = 1$ in (2.29) leads to

$$\begin{aligned} & Le_{n+1}Le_m - Le_nLe_{m+1} \\ & = (-1)^{n+1}(Le_0 + 1) \left[\frac{1}{2}(-1)^{n+1}(Le_{m-1} - Le_{n-1}) + Le_{m-n-1} + 1 \right] \\ & \quad + \frac{1}{2}(Le_1 - 1)(Le_m - Le_n) \\ & = Le_{m-1} - Le_{n-1} + 2(-1)^{n+1}Le_{m-n-1} + 2(-1)^{n+1}, \end{aligned}$$

which is easily seen to be equal to the right hand side of (2.31). □

Corollary 2.24 (Cassini’s identity). For $n \in \mathbb{Z}$,

$$Le_{n-1}Le_{n+1} - Le_n^2 = -Le_{n-1} + Le_{n-2} - 4(-1)^n. \tag{2.32}$$

Proof. With $r = -k = 1$ in (2.29), we get

$$\begin{aligned}
 & Le_{n-1}Le_{n+1} - Le_n^2 \\
 &= (-1)^{n+1}(Le_{-2} + 1) \left[\frac{1}{2}(-1)^{n+1}(Le_n - Le_{n-1}) + Le_0 + 1 \right] \\
 &\quad + \frac{1}{2}(Le_{-1} - 1)(Le_{n+1} - Le_n) \\
 &= Le_n - Le_{n-1} + 4(-1)^{n+1} - Le_{n+1} + Le_n \\
 &= -4(-1)^n + Le_{n-2} - Le_{n-1},
 \end{aligned}$$

and (2.32) follows. □

3. The Leonardo quaternions sequence

In order to introduce a new quaternion sequence involving Leonardo’s numbers, we begin this section by presenting the definition of a general element of this sequence.

Definition 3.1. The s th element of the Leonardo quaternions sequence $\{QLe_n\}_{n=0}^\infty$ is given by

$$QLe_s = Le_s + Le_{s+1}\mathbf{i} + Le_{s+2}\mathbf{j} + Le_{s+3}\mathbf{k}, \quad s \in \mathbb{N}_0. \tag{3.1}$$

If we denote the s th element of the the complex sequence with Leonardo’ numbers by

$$QLe_s^c = Le_s + Le_{s+1}\mathbf{i}, \tag{3.2}$$

one can rewrite the s th element of $\{QLe_n\}_{n=0}^\infty$ as

$$QLe_s = Le_s + Le_{s+1}\mathbf{i} + (Le_{s+2} + Le_{s+3}\mathbf{i})\mathbf{j} = QLe_s^c + QLe_{s+2}^c\mathbf{j}.$$

Taking into account (2.22), through easy calculations, the s th element of $\{QLe_n\}_{n=0}^\infty$ can also be written as

$$QLe_s = Le_s(1 + \mathbf{j} + \mathbf{k}) + Le_{s+1}(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + \mathbf{j} + 2\mathbf{k}. \tag{3.3}$$

Binet’s formula for Leonardo quaternions is given in the the following result.

Theorem 3.2 (Binet’s formula). *For $n \in \mathbb{N}_0$,*

$$QLe_n = 2 \left(\frac{\phi^{n+1}\widehat{\phi} - \psi^{n+1}\widehat{\psi}}{\phi - \psi} \right) - q_u, \tag{3.4}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$, $\widehat{\phi} = 1 + \phi\mathbf{i} + (1 + \phi)\mathbf{j} + (1 + 2\phi)\mathbf{k}$, $\widehat{\psi} = 1 + \psi\mathbf{i} + (1 + \psi)\mathbf{j} + (1 + 2\psi)\mathbf{k}$ and $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Proof. From (3.3), applying Binet’s formula (2.27) for Leonardo numbers in Theorem 2.19, we obtain

$$\begin{aligned}
 & QLe_n \\
 &= 2 \frac{\phi^{n+1} - \psi^{n+1}}{\phi - \psi} (1 + \mathbf{j} + \mathbf{k}) + 2 \frac{\phi^{n+2} - \psi^{n+2}}{\phi - \psi} (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - 1 - \mathbf{j} - \mathbf{j} - \mathbf{k} \\
 &= 2 \frac{\phi^{n+1}(1 + \phi\mathbf{i} + (1 + \phi)\mathbf{j} + (1 + 2\phi)\mathbf{k}) - \psi^{n+1}(1 + \psi\mathbf{i} + (1 + \psi)\mathbf{j} + (1 + 2\psi)\mathbf{k})}{\phi - \psi} \\
 &\quad - q_u.
 \end{aligned}$$

Hence, this leads to (3.4). □

Assuming that Binet’s formula (3.4) holds for negative subscripts and taking into account Definition 2.20, it is possible to extend Definition 3.1 to negative subscripts as follows.

Definition 3.3. For $s \in \mathbb{N}$,

$$QLe_{-s} = Le_{-s} + Le_{-s+1}\mathbf{i} + Le_{-s+2}\mathbf{j} + Le_{-s+3}\mathbf{k}. \tag{3.5}$$

3.1. Basic properties

In this subsection, several properties of the Leonardo quaternions sequence are studied.

Proposition 3.4. *Let $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. The following identities hold:*

(a)
$$QLe_n + \overline{QLe_n} = 2Le_n; \tag{3.6}$$

(b)
$$QLe_{n+2} = QLe_{n+1} + QLe_n + q_u; \tag{3.7}$$

(c)
$$QLe_n = 2QF_{n+1} - q_u; \tag{3.8}$$

(d)
$$QLe_{n+1} - QLe_n = 2QF_n; \tag{3.9}$$

(e)
$$q_u QLe_s - QLe_s q_u = 2[(Le_{s+1} + 1)\mathbf{i} - (Le_{s+2} + 1)\mathbf{j} + (Le_s + 1)\mathbf{k}]. \tag{3.10}$$

Proof. As $QLe_n = Le_n + \widetilde{QLe_n}$ and $\overline{QLe_n} = Le_n - \widetilde{QLe_n}$, it is straightforward to get (3.6).

Since $QLe_{n+1} + QLe_n = Le_{n+2} + Le_{n+3}\mathbf{i} + Le_{n+4}\mathbf{j} + Le_{n+5}\mathbf{k} - (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$, we obtain (3.7).

Concerning (3.8), as $QLe_n = Le_n + Le_{n+1}\mathbf{i} + Le_{n+2}\mathbf{j} + Le_{n+3}\mathbf{k}$, then, by (2.23) in Proposition 2.17, we get

$$QLe_n = 2(F_{n+1} + F_{n+2}\mathbf{i} + F_{n+3}\mathbf{j} + F_{n+4}\mathbf{k}) - (1 + \mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Taking into account that the recurrence relation (3.7) is inhomogeneous, Proposition 2.17 leads to (3.9) as

$$QLe_{n+1} - QLe_n = QLe_{n-1} + q_u = 2(F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k}) = 2QF_n.$$

Finally, in order to prove (3.10), observe that

$$\begin{aligned} q_u QLe_s &= Le_s - Le_{s+1} - Le_{s+2} - Le_{s+3} + (Le_{s+1} + Le_s + Le_{s+3} - Le_{s+2})\mathbf{i} \\ &\quad + (Le_{s+2} - Le_{s+3} + Le_s + Le_{s+1})\mathbf{j} \\ &\quad + (Le_{s+3} + Le_{s+2} - Le_{s+1} + Le_s)\mathbf{k} \end{aligned}$$

and

$$\begin{aligned} QLe_s q_u &= Le_s - Le_{s+1} - Le_{s+2} - Le_{s+3} + (Le_s + Le_{s+1} + Le_{s+2} - Le_{s+3})\mathbf{i} \\ &\quad + (Le_s - Le_{s+1} + Le_{s+2} + Le_{s+3})\mathbf{j} \\ &\quad + (Le_s + Le_{s+1} - Le_{s+2} + Le_{s+3})\mathbf{k}. \end{aligned}$$

Hence,

$$q_u QLe_s - QLe_s q_u = 2(Le_{s+3} - Le_{s+2})\mathbf{i} + 2(Le_{s+1} - Le_{s+3})\mathbf{j} + 2(Le_{s+2} - Le_{s+1})\mathbf{k},$$

which is easily seen to be equal to the second member in (3.10). □

In the next result, properties related to the norm of a Leonardo quaternion, and some relations with the Fibonacci sequence, the Fibonacci quaternions sequence and the Lucas sequences, are presented.

Proposition 3.5. *For $n \in \mathbb{N}_0$,*

(a)
$$N(QLe_n) = 4(N(QF_{n+1}) - L_{n+4} + 1), \tag{3.11}$$

where $N(QF_{n+1}) = 3F_{2n+5}$;

(b)
$$N(QLe_n) - N(QLe_{n+1}) = 4L_{n+3}(1 - 3F_{n+3}); \tag{3.12}$$

(c)

$$N(QLe_n) = (Le_{n+3} - 1)(Le_{n+4} - 1) - (Le_{n-1} - 1)(Le_n - 1) + 4. \tag{3.13}$$

Proof. One starts by proving (a) applying (2.23) in Proposition 2.17, which leads to

$$\begin{aligned} N(QLe_n) &= Le_n^2 + Le_{n+1}^2 + Le_{n+2}^2 + Le_{n+3}^2 \\ &= (2F_{n+1} - 1)^2 + (2F_{n+2} - 1)^2 + (2F_{n+3} - 1)^2 + (2F_{n+4} - 1)^2 \\ &= 4[(F_{n+1}^2 + F_{n+2}^2 + F_{n+3}^2 + F_{n+4}^2) \\ &\quad - (F_{n+1} + F_{n+2} + F_{n+3} + F_{n+4}) + 1] \\ &= 4[N(QF_{n+1}) - (2F_{n+3} + F_{n+4}) + 1] \\ &= 4[N(QF_{n+1}) - (Le_{n+2} + 1 + F_{n+4}) + 1]. \end{aligned}$$

From here and (2.26) in Proposition 2.18, the right-hand side in (3.11) is obtained. Moreover, by [32], $N(QF_{n+1})$ can be written as stated.

In what concerns (3.12), taking into account (a),

$$\begin{aligned} N(QLe_n) - N(QLe_{n+1}) &= 4(N(QF_{n+1}) - N(QF_{n+2}) + L_{n+5} - L_{n+4}) \\ &= 4[3(F_{2n+5} - F_{2n+7}) + L_{n+3}], \end{aligned}$$

where, by (2.10) in Proposition 2.6, $F_{2n+5} - F_{2n+7} = -F_{2n+6} = -F_{n+3}L_{n+3}$.

As

$$\begin{aligned} N(QLe_n) &= QLe_n \overline{QLe_n} \\ &= Le_n^2 + Le_{n+1}^2 + Le_{n+2}^2 + Le_{n+3}^2 \\ &= \sum_{s=0}^{n+3} Le_s^2 - \sum_{s=0}^{n-1} Le_s^2, \end{aligned}$$

then, invoking [15, Proposition 3.3], the latter expression is precisely (3.13). □

3.2. Sum and binomial-sum identities

In the next result, formulae for the sum of the first n Leonardo quaternions, of the first n Leonardo quaternions with odd subscripts, and of the first n Leonardo quaternions with even subscripts, are presented.

Theorem 3.6. *Let $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. For $n \in \mathbb{N}_0$,*

$$\sum_{s=1}^n QLe_s = QLe_{n+2} - QLe_2 - nq_u, \tag{3.14}$$

$$\sum_{s=1}^n QLe_{2s-1} = QLe_{2n} - QLe_0 - nq_u, \tag{3.15}$$

$$\sum_{s=1}^n QLe_{2s} = QLe_{2n+1} - QLe_1 - nq_u, \tag{3.16}$$

Proof. For (3.14), we use induction on n . From (3.7), it is straightforward to see that the base case holds. As for the induction step, we have

$$\begin{aligned} \sum_{s=1}^{n+1} QLe_s &= \sum_{s=1}^n QLe_s + QLe_{n+1} \\ &= QLe_{n+2} - QLe_2 - nq_u + QLe_{n+1} \\ &= QLe_{n+3} - QLe_2 - (n + 1)q_u. \end{aligned}$$

In a similar way, (3.15) and (3.16) can be proved by induction on n . □

In the following result, certain binomial-sum identities are established.

Theorem 3.7. Let $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. For $n \in \mathbb{N}_0$,

$$QLe_{2n} = \sum_{i=0}^n \binom{n}{i} (QLe_i + q_u) - q_u, \tag{3.17}$$

$$QLe_{2n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} (QLe_{i-1} + q_u) - q_u. \tag{3.18}$$

Proof. Applying the Binet’s formula (3.4) for the Leonardo quaternions in Theorem 3.2, we arrive at

$$\begin{aligned} QLe_{2n} &= 2 \frac{\phi^{2n+1} \widehat{\phi} - \psi^{2n+1} \widehat{\psi}}{\phi - \psi} - q_u \\ &= 2 \frac{(1 + \phi)^n \phi \widehat{\phi} - (1 + \psi)^n \psi \widehat{\psi}}{\phi - \psi} - q_u \\ &= \frac{2}{\phi - \psi} \left(\sum_{i=0}^n \binom{n}{i} \phi^i \phi \widehat{\phi} - \sum_{i=0}^n \binom{n}{i} \psi^i \psi \widehat{\psi} \right) - q_u \\ &= \sum_{i=0}^n \binom{n}{i} \left(2 \frac{\phi^{i+1} \widehat{\phi} - \psi^{i+1} \widehat{\psi}}{\phi - \psi} - q_u \right) + \sum_{i=0}^n \binom{n}{i} q_u - q_u, \end{aligned}$$

and (3.17) follows. Using a similar reasoning, we get (3.18). □

3.3. Tagiuri-Vajda’s identity and related identities

The first theorem of this section, preceded by an auxiliary result, establishes Tagiuri-Vajda’s identity for the Leonardo quaternions.

Lemma 3.8. For $r \in \mathbb{Z}$,

$$\psi^r \widehat{\phi} \widehat{\psi} - \phi^r \widehat{\psi} \widehat{\phi} = -(\phi - \psi) [QLe_{r-1} + q_u - (6F_{r+1} - 3F_r)\mathbf{k}]$$

where $\phi = \frac{1 + \sqrt{5}}{2}$, $\psi = \frac{1 - \sqrt{5}}{2}$, $\widehat{\phi} = 1 + \phi\mathbf{i} + (1 + \phi)\mathbf{j} + (1 + 2\phi)\mathbf{k}$, $\widehat{\psi} = 1 + \psi\mathbf{i} + (1 + \psi)\mathbf{j} + (1 + 2\psi)\mathbf{k}$ and $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Proof. As $\widehat{\phi}\widehat{\psi} = 2 + 2\psi\mathbf{i} + (2 + 2\psi)\mathbf{j} + (5 - 2\psi)\mathbf{k}$ and $\widehat{\psi}\widehat{\phi} = 2 + 2\phi\mathbf{i} + (2 + 2\phi)\mathbf{j} + (5 - 2\phi)\mathbf{k}$, then Binet’s formula (2.6) for the Fibonacci numbers in Theorem 2.4 allows to arrive at

$$\begin{aligned} \psi^r \widehat{\phi} \widehat{\psi} - \phi^r \widehat{\psi} \widehat{\phi} &= 2(\psi^r - \phi^r) \\ &\quad + 2(\psi^{r+1} - \phi^{r+1})\mathbf{i} + 2[(\psi^r - \phi^r) + (\psi^{r+1} - \phi^{r+1})]\mathbf{j} \\ &\quad + [5(\psi^r - \phi^r) - 2(\psi^{r+1} - \phi^{r+1})]\mathbf{k} \\ &= -2(\phi - \psi)F_r - 2(\phi - \psi)F_{r+1}\mathbf{i} - 2(\phi - \psi)(F_r + F_{r+1})\mathbf{j} \\ &\quad + (\phi - \psi)(-5F_r + 2F_{r+1})\mathbf{k} \\ &= -(\phi - \psi)[2(F_r + F_{r+1})\mathbf{i} + F_{r+2}\mathbf{j} + F_{r+3}\mathbf{k}] \\ &\quad - (-5F_r + 2F_{r+1})\mathbf{k} - 2F_{r+3}\mathbf{k} \\ &= -(\phi - \psi)[2QF_r - (-5F_r + 2F_{r+1} + 2F_{r+3})\mathbf{k}] \\ &= -(\phi - \psi)[2QF_r - 3(2F_{r+1} - F_r)\mathbf{k}] \\ &= -(\phi - \psi)[QLe_{r-1} + q_u - (6F_{r+1} - 3F_r)\mathbf{k}]. \end{aligned}$$

□

Theorem 3.9 (Tagiuri-Vajda’s identity). For $n, k, r \in \mathbb{Z}$,

$$\begin{aligned} &QLe_{n+k}QLe_{n+r} - QLe_nQLe_{n+k+r} \\ &= 2(-1)^{n+1}(Le_{k-1} + 1)(QLe_{r-1} - 3(2F_{r+1} - F_r)\mathbf{k} + q_u) \\ &\quad + q_u(QLe_{n+k+r} - QLe_{n+r}) + (QLe_n - QLe_{n+k})q_u. \end{aligned} \tag{3.19}$$

Proof. On the one hand, from the Binet’s formula (3.4) for the Leonardo quaternions in Theorem 3.2,

$$\begin{aligned} & QLe_{n+k}QLe_{n+r} \\ &= \left(2\frac{\phi^{n+k+1}\widehat{\phi} - \psi^{n+k+1}\widehat{\psi}}{\phi - \psi} - q_u\right) \left(2\frac{\phi^{n+r+1}\widehat{\phi} - \psi^{n+r+1}\widehat{\psi}}{\phi - \psi} - q_u\right) \\ &= \frac{4}{(\phi - \psi)^2} \left(\phi^{2n+k+r+2}\widehat{\phi}^2 - \phi^{n+k+1}\psi^{n+r+1}\widehat{\phi}\widehat{\psi} - \psi^{n+k+1}\phi^{n+r+1}\widehat{\psi}\widehat{\phi} + \psi^{2n+k+r+2}\widehat{\psi}^2\right) \\ &\quad - \frac{2}{\phi - \psi} \left(\phi^{n+k+1}\widehat{\phi}q_u - \psi^{n+k+1}\widehat{\psi}q_u + q_u\phi^{n+r+1}\widehat{\phi} - q_u\psi^{n+r+1}\widehat{\psi}\right) + q_u^2. \end{aligned}$$

On the other hand, once again by Binet’s formula (3.4) in Theorem 3.2,

$$\begin{aligned} & QLe_nQLe_{n+k+r} \\ &= \left(2\frac{\phi^{n+1}\widehat{\phi} - \psi^{n+1}\widehat{\psi}}{\phi - \psi} - q_u\right) \left(2\frac{\phi^{n+k+r+1}\widehat{\phi} - \psi^{n+k+r+1}\widehat{\psi}}{\phi - \psi} - q_u\right) \\ &= \frac{4}{(\phi - \psi)^2} \left(\phi^{2n+k+r+2}\widehat{\phi}^2 - \phi^{n+1}\psi^{n+k+r+1}\widehat{\phi}\widehat{\psi} - \psi^{n+1}\phi^{n+k+r+1}\widehat{\psi}\widehat{\phi} + \psi^{2n+k+r+2}\widehat{\psi}^2\right) \\ &\quad - \frac{2}{\phi - \psi} \left(\phi^{n+1}\widehat{\phi}q_u - \psi^{n+1}\widehat{\psi}q_u + q_u\phi^{n+k+r+1}\widehat{\phi} - q_u\psi^{n+k+r+1}\widehat{\psi}\right) + q_u^2. \end{aligned}$$

Hence, from Binet’s formula (2.27) for the Leonardo quaternions in Proposition 2.19,

$$\begin{aligned} & QLe_{n+k}QLe_{n+r} - QLe_nQLe_{n+k+r} \\ &= \frac{4(\phi\psi)^{n+1}}{(\phi - \psi)^2} [(\phi^k - \psi^k)(-\psi^r\widehat{\phi}\widehat{\psi} + \phi^r\widehat{\psi}\widehat{\phi})] \\ &\quad - \frac{2}{\phi - \psi} [(\phi^{n+k+1}\widehat{\phi} - \psi^{n+k+1}\widehat{\psi})q_u + q_u(\phi^{n+r+1}\widehat{\phi} - \psi^{n+r+1}\widehat{\psi}) \\ &\quad - (\phi^{n+1}\widehat{\phi} - \psi^{n+1}\widehat{\psi})q_u - q_u(\phi^{n+k+r+1}\widehat{\phi} - \psi^{n+k+r+1}\widehat{\psi})] \\ &= \frac{2(-1)^{n+1}}{\phi - \psi} [-(Le_{k-1} + 1)(\psi^r\widehat{\phi}\widehat{\psi} - \phi^r\widehat{\psi}\widehat{\phi})] \\ &\quad - [(QLe_{n+k} - QLe_n)q_u + q_u(QLe_{n+r} - QLe_{n+k+r})], \end{aligned}$$

which, by Lemma 3.8, leads to (3.19). □

As consequences of Tagiuri-Vajda’s identity, Catalan’s identity, d’Ocagne’s identity and Cassini’s identity for Leonardo quaternions are, respectively, obtained in the following results.

Corollary 3.10 (Catalan’s identity). *For $n, r \in \mathbb{Z}$,*

$$\begin{aligned} & QLe_{n-r}QLe_{n+r} - QLe_n^2 \\ &= 2(-1)^{n+r}(Le_{r-1} + 1)(QLe_{r-1} - 3(2F_{r+1} - F_r)\mathbf{k} + q_u) \\ &\quad + q_u(QLe_n - QLe_{n+r}) + (QLe_n - QLe_{n-r})q_u. \end{aligned} \tag{3.20}$$

Proof. In order to arrive at (3.20), take $k = -r$ in (3.19). More concretely,

$$\begin{aligned} & QLe_{n-r}QLe_{n+r} - QLe_n^2 \\ &= 2(-1)^{n+1}[Le_{-r-1} + 1][QLe_{r-1} - 3(2F_{r+1} - F_r)\mathbf{k} + q_u] \\ &\quad + q_u(QLe_n - QLe_{n+r}) + (QLe_n - QLe_{n-r})q_u \\ &= 2(-1)^{n+1}[(-1)^{r+1}(Le_{r-1} + 1)][QLe_{r-1} - 3(2F_{r+1} - F_r)\mathbf{k} + q_u] \\ &\quad + q_u(QLe_n - QLe_{n+r}) + (QLe_n - QLe_{n-r})q_u \end{aligned}$$

□

Corollary 3.11 (d’Ocagne’s identity). For $m, n \in \mathbb{Z}$,

$$\begin{aligned} &QL_{e_{n+1}}QL_{e_m} - QL_{e_n}QL_{e_{m+1}} \\ &= 4(-1)^{n+1}[QL_{e_{m-n-1}} - 3(2F_{m-n+1} - F_{m-n})\mathbf{k} + q_u] \\ &\quad + q_u(QL_{e_{m+1}} - QL_{e_m}) + (QL_{e_n} - QL_{e_{n+1}})q_u. \end{aligned} \tag{3.21}$$

Proof. Taking $r = m - n$ and $k = 1$ in (3.19) leads to (3.21), since

$$\begin{aligned} &QL_{e_{n+1}}QL_{e_m} - QL_{e_n}QL_{e_{m+1}} \\ &= 2(-1)^{n+1}[Le_0 + 1][QL_{e_{m-n-1}} - 3(2F_{m-n+1} - F_{m-n})\mathbf{k} + q_u] \\ &\quad + q_u(QL_{e_{m+1}} - QL_{e_m}) + (QL_{e_n} - QL_{e_{n+1}})q_u. \end{aligned}$$

□

Corollary 3.12 (Cassini’s identity). For $n \in \mathbb{Z}$,

$$QL_{e_{n-1}}QL_{e_{n+1}} - QL_{e_n}^2 = 4(-1)^{n+1}(QL_{e_0} - 3\mathbf{k} + q_u) - q_uQL_{e_{n-1}} + QL_{e_{n-2}}q_u, \tag{3.22}$$

Proof. With $r = -k = 1$ in (3.19), we get

$$\begin{aligned} &QL_{e_{n-1}}QL_{e_{n+1}} - QL_{e_n}^2 \\ &= 2(-1)^{n+1}(Le_{-2} + 1)[QL_{e_0} - 3(2F_2 - F_1)\mathbf{k} + q_u] \\ &\quad + q_u(QL_{e_n} - QL_{e_{n+1}}) + (QL_{e_n} - QL_{e_{n-1}})q_u \end{aligned}$$

Hence, applying (3.7) of Proposition 3.4, (3.22) holds. □

Partially following the ideas of Daşdemir in [24], where he presented Gelin-Cesàro’s identity for Fibonacci quaternions and Gelin-Cesàro’s identity for Lucas quaternions, Gelin-Cesàro’s identity for Leonardo quaternions is deduced in the following theorem which is preceded by auxiliary results.

Lemma 3.13. Let $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. For $s, n \in \mathbb{Z}$, the following equalities hold:

(a)

$$\begin{aligned} \widetilde{QL_{e_s}} \times \widetilde{QL_{e_{n+2}}} &= (Le_{s+2}Le_{n+5} - Le_{s+3}Le_{n+4})\mathbf{i} \\ &\quad - (Le_{s+1}Le_{n+5} - Le_{s+3}Le_{n+3})\mathbf{j} \\ &\quad + (Le_{s+1}Le_{n+4} - Le_{s+2}Le_{n+3})\mathbf{k}; \end{aligned}$$

(b)

$$(\widetilde{QL_{e_{-2}}} + 3\mathbf{k} + \widetilde{q_u}) \times \widetilde{QL_{e_{n+2}}} = (2Le_{n+5} - 5Le_{n+4})\mathbf{i} + 5Le_{n+3}\mathbf{j} - 2Le_{n+3}\mathbf{k};$$

(c)

$$\widetilde{q_u} \times \widetilde{QL_{e_s}} = (Le_{s+1} + 1)\mathbf{i} - (Le_{s+2} + 1)\mathbf{j} + (Le_s + 1)\mathbf{k};$$

(d)

$$\begin{aligned} &[(Le_{n+1} + 1)\mathbf{i} - (Le_{n+2} + 1)\mathbf{j} + (Le_n + 1)\mathbf{k}] \times \widetilde{QL_{e_{n+2}}} \\ &= [-(Le_{n+2} + 1)Le_{n+5} - (Le_n + 1)Le_{n+4}]\mathbf{i} \\ &\quad - [(Le_{n+1} + 1)Le_{n+5} - (Le_n + 1)Le_{n+3}]\mathbf{j} \\ &\quad + [(Le_{n+1} + 1)Le_{n+4} + (Le_{n+2} + 1)Le_{n+3}]\mathbf{k}. \end{aligned}$$

Proof. The equalities in (a), (b) and (d) come straightforward from the vector cross product in each statement. For (b), observe also that

$$\widetilde{QL_{e_{-2}}} + 3\mathbf{k} + \widetilde{q_u} = 2\mathbf{j} + 5\mathbf{k}.$$

As far as (c), notice that the associated vector cross product gives $(Le_{s+3} - Le_{s+2})\mathbf{i} - (Le_{s+3} - Le_{s+1})\mathbf{j} + (Le_{s+2} - Le_{s+1})\mathbf{k}$. □

Lemma 3.14. For $k \in \mathbb{Z}$, let $x_k = Le_k + 1$. For $n \in \mathbb{Z}$, the following equalities hold:

(a)

$$Le_{n+2}Le_{n+5} - Le_{n+3}Le_{n+4} = -x_{n+2} + 4(-1)^n;$$

$$\begin{aligned}
(b) \quad & Le_{n+1}Le_{n+5} - Le_{n+3}^2 = x_{n+2} - x_{n+4} + 4(-1)^{n+1}; \\
(c) \quad & Le_{n+1}Le_{n+4} - Le_{n+2}Le_{n+3} = -x_{n+1} + 4(-1)^{n+3}; \\
(d) \quad & 2Le_{n+5} - 5Le_{n+4} = -3(x_{n+4} - 1) + 2x_{n+3}; \\
(e) \quad & Le_{n-1}Le_{n+5} - Le_nLe_{n+4} = -3x_{n+1} + x_n + 20(-1)^{n+1}; \\
(f) \quad & Le_{n-2}Le_{n+5} - Le_nLe_{n+3} = -2x_{n+2} - x_n + 20(-1)^n; \\
(g) \quad & Le_{n-2}Le_{n+4} - Le_{n-1}Le_{n+3} = -x_{n+2} + x_{n-3} + 20(-1)^n.
\end{aligned}$$

Proof. Applying Tagiuri-Vajda's identity (2.29) in Theorem 2.21, with $k = 1$ and $r = 2$, leads to

$$\begin{aligned}
& Le_{n+2}Le_{n+5} - Le_{n+3}Le_{n+4} \\
&= (-1)^{n+4}(Le_0 + 1) \left[\frac{1}{2}(-1)^{n+3}(Le_{n+3} - Le_{n+1}) + Le_1 + 1 \right] \\
&\quad - \frac{1}{2}(Le_1 - 1)(Le_{n+4} - Le_{n+2}) \\
&= (-1)^{2n+7}(Le_{n+2} + 1) + 4(-1)^{n+4},
\end{aligned}$$

and, as easily seen, the latter expression is equal to the right-hand side of the equality in (a).

Using Catalan's identity (2.30) in Corollary 2.22, with $r = 2$, allows us to arrive at (b):

$$\begin{aligned}
Le_{n+1}Le_{n+5} - Le_{n+3}^2 &= -Le_{n+1} - Le_{n+5} + 2Le_{n+3} + (-1)^{n+1}(Le_1 + 1)^2 \\
&= -Le_{n+1} - Le_{n+4} + Le_{n+3} - 1 + 4(-1)^{n+1} \\
&= Le_{n+2} + 1 - (Le_{n+4} + 1) + 4(-1)^{n+1} \\
&= x_{n+2} - x_{n+4} + 4(-1)^{n+1}.
\end{aligned}$$

As far as (c), once again invoking Tagiuri-Vajda's identity (2.29) in Theorem 2.21, with $k = 1$ and $r = 2$, observe that

$$\begin{aligned}
& Le_{n+1}Le_{n+4} - Le_{n+2}Le_{n+3} \\
&= -(Le_{n+2}Le_{n+3} - Le_{n+1}Le_{n+4}) \\
&= (-1)^{n+3}(Le_0 + 1) \left[\frac{1}{2}(-1)^{n+2}(Le_{n+2} - Le_n) + Le_1 + 1 \right] \\
&\quad - \frac{1}{2}(Le_1 - 1)(Le_{n+3} - Le_{n+1}) \\
&= (-1)^{2n+5}(Le_{n+2} - Le_n) + 4(-1)^{n+3} \\
&= (-1)^{2n+5}(Le_{n+1} + 1) + 4(-1)^{n+3}.
\end{aligned}$$

For (d), notice that

$$\begin{aligned}
2Le_{n+5} - 5Le_{n+4} &= -3Le_{n+4} + 2(Le_{n+3} + 1) \\
&= -3(x_{n+4} - 1) + 2x_{n+3}.
\end{aligned}$$

Tagiuri-Vajda's identity (2.29) in Theorem 2.21, with $k = -1$ and $r = 5$, allows to write

$$\begin{aligned}
& Le_{n-1}Le_{n+5} - Le_nLe_{n+4} \\
&= (-1)^{n+1}(Le_{-2} + 1) \left[\frac{1}{2}(-1)^{n+1}(Le_{n+4} - Le_{n-1}) + Le_4 + 1 \right] \\
&\quad + \frac{1}{2}(Le_{-1} - 1)(Le_{n+5} - Le_n) \\
&= Le_{n+4} - Le_{n-1} + 20(-1)^{n+1} + Le_n - Le_{n+5} \\
&= -3Le_{n+1} + Le_n - 2 + 20(-1)^{n+1},
\end{aligned}$$

and (e) follows.

In what concerns (f), Tagiuri-Vajda's identity (2.29) in Theorem 2.21, with $k = 5$ and $r = 2$, implies

$$\begin{aligned} & Le_{n-2}Le_{n+5} - Le_nLe_{n+3} \\ &= -(Le_{n+3}Le_n - Le_{n-2}Le_{n+5}) \\ &= -\left\{(-1)^{n-1}(Le_4 + 1) \left[\frac{1}{2}(-1)^{n-1}(Le_{n-1} - Le_{n-3} + Le_1 + 1) \right] \right. \\ &\quad \left. + \frac{1}{2}(Le_5 - 1)(Le_n - Le_{n-2}) \right\} \\ &= -5Le_{n-1} + 5Le_{n-3} + 20(-1)^n - 7Le_n + 7Le_{n-2} \\ &= 2Le_{n-2} - 7Le_n - 5 + 20(-1)^n \\ &= -2Le_{n+1} - 3Le_n - 5 + 20(-1)^n \\ &= -2Le_{n+2} - Le_n - 3 + 20(-1)^n, \end{aligned}$$

leading to the result.

Finally, as follows, d'Ocagne's identity (2.31) in Corollary 2.23 is applied in order to arrive at (g).

$$\begin{aligned} Le_{n-2}Le_{n+4} - Le_{n-1}Le_{n+3} &= -(Le_{n-1}Le_{n+3} - Le_{n-2}Le_{n+4}) \\ &= -[2(-1)^{n-1}(Le_4 + 1) + Le_{n+2} - Le_{n-3}] \\ &= 20(-1)^n - Le_{n+2} + Le_{n-3} \end{aligned}$$

□

Theorem 3.15 (Gelin-Cesàro's identity). *Let $a = 2 + 2\mathbf{j} + 5\mathbf{k}$ and $b = -2 + 2\mathbf{i} - 7\mathbf{k}$. For $k \in \mathbb{Z}$, let $x_k = Le_k + 1$. For $n \in \mathbb{Z}$, let*

$$\begin{aligned} A_n &= 2[(Le_{n-1} + 1)\mathbf{i} - (Le_n + 1)\mathbf{j} + (Le_{n-2} + 1)\mathbf{k}] - QLe_{n-3}q_u - q_u^2, \\ B_n &= 2[(Le_n + 1)\mathbf{i} - (Le_{n+1} + 1)\mathbf{j} + (Le_{n-1} + 1)\mathbf{k}] - QLe_nq_u - q_u^2. \end{aligned}$$

For $n \in \mathbb{Z}$,

$$QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2} - QLe_n^4 = C_n - D_n, \tag{3.23}$$

where

$$C_n = (4(-1)^{n-1}a + A_n)(QLe_n^2 + B_n) + QLe_n^2(4(-1)^{n-1}b + B_n) + (4(-1)^{n-1}A_n + 16a)b$$

and

$$\begin{aligned} D_n &= \{4(x_n - 1)[(-x_{n+2} + 4(-1)^n)\mathbf{i} - (x_{n+2} - x_{n+4} + 4(-1)^{n+1})\mathbf{j} \\ &\quad + (-x_{n+1} + 4(-1)^{n+3})\mathbf{k}] \\ &\quad + 8(-1)^{n-1}[(-3(x_{n+4} - 1) + 2x_{n+3})\mathbf{i} + 5(x_{n+3} - 1)\mathbf{j} - 2(x_{n+3} - 1)\mathbf{k}] \\ &\quad - 2[(-3x_{n+1} + x_n + 20(-1)^{n+1})\mathbf{i} \\ &\quad - (-2x_{n+2} - x_n + 20(-1)^n)\mathbf{j} + (-x_{n+2} + x_{n-3} + 20(-1)^n)\mathbf{k}] \\ &\quad - 2(x_{n-3} + 1)[x_{n+3}\mathbf{i} - x_{n+4}\mathbf{j} + x_{n+2}\mathbf{k}] \\ &\quad + 2[(-x_{n+2}(x_{n+5} - 1) - x_n(x_{n+4} - 1))\mathbf{i} - (x_{n+1}(x_{n+5} - 1) - x_n(x_{n+3} - 1))\mathbf{j} \\ &\quad + (x_{n+1}(x_{n+4} - 1) + x_{n+2}(x_{n+3} - 1))\mathbf{k}]\}QLe_{n-2}. \end{aligned}$$

Proof. In order to arrive at (3.23), we develop $QLe_{n+1}QLe_{n-1}QLe_{n+2}QLe_{n-2}$ in two different ways. On the one hand, Catalan's identity (3.20) for Leonardo quaternions with $r = -1$ and $r = -2$ leads, respectively, to

$$\begin{aligned} & QLe_{n+1}QLe_{n-1} \\ &= QLe_n^2 + 2(-1)^{n-1}(Le_{-2} + 1)(QLe_{-2} - 3(2F_0 - F_{-1})\mathbf{k} + q_u) \\ &\quad + q_u(QLe_n - QLe_{n-1}) + (QLe_n - QLe_{n+1})q_u \\ &= QLe_n^2 + 4(-1)^{n-1}(QLe_{-2} + 3\mathbf{k} + q_u) + q_uQLe_{n-2} - QLe_{n-1}q_u \end{aligned}$$

and

$$\begin{aligned} & QLe_{n+2}QLe_{n-2} \\ &= QLe_n^2 + 2(-1)^{n-2}(Le_{-3} + 1)(QLe_{-3} - 3(2F_{-1} - F_{-2})\mathbf{k} + q_u) \\ &\quad + q_u(QLe_n - QLe_{n-2}) + (QLe_n - QLe_{n+2})q_u \\ &= QLe_n^2 + 4(-1)^{n-1}(QLe_{-3} - 9\mathbf{k} + q_u) + q_uQLe_{n-1} - QLe_{n+1}q_u. \end{aligned}$$

From these equalities, also applying (3.10) in Proposition 3.4,

$$\begin{aligned}
 & QLe_{n+1}QLe_{n-1}QLe_{n+2}QLe_{n-2} \\
 &= QLe_n^4 + 4(-1)^{n-1}[(QLe_{-2} + 3\mathbf{k} + q_u)(QLe_n^2 + q_uQLe_{n-1} - QLe_{n+1}q_u) \\
 &\quad + (QLe_n^2 + q_uQLe_{n-2} - QLe_{n-1}q_u)(QLe_{-3} - 9\mathbf{k} + q_u)] \\
 &\quad + QLe_n^2(q_uQLe_{n-1} - QLe_{n+1}q_u) + (q_uQLe_{n-2} - QLe_{n-1}q_u)QLe_n^2 \\
 &\quad + (QLe_{n-1}q_u - q_uQLe_{n-2})(QLe_{n+1}q_u - q_uQLe_{n-1}) \\
 &\quad + 16(QLe_{-2} + 3\mathbf{k} + q_u)(QLe_{-3} - 9\mathbf{k} + q_u) \\
 &= QLe_n^4 + 4(-1)^{n-1}[a(QLe_n^2 + q_uQLe_{n-1} - QLe_{n+1}q_u) \\
 &\quad + (QLe_n^2 + q_uQLe_{n-2} - QLe_{n-1}q_u)b] \\
 &\quad + QLe_n^2(q_uQLe_{n-1} - QLe_{n+1}q_u) + (q_uQLe_{n-2} - QLe_{n-1}q_u)QLe_n^2 \\
 &\quad + (QLe_{n-1}q_u - q_uQLe_{n-2})(QLe_{n+1}q_u - q_uQLe_{n-1}) + 16ab \\
 &= QLe_n^4 + 4(-1)^{n-1}[a(QLe_n^2 + q_uQLe_{n-1} - QLe_{n+1}q_u) \\
 &\quad + (QLe_n^2 + q_uQLe_{n-2} - QLe_{n-1}q_u)b] \\
 &\quad + (QLe_n^2 + q_uQLe_{n-2} - QLe_{n-1}q_u)(q_uQLe_{n-1} - QLe_{n+1}q_u) \\
 &\quad + (q_uQLe_{n-2} - QLe_{n-1}q_u)QLe_n^2 + 16ab \\
 &= QLe_n^4 + (4(-1)^{n-1}a + A_n)(QLe_n^2 + B_n) + QLe_n^2(4(-1)^{n-1}b + B_n) \\
 &\quad + (4(-1)^{n-1}A_n + 16a)b \\
 &= QLe_n^4 + C_n.
 \end{aligned}$$

where $a = QLe_{-2} + 3\mathbf{k} + q_u = 2 + 2\mathbf{j} + 5\mathbf{k}$, $b = QLe_{-3} - 9\mathbf{k} + q_u = -2 + 2\mathbf{i} - 7\mathbf{k}$ and, by (3.10) in Proposition 3.4,

$$\begin{aligned}
 A_n &= q_uQLe_{n-2} - QLe_{n-1}q_u \\
 &= q_uQLe_{n-2} - QLe_{n-2}q_u - QLe_{n-3}q_u - q_u^2 \\
 &= 2[(Le_{n-1} + 1)\mathbf{i} - (Le_n + 1)\mathbf{j} + (Le_{n-2} + 1)\mathbf{k}] - QLe_{n-3}q_u - q_u^2
 \end{aligned}$$

and

$$\begin{aligned}
 B_n &= q_uQLe_{n-1} - QLe_{n+1}q_u \\
 &= q_uQLe_{n-1} - QLe_nq_u - QLe_{n-1}q_u - q_u^2 \\
 &= 2[(Le_n + 1)\mathbf{i} - (Le_{n+1} + 1)\mathbf{j} + (Le_{n-1} + 1)\mathbf{k}] - QLe_nq_u - q_u^2.
 \end{aligned}$$

On the other hand, invoking (2.4) in Proposition 2.2,

$$\begin{aligned}
 QLe_{n+1}QLe_{n-1}QLe_{n+2}QLe_{n-2} &= \overbrace{QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2}} \\
 &\quad + 2(\overbrace{QLe_{n+1}QLe_{n-1}} \times \overbrace{QLe_{n+2}})QLe_{n-2}.
 \end{aligned}$$

By Theorem 2.1,

$$\begin{aligned}
 & \overbrace{QLe_{n+1}QLe_{n-1}} \\
 &= 2Le_n\overbrace{QLe_n} + 4(-1)^{n-1}(\overbrace{QLe_{-2} + 3\mathbf{k} + \widetilde{q}_u} + \overbrace{q_uQLe_{n-2}} - \overbrace{QLe_{n-1}q_u}) \\
 &= 2Le_n\overbrace{QLe_n} + 4(-1)^{n-1}(\overbrace{QLe_{-2} + 3\mathbf{k} + \widetilde{q}_u}) \\
 &\quad + \overbrace{QLe_{n-2}} + Le_{n-2}\widetilde{q}_u + \widetilde{q}_u \times \overbrace{QLe_{n-2}} - Le_{n-1}\widetilde{q}_u - \overbrace{QLe_{n-1}} - \overbrace{QLe_{n-1}} \times \widetilde{q}_u \\
 &= 2Le_n\overbrace{QLe_n} + 4(-1)^{n-1}(\overbrace{QLe_{-2} + 3\mathbf{k} + \widetilde{q}_u}) \\
 &\quad - (\overbrace{QLe_{n-3} + \widetilde{q}_u}) - (Le_{n-3} + 1)\widetilde{q}_u + \widetilde{q}_u \times (\overbrace{QLe_n} - \widetilde{q}_u) \\
 &= 2Le_n\overbrace{QLe_n} + 4(-1)^{n-1}(\overbrace{QLe_{-2} + 3\mathbf{k} + \widetilde{q}_u}) - \overbrace{QLe_{n-3}} - (Le_{n-3} + 2)\widetilde{q}_u \\
 &\quad + \widetilde{q}_u \times \overbrace{QLe_n} \\
 &= 2Le_n\overbrace{QLe_n} + 4(-1)^{n-1}(\overbrace{QLe_{-2} + 3\mathbf{k} + \widetilde{q}_u}) - \overbrace{QLe_{n-3}} - (Le_{n-3} + 2)\widetilde{q}_u \\
 &\quad + (Le_{n+1} + 1)\mathbf{i} - (Le_{n+2} + 1)\mathbf{j} + (Le_n + 1)\mathbf{k}.
 \end{aligned}$$

From this equality, invoking Lemma 3.13 and Lemma 3.14, and denoting $Le_k + 1$, for $k \in \mathbb{Z}$, by x_k ,

$$\begin{aligned}
 & QLe_{n+1}QLe_{n-1}QLe_{n+2}QLe_{n-2} \\
 &= QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2} + \left\{ 4Le_n(\widetilde{QLe_n} \times \widetilde{QLe_{n+2}}) \right. \\
 &\quad + 8(-1)^{n-1}[(\widetilde{QLe_{-2}} + 3\mathbf{k} + \widetilde{q_u}) \times \widetilde{QLe_{n+2}}] \\
 &\quad - 2(\widetilde{QLe_{n-3}} \times \widetilde{QLe_{n+2}}) - 2(Le_{n-3} + 2)(\widetilde{q_u} \times \widetilde{QLe_{n+2}}) \\
 &\quad \left. + 2[(Le_{n+1} + 1)\mathbf{i} - (Le_{n+2} + 1)\mathbf{j} + (Le_n + 1)\mathbf{k}] \times \widetilde{QLe_{n+2}} \right\} QLe_{n-2} \\
 &= QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2} \\
 &\quad + \left\{ 4Le_n[(Le_{n+2}Le_{n+5} - Le_{n+3}Le_{n+4})\mathbf{i} - (Le_{n+1}Le_{n+5} - Le_{n+3}^2)\mathbf{j} \right. \\
 &\quad + (Le_{n+1}Le_{n+4} - Le_{n+2}Le_{n+3})\mathbf{k}] \\
 &\quad + 8(-1)^{n-1}[(2Le_{n+5} - 5Le_{n+4})\mathbf{i} + 5Le_{n+3}\mathbf{j} - 2Le_{n+3}\mathbf{k}] \\
 &\quad - 2[(Le_{n-1}Le_{n+5} - Le_nLe_{n+4})\mathbf{i} - (Le_{n-2}Le_{n+5} - Le_nLe_{n+3})\mathbf{j} \\
 &\quad + (Le_{n-2}Le_{n+4} - Le_{n-1}Le_{n+3})\mathbf{k}] \\
 &\quad - 2(Le_{n-3} + 2)[(Le_{n+3} + 1)\mathbf{i} - (Le_{n+4} + 1)\mathbf{j} + (Le_{n+2} + 1)\mathbf{k}] \\
 &\quad + 2[-(Le_{n+2} + 1)Le_{n+5} - (Le_n + 1)Le_{n+4}]\mathbf{i} \\
 &\quad - ((Le_{n+1} + 1)Le_{n+5} - (Le_n + 1)Le_{n+3})\mathbf{j} \\
 &\quad \left. + ((Le_{n+1} + 1)Le_{n+4} + (Le_{n+2} + 1)Le_{n+3})\mathbf{k} \right\} QLe_{n-2} \\
 &= QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2} \\
 &\quad + \left\{ 4(x_n - 1)[(-x_{n+2} + 4(-1)^n)\mathbf{i} - (x_{n+2} - x_{n+4} + 4(-1)^{n+1})\mathbf{j} \right. \\
 &\quad + (-x_{n+1} + 4(-1)^{n+3})\mathbf{k}] \\
 &\quad + 8(-1)^{n-1}[(-3(x_{n+4} - 1) + 2x_{n+3})\mathbf{i} + 5(x_{n+3} - 1)\mathbf{j} - 2(x_{n+3} - 1)\mathbf{k}] \\
 &\quad - 2[(-3x_{n+1} + x_n + 20(-1)^{n+1})\mathbf{i} - (-2x_{n+2} - x_n + 20(-1)^n)\mathbf{j} \\
 &\quad + (-x_{n+2} + x_{n-3} + 20(-1)^n)\mathbf{k}] \\
 &\quad - 2(x_{n-3} + 1)[x_{n+3}\mathbf{i} - x_{n+4}\mathbf{j} + x_{n+2}\mathbf{k}] \\
 &\quad + 2[(-x_{n+2}(x_{n+5} - 1) - x_n(x_{n+4} - 1))\mathbf{i} \\
 &\quad - (x_{n+1}(x_{n+5} - 1) - x_n(x_{n+3} - 1))\mathbf{j} \\
 &\quad \left. + (x_{n+1}(x_{n+4} - 1) + x_{n+2}(x_{n+3} - 1))\mathbf{k} \right\} QLe_{n-2} \\
 &= QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2} + D_n.
 \end{aligned}$$

Therefore, $QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2} - QLe_n^4 = C_n - D_n$. □

3.4. Generating functions

In the first result, the generating function of the Leonardo quaternions sequence $\{QLe_n\}_{n=0}^\infty$ is presented.

Theorem 3.16. *The generating function of $\{QLe_n\}_{n=0}^\infty$ is given by*

$$G(t) = \frac{1 + \mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + (-1 + \mathbf{i} - \mathbf{j} - \mathbf{k})t + (1 - \mathbf{i} - \mathbf{j} - 3\mathbf{k})t^2}{1 - 2t + t^3}. \quad (3.24)$$

Proof. Let $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. As $G(t) - G(t)t - G(t)t^2 = QLe_0(1-t) + QLe_1t + q_u \sum_{n=0}^\infty t^n - q_u - q_ut$, where $\sum_{n=0}^\infty t^n$ is a formal power series in t whose closed form is $\frac{1}{1-t}$, then

$$G(t) = \frac{QLe_0(1-t) + QLe_1t + q_u \frac{t^2}{1-t}}{1-t-t^2}.$$

The development of the right hand side of this equality leads to (3.24). □

In the subsequent result, for each $m \in \mathbb{N}$, the generating function of the sequence $\{QLe_{m+n}\}_{n=0}^\infty$ is obtained.

Theorem 3.17. Let $m \in \mathbb{N}$. The generating function of $\{QLe_{m+n}\}_{n=0}^\infty$ is given by

$$\sum_{n=0}^\infty QLe_{m+n}t^n = \frac{QLe_m + (QLe_{m-1} - QLe_m + q_u)t - QLe_{m-1}t^2}{1 - 2t + t^3}, \tag{3.25}$$

where $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Proof. Applying Binet’s formula (3.4) for the Leonardo quaternions in Theorem 3.2, we obtain

$$\begin{aligned} \sum_{n=0}^\infty QLe_{m+n}t^n &= \sum_{n=0}^\infty \left(2 \frac{\phi^{m+n+1}\widehat{\phi} - \psi^{m+n+1}\widehat{\psi}}{\phi - \psi} - q_u \right) t^n \\ &= \frac{2}{\phi - \psi} \sum_{n=0}^\infty \left(\phi^{m+1}\phi^n t^n \widehat{\phi} - \psi^{m+1}\psi^n t^n \widehat{\psi} \right) - q_u \sum_{n=0}^\infty t^n \\ &= \frac{2}{\phi - \psi} \left(\phi^{m+1} \frac{1}{1 - \phi t} \widehat{\phi} - \psi^{m+1} \frac{1}{1 - \psi t} \widehat{\psi} \right) - q_u \frac{1}{1 - t} \\ &= \frac{2}{\phi - \psi} \frac{\phi^{m+1}\widehat{\phi} - \psi^{m+1}\widehat{\psi} + (\phi\psi\psi^m\widehat{\psi} - \phi^m\widehat{\phi}\phi\psi)t}{(1 - \phi t)(1 - \psi t)} - q_u \frac{1}{1 - t} \\ &= \frac{2 \frac{\phi^{m+1}\widehat{\phi} - \psi^{m+1}\widehat{\psi}}{\phi - \psi} - q_u + \left(2 \frac{\phi^m\widehat{\phi} - \psi^m\widehat{\psi}}{\phi - \psi} - q_u \right) t}{1 - t - t^2} \\ &= \frac{1 + t}{1 - t - t^2} + q_u \frac{1}{1 - t} - q_u \frac{1}{1 - t} \\ &= \frac{QLe_m + QLe_{m-1}t}{1 - t - t^2} + q_u \frac{1 + t}{1 - t - t^2} - q_u \frac{1}{1 - t} \end{aligned}$$

which allows us to arrive at (3.25). □

The exponential generating function that generates the numbers $\frac{QLe_n}{n!}$ is presented in the last theorem of the section. As a consequence, the corresponding Poisson generating function is obtained.

Theorem 3.18. The exponential generating function of $\{QLe_n\}_{n=0}^\infty$ is given by

$$G_E(t) = 2 \frac{\phi\widehat{\phi}e^{\phi t} - \psi\widehat{\psi}e^{\psi t}}{\phi - \psi} - q_u e^t, \tag{3.26}$$

where $\phi = \frac{1 + \sqrt{5}}{2}$, $\psi = \frac{1 - \sqrt{5}}{2}$, $\widehat{\phi} = 1 + \phi\mathbf{i} + (1 + \phi)\mathbf{j} + (1 + 2\phi)\mathbf{k}$, $\widehat{\psi} = 1 + \psi\mathbf{i} + (1 + \psi)\mathbf{j} + (1 + 2\psi)\mathbf{k}$ and $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Proof. Since $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$, then

$$\begin{aligned} G_E(t) &= \sum_{n=0}^\infty QLe_n \frac{t^n}{n!} \\ &= \sum_{n=0}^\infty \left(2 \frac{\phi^{n+1}\widehat{\phi} - \psi^{n+1}\widehat{\psi}}{\phi - \psi} - q_u \right) \frac{t^n}{n!} \\ &= \frac{2}{\phi - \psi} \left(\phi\widehat{\phi} \sum_{n=0}^\infty \frac{(\phi t)^n}{n!} - \psi\widehat{\psi} \sum_{n=0}^\infty \frac{(\psi t)^n}{n!} \right) - q_u \sum_{n=0}^\infty \frac{t^n}{n!} \\ &= \frac{2}{\phi - \psi} \left(\phi\widehat{\phi}e^{\phi t} - \psi\widehat{\psi}e^{\psi t} \right) - q_u e^t, \end{aligned}$$

which leads to (3.26). □

Corollary 3.19. *The Poisson generating function of $\{QLe_n\}_{n=0}^\infty$ is given by*

$$G_P(t) = 2 \frac{\phi \hat{\phi} e^{\phi t} - \psi \hat{\psi} e^{\psi t}}{(\phi - \psi) e^t} - q_u, \quad (3.27)$$

where $\phi = \frac{1 + \sqrt{5}}{2}$, $\psi = \frac{1 - \sqrt{5}}{2}$, $\hat{\phi} = 1 + \phi \mathbf{i} + (1 + \phi) \mathbf{j} + (1 + 2\phi) \mathbf{k}$, $\hat{\psi} = 1 + \psi \mathbf{i} + (1 + \psi) \mathbf{j} + (1 + 2\psi) \mathbf{k}$ and $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Proof. As $G_P(t) = e^{-t} G_E(t)$, then, invoking (3.26), (3.27) follows. \square

Acknowledgment. P. D. Beites was supported by FCT (Fundação para a Ciência e a Tecnologia, Portugal), project UIDB/00212/2020 of CMA-UBI (Centro de Matemática e Aplicações da Universidade da Beira Interior, Portugal), and by project MTM2017-83506-C2-2-P (Spain). The author P. Catarino was supported by FCT, projects UIDB/00013/2020, UIDP/00013/2020 and UID/CED/00194/2020.

References

- [1] B. Aloui and A. Boussayoud, *Generating functions of the product of the k -Fibonacci and k -Pell numbers and Chebyshev polynomials of the third and fourth kind*, Math. Eng. Sci. Aerosp. **12**, 245-257, 2021.
- [2] F. Alves and R. Vieira, *The Newton fractal's Leonardo sequence study with the Google Colab*, Int. Electron. J. Math. Educ. **15**, em0575, 2020.
- [3] M. Akyiğit, H.H. Kösal and M. Tosun, *Split Fibonacci quaternions*, Adv. Appl. Clifford Algebras **23**, 535-545, 2013.
- [4] M. Akyiğit, H.H. Kösal and M. Tosun, *Fibonacci generalized quaternions*, Adv. Appl. Clifford Algebras **24**, 631-641, 2014.
- [5] Y. Alp and E.G. Koçer, *Some properties of Leonardo numbers*, Konuralp J. Math. **9**, 183-189, 2021.
- [6] P.D. Beites and A.P. Nicolás, *An associative triple system of the second kind*, Commun. Algebra **44**, 5027-5043, 2016.
- [7] P.D. Beites and A.P. Nicolás, *A note on standard composition algebras of types II and III*, Adv. Appl. Clifford Algebras **27**, 955-964, 2017.
- [8] P.D. Beites, A.P. Nicolás, P. Saraiva and J. Vitória, *Vector cross product differential and difference equations in \mathbb{R}^3 and in \mathbb{R}^7* , Electron. J. Linear Algebra **34**, 675-686, 2018.
- [9] G. Bilgici and P. Catarino, *Unrestricted pell and pell-lucas quaternions*, Int. J. Math. Syst. Sci. **1**, Article 816, 2018.
- [10] G. Bilgici, U. Tokeşer and Z. Ünal, *k -Fibonacci and k -Lucas generalized quaternions*, Konuralp J. Math. **5**, 102-113, 2017.
- [11] T.R. Blackman and S. Lemurell, *Spectral correspondences for Maass waveforms on quaternion groups*, J. Number Theory **158**, 1-22, 2016.
- [12] I. Cação, H.R. Malonek and G. Tomaz, *Shifted Generalized Pascal Matrices in the Context of Clifford Algebra-Valued Polynomial Sequences*, in: Computational Science and its Applications, Lecture Notes in Computer Science **10405**, Springer, 2017.
- [13] N.D. Cahill, J.R. D'Errico, D.A. Narayan and J.Y. Narayan, *Fibonacci determinants*, Coll. Math. J. **33**, 221-225, 2002.
- [14] P. Catarino, *The modified Pell and the modified k -Pell quaternions and octonions*, Adv. Appl. Clifford Algebras **26**, 577-590, 2016.
- [15] P. Catarino and A. Borges, *On Leonardo numbers*, Acta Math. Univ. Comen. **89**, 75-86, 2020.
- [16] P. Catarino and A. Borges, *A note on incomplete Leonardo numbers*, Integers **20**, article A43, 2020.

- [17] P. Catarino and H. Campos, *Incomplete k -Pell, k -Pell-Lucas and modified k -Pell numbers*, Hacet. J. Math. Stat. **46**, 361-372, 2017.
- [18] P. Catarino and R. de Almeida, *On a quaternionic sequence with Vietoris' numbers*, Filomat **35**, 1065-1086, 2021.
- [19] P. Catarino and R. de Almeida, *A note on Vietoris' number sequence*, Mediterr. J. Math. **19**, 1-19, 2022.
- [20] Y. Choo, *A generalized quaternion with generalized Fibonacci number components*, Appl. Math. Sci. **14**, 31-38, 2020.
- [21] C.B. Çimen and A. İpek, *On Pell quaternions and Pell-Lucas quaternions*, Adv. Appl. Clifford Algebras **26**, 39-51, 2016.
- [22] J.H. Conway and D.A. Smith, *On Quaternions and Octonions: their Geometry, Arithmetic and Symmetry*, A K Peters/CRC Press, New York, 2003.
- [23] N. Correia and R. Pacheco, *Harmonic maps of finite uniton number and their canonical elements*, Ann. Glob. Anal. Geom. **47**, 335-358, 2015.
- [24] A. Daşdemir, *Gelin-Cesàro identities for Fibonacci and Lucas quaternions*, Ann. Univ. Paedagog. Crac. Stud. Math. **XVIII**, 137-144, 2019.
- [25] E.W. Dijkstra, *Smoothsort, an alternative for sorting in situ*, Sci. Comput. Program. **1**, 223-233, 1982.
- [26] G.B. Djordjević and H.M. Srivastava, *Some generalizations of certain sequences associated with the Fibonacci numbers*, J. Indones. Math. Soc. **12**, 99-112, 2006.
- [27] M.I. Falcão, F. Miranda, R. Severino and M.J. Soares, *Evaluation schemes in the ring of quaternionic polynomials*, BIT Numer. Math. **58**, 51-72, 2018.
- [28] N. Jacobson, *Composition algebras and their automorphisms*, Rend. Circ. Mat. Palermo **7**, 55-80, 1958.
- [29] S. Halici, *On Fibonacci quaternions*, Adv. Appl. Clifford Algebras **22**, 321-327, 2012.
- [30] S. Halici and G. Cerda-Morales, *On Quaternion-Gaussian Fibonacci numbers and their properties*, An. St. Univ. Ovidius Constanta, Ser. Mat. **29**, 71-82, 2021.
- [31] S. Halici and A. Karataş, *On a generalization for Fibonacci quaternions*, Chaos Soliton Fract. **98**, 178-182, 2017.
- [32] A.F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, Am. Math. Mon. **70**, 289-291, 1963.
- [33] A. İpek, *On (p, q) -Fibonacci quaternions and their Binet formulas, generating functions and certain binomial sums*, Adv. Appl. Clifford Algebras **27**, 1343-1351, 2017.
- [34] E. Kilic, D. Tasci and P. Haukkanen, *On the generalized Lucas sequences by Hessenberg matrices*, Ars Combinatoria **95**, 383-395, 2010.
- [35] D. Knuth, *The art of computer programming*, Addison Wesley Longman, 1997.
- [36] T. Koshy, *Fibonacci and Lucas numbers with applications*, Wiley, 2018.
- [37] F.S. Leite, *The geometry of hypercomplex matrices*, Linear Multilinear Algebra **34**, 123-132, 1993.
- [38] J. Morais and I. Cação, *Quaternion Zernike spherical polynomials*, Math. Comput. **84**, 1317-1337, 2015.
- [39] B.K. Patel and P.K. Ray, *On the properties of (p, q) -Fibonacci and (p, q) -Lucas quaternions*, Mathematical Reports **21**, 15-25, 2019.
- [40] E. Polatlı and S. Kesim, *A note on Catalan's identity for the k -Fibonacci quaternions*, J. Integer Seq. **18**, article 15.8.2, 2015.
- [41] E. Polatlı, C. Kizilates and S. Kesim, *On split k -Fibonacci and k -Lucas quaternions*, Adv. Appl. Clifford Algebras **26**, 353-362, 2016.
- [42] E. Polatlı and Y. Soykan, *On Generalized Third-order Jacobsthal numbers*, Asian Res. J. Math. **17**, 1-19, 2021.
- [43] J.L. Ramírez, *Some combinatorial properties of the k -Fibonacci and the k -Lucas quaternions*, An. St. Univ. Ovidius Constanta, Ser. Mat. **23**, 201-212, 2015.

- [44] N. Saba, A. Boussayoud and K. V. Kanuri, *Mersenne Lucas numbers and complete homogeneous symmetric functions*, J. Math. Comput. Sci. **24**, 127-139, 2022.
- [45] R. Serôdio, P.D. Beites and J. Vitória, *Intersection of a double cone and a line in the split-quaternions context*, Adv. Appl. Clifford Algebras **27**, 2795-2803, 2017.
- [46] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, The OEIS Foundation, <https://oeis.org>, 2021.
- [47] A. Szynal-Liana and I. Włoch, *The Pell quaternions and the Pell octonions*, Adv. Appl. Clifford Algebras **26**, 435-440, 2016.
- [48] A. Szynal-Liana and I. Włoch, *A note on Jacobsthal quaternions*, Adv. Appl. Clifford Algebras **26**, 441-447, 2016.
- [49] E. Tan and H.-H. Leung, *Some results on Horadam quaternions*, Chaos Soliton Fract. **138**, article 109961, 2020.
- [50] D. Tasci, *On k -Jacobsthal and k -Jacobsthal-Lucas quaternions*, Journal of Science and Arts **3**, 469-476, 2017.
- [51] U. Tokeşer, Z. Ünal and G. Bilgici, *Split Pell and Pell-Lucas quaternions*, Adv. Appl. Clifford Algebras **27**, 1881-1893, 2017.
- [52] S. Vajda, *Fibonacci & Lucas numbers, and the golden section*, Ellis Horwood Ltd., Chichester, England, 1989.
- [53] R. Vieira, F. Alves and P. Catarino, *Relações bidimensionais e identidades da sequência de Leonardo*, Revista Sergipana de Matemática e Educação Matemática **4**, 156-173, 2019.
- [54] R. Vieira, M. Mangueira, F. Alves and P. Catarino, *A forma matricial dos números de Leonardo*, Ciência e Natura **42**, article e100, 2020.
- [55] T. Yağmur, *Split Jacobsthal and Jacobsthal-Lucas quaternions*, Commun. Math. Appl. **10**, 429-438, 2019.