Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 20
DOI : 10.15672/hujms.xx

Research Article

Iterations and unions of star selection properties on topological spaces

Javier Casas-de la Rosa*, William Chen-Mertens, Sergio A. Garcia-Balan

Department of Mathematics and Statistics, York University, 4700 Keele St. Toronto, ON M3J 1P3 Canada

Abstract

In this paper, we investigate what selection principles properties are possessed by small (with respect to the bounding and dominating numbers) unions of spaces with certain (star) selection principles. Furthermore, we give several results about iterations of these properties and weaker properties than paracompactness. In addition, we study the behaviour of these iterated properties on Ψ-spaces. Finally, we show that, consistently, there is a normal star-Menger space that is not strongly star-Menger; this example answers a couple of questions posed in [J. Casas-de la Rosa, S. A. Garcia-Balan, P. J. Szeptycki, *Some star and strongly star selection principles*, Topology Appl. **258**, 572-587, 2019].

Mathematics Subject Classification (2020). 54D20, 54A35

Keywords. Menger, star-Menger, strongly star-Menger, Hurewicz, star-Hurewicz, strongly star-Hurewicz, star selection principles, Ψ-spaces, iterated stars

1. Introduction

1.1. Notation and terminology

Let X be a set and let $\mathcal{U}$ be a collection of subsets of X. If A is a subset of X, then the star of A with respect to $\mathcal{U}$, denoted by $St(A, \mathcal{U})$, is the set $\bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$; for $A = \{x\}$ with $x \in X$, we write $St(x, \mathcal{U})$ instead of $St(\{x\}, \mathcal{U})$. We denote by $[X]^{<\omega}$ the collection of all finite subsets of X. Throughout this paper, all spaces are assumed to be regular, unless a specific separation axiom is indicated. For notation and terminology, we refer to [12].

We recall some classical star covering properties following the terminology of [11]. A space X is said to be strongly starcompact (strongly star-Lindelöf), briefly SSC (SSL), if for every open cover $\mathcal{U}$ of X there exists a finite (countable) subset F of X such that $St(F, \mathcal{U}) = X$. A space X is starcompact (star-Lindelöf), briefly SC (SL), if for every open cover $\mathcal{U}$ of X there exists a finite (countable) subset $\mathcal{V}$ of $\mathcal{U}$ such that $St(\bigcup\mathcal{V}, \mathcal{U}) = X$. It is well-known that countable compactness and strongly starcompactness are equivalent

*Corresponding Author.
Email addresses: olimpico.25@hotmail.com (J. Casas-de la Rosa), chenwb@gmail.com (W. Chen-Mertens), sergiogb@yorku.ca (S. A. Garcia-Balan)
Received: 07.11.2022; Accepted: 16.02.2024

for Hausdorff spaces (see [11]). We refer the reader to the survey of Matveev [20] for a more detailed treatment of these star covering properties.

Recall that a space X is said to be *metacompact* (*metaLindelöf*) if every open cover $\mathcal{U}$ of X has a point-finite (point-countable) open refinement $\mathcal{V}$. Further, a space X is said to be *paracompact* (*paraLindelöf*) if every open cover $\mathcal{U}$ of X has a locally-finite (locally-countable) open refinement $\mathcal{V}$. For more information about the relationships among these covering properties (and others), we refer the reader to [6].

Recall that for $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n (and $f \leq g$ means that $f(n) \leq g(n)$ for all n). A subset B of ω^ω is *bounded* if there is $g \in \omega^\omega$ such that $f \leq^* g$ for each $f \in B$. A subset D of ω^ω is *dominating* if for each $g \in \omega^\omega$ there is $f \in D$ such that $g \leq^* f$. The minimal cardinality of an unbounded subset of ω^ω is denoted by $\mathfrak{b}$, and the minimal cardinality of a dominating subset of ω^ω is denoted by $\mathfrak{d}$. The family of all meager subsets of $\mathbb{R}$ is denoted by $\mathcal{M}$ and the minimum of the cardinalities of subfamilies $\mathcal{U} \subset \mathcal{M}$ such that $\bigcup \mathcal{U} = \mathbb{R}$ is denoted by $cov(\mathcal{M})$.

Recall that a family $\mathcal{A}$ of infinite subsets of ω is almost disjoint (a.d., for short) if the intersection of any two distinct sets in $\mathcal{A}$ is finite. Let $\mathcal{A}$ be an a.d. family, we consider $\Psi(\mathcal{A}) = \mathcal{A} \cup \omega$ with the following topology: the points of ω are isolated and a basic neighbourhood of a point $a \in \mathcal{A}$ is of the form $\{a\} \cup (a \setminus F)$, where F is a finite subset of ω. Then $\Psi(\mathcal{A})$ is called a Ψ-space (see [14]).

1.2. Classical (star) selection principles

Likely, among classical selection principles, the most well-known are the Menger, Rothberger and Hurewicz properties. Let us recall those notions and its star versions. Given a topological space X, we denote by $\mathcal{O}$ the collection of all open covers of X and by Γ the collection of all γ-covers of X. Recall that an open cover $\mathcal{U}$ of X is a γ-cover if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. A space X is Menger (M) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is an open cover of X (see [21]). A space X is Rothberger (R) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a sequence $\{U_n : n \in \omega\}$ such that for each $n \in \omega$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \omega\}$ is an open cover of X (see [23]). A space X is Hurewicz (H) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \bigcup \mathcal{V}_n$ for all but finitely many n (see [16]). The following star versions for the cases Menger and Rothberger were introduced in [17] and the star versions for the Hurewicz case were defined in [2].

Definition 1.1. A space X is:

(1) star-Menger (SM) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.
(2) strongly star-Menger (SSM) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there exists a sequence $\{F_n : n \in \omega\}$ of finite subsets of X such that $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.
(3) star-Rothberger (SR) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there are $U_n \in \mathcal{U}_n$, $n \in \omega$, such that $\{St(U_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.
(4) strongly star-Rothberger (SSR) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there exists a sequence $\{x_n : n \in \omega\}$ of elements of X such that $\{St(x_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.

(5) star-Hurewicz (SH) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in St(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n.

(6) strongly star-Hurewicz (SSH) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there exists a sequence $\{F_n : n \in \omega\}$ of finite subsets of X such that for each $x \in X$, $x \in St(F_n, \mathcal{U}_n)$ for all but finitely many n.

It is worth to mention that for paracompact Hausdorff spaces the three Menger-type properties, SM, SSM and M are equivalent and the same situation holds for the three Rothberger-type properties and the three Hurewicz-type properties (see [17] and [2]). Even more, those equivalences still true for paraLindelöf spaces (see [8]).

Figure 1 shows the relationships among these properties (in the diagram C and L are used to denote compactness and the Lindelöf property, respectively). We mention that none of the arrows in the following diagram reverse. We refer the reader to [19] to see the current state of knowledge about these relationships with others.

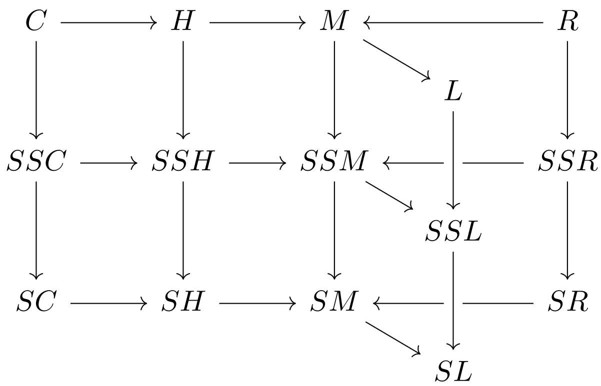


Figure 1. Star selection principles.

2. Small unions of some star spaces

In [30] Tall proved that if a space X is Lindelöf and it can be written as a union of less than $\mathfrak{d}$ compact spaces, then X is Menger. It turns out that we can replace "compact" by "star-Hurewicz" in Tall's result (see Proposition 2.3 below)[†], or we can replace "$\mathfrak{d}$" and "compact" by "$\mathfrak{b}$" and "star-Menger" (Proposition 2.7). Furthermore, a Lindelöf space that can be written as a union of less than $\mathfrak{b}$ star-Hurewicz spaces, is Hurewicz (Proposition 2.4). These results are contained in Theorem 2.2 below.

In addition, we investigate what happens if instead of starting with a Lindelöf space that can be written as some small union, we consider a star-Lindelöf space or a strongly star-Lindelöf space or an absolutely strongly star-Lindelöf space (see Definition 2.17 below). Some other interesting relationships are obtained and described in Theorem 2.10 and Theorem 2.19 below. Let us first introduce some notation that allows to present these results in an organized manner[‡].

[†] In [9], the authors also use the idea of unions of size less than $\mathfrak{d}$ many Hurewicz-type spaces to obtain some results about star-Scheepers spaces.

[‡] Preliminary versions of some results in this section are contained in the PhD Dissertation of the third listed author (see [13]).

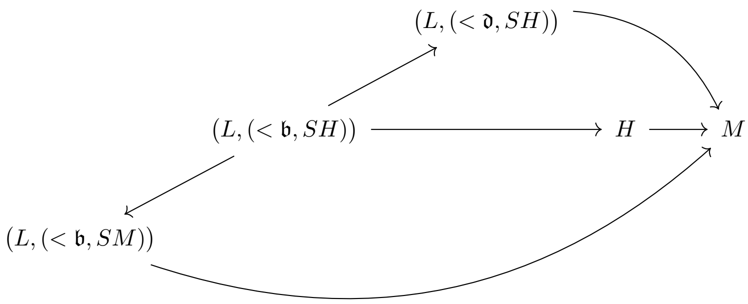
Definition 2.1. Let X be any space, A and B denote some properties and κ is some cardinal.

$$(A, (< \kappa, B))$$

stands for "X satisfies property A and it can be written as a union of less than κ spaces each of them satisfying property B".

For instance, if L, C and M denote Lindelöf, compact and Menger, respectively, then Tall's result can be written as "$(L, (< \mathfrak{d}, C)) \to M$". More in general, we have:

Theorem 2.2. *For any space X the following holds:*



The proof is divided as Propositions 2.3, 2.4 and 2.7.

Proposition 2.3. *If X is a Lindelöf space and X is the union of less than $\mathfrak{d}$ star-Hurewicz spaces, then X is Menger.*

Proof. Let κ be a cardinal smaller than $\mathfrak{d}$ and put $X = \bigcup_{\alpha<\kappa} Y_\alpha$ with each Y_α being a star-Hurewicz space. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is Lindelöf, we can assume that for each $n \in \omega$, $\mathcal{U}_n$ is countable and put $\mathcal{U}_n = \{U_n^i : i \in \omega\}$. Since each Y_α is star-Hurewicz, for each $\alpha < \kappa$, there exists a finite subset $\mathcal{V}_n^\alpha$ of $\mathcal{U}_n$ such that $\{St(\bigcup \mathcal{V}_n^\alpha, \mathcal{U}_n) : n \in \omega\}$ is a collection of open sets in X that covers Y_α and satisfies that for each $x \in Y_\alpha$, $x \in St(\bigcup \mathcal{V}_n^\alpha, \mathcal{U}_n)$ for all but finitely many n. Define, for each $\alpha < \kappa$, a function f_α as follows: for each $n \in \omega$, let $f_\alpha(n) = min\{i \in \omega : \mathcal{V}_n^\alpha \subseteq \{U_n^j : j \leq i\}\}$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{d}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $g \not\leq^* f_\alpha$. For each $n \in \omega$, let $\mathcal{W}_n = \{U_n^i : i \leq g(n)\}$.
Claim: $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.
Let $x \in X$. Then, there exists $\alpha < \kappa$ such that $x \in Y_\alpha$. Hence, there is $n_0 \in \omega$ so that for every $n \geq n_0$, $x \in St(\bigcup \mathcal{V}_n^\alpha, \mathcal{U}_n)$. Since $g \not\leq^* f_\alpha$, we can take $n > n_0$ such that $g(n) > f_\alpha(n)$. Then $x \in St(\bigcup \mathcal{V}_n^\alpha, \mathcal{U}_n) \subseteq St(\bigcup_{j \leq f_\alpha(n)} U_n^j, \mathcal{U}_n) \subseteq St(\bigcup_{j \leq g(n)} U_n^j, \mathcal{U}_n) = St(\bigcup \mathcal{W}_n, \mathcal{U}_n)$. Therefore, the collection $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Thus, X is star-Menger. Finally, since X is Lindelöf, X is a paracompact space and this allow us to conclude that X is Menger. $\square$

Proposition 2.4. *If X is a Lindelöf space and X is the union of less than $\mathfrak{b}$ star-Hurewicz spaces, then X is Hurewicz.*

Proof. By mimicking the first part of the proof of Theorem 2.3, since the collection of $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{b}$, we can define a function $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $f_\alpha \leq^* g$. For each $n \in \omega$, let $\mathcal{W}_n = \{U_n^i : i \leq g(n)\}$. It is not hard to show that $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X. $\square$

In this article, by large cover we mean the following:

Definition 2.5. A cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of a space X is called large if for every $\alpha < \kappa$, $\{U_\beta : \alpha \leq \beta < \kappa\}$ is a cover of X. We denote the class of large covers of X by $\mathcal{L}(X)$.

Observe that when we consider countable covers, the previous definition and the one given in [25] coincide.

For the following lemma we recall some classical notation of (star) selection principles introduced by M. Scheepers (Kočinac) in [25] ([17]). Let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of sets.

$S_{fin}(\mathcal{A}, \mathcal{B})$: For each sequence $\{A_n : n \in \omega\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n : n \in \omega\}$ such that for each $n \in \omega$, $B_n \in [A_n]^{<\omega}$ and $\bigcup\{B_n : n \in \omega\}$ is an element of $\mathcal{B}$.

$U_{fin}(\mathcal{A}, \mathcal{B})$: For each sequence $\{A_n : n \in \omega\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n : n \in \omega\}$ such that for each $n \in \omega$, $B_n \in [A_n]^{<\omega}$ and $\{\bigcup B_n : n \in \omega\}$ is an element of $\mathcal{B}$.

$S^*_{fin}(\mathcal{A}, \mathcal{B})$: For each sequence $\{A_n : n \in \omega\}$ of elements of $\mathcal{A}$ there exists a sequence $\{B_n : n \in \omega\}$ such that for each $n \in \omega$, $B_n \in [A_n]^{<\omega}$ and $\{St(\bigcup B_n, A_n) : n \in \omega\}$ is an element of $\mathcal{B}$.

The following Lemma will be useful to clarify some steps in the proofs of Propositions 2.7 and Lemma 2.9.

Lemma 2.6 (Folklore)**.** *For any space X:*

(1) $S_{fin}(\mathcal{O}, \mathcal{O}) \leftrightarrow U_{fin}(\mathcal{O}, \mathcal{L})$.
(2) $S^*_{fin}(\mathcal{O}, \mathcal{O}) \leftrightarrow S^*_{fin}(\mathcal{O}, \mathcal{L})$.

Proof. Let X be any space. Observe that $U_{fin}(\mathcal{O}, \mathcal{L}) \to S_{fin}(\mathcal{O}, \mathcal{O})$ and $S^*_{fin}(\mathcal{O}, \mathcal{L}) \to S^*_{fin}(\mathcal{O}, \mathcal{O})$ are immediate. Now, assume $S_{fin}(\mathcal{O}, \mathcal{O})$ ($S^*_{fin}(\mathcal{O}, \mathcal{O})$ respectively) holds. Let $\{\mathcal{U}_n : n \in \omega\}$ be any sequence of open covers of X and let $m \in \omega$. Since the collection $\{\mathcal{U}_n : m \leq n < \omega\}$ is a sequence of open covers of X, then for each $n \geq m$ there exists a finite subset $\mathcal{V}^m_n$ of $\mathcal{U}_n$ such that $\bigcup\{\mathcal{V}^m_n : m \leq n < \omega\}$ ($\{St(\bigcup \mathcal{V}^m_n, \mathcal{U}_n) : m \leq n < \omega\}$, resp.) is an open cover of X. So, for each $n \in \omega$ we define $\mathcal{W}_n = \bigcup_{m \leq n} \mathcal{V}^m_n$. Hence, for each $m \in \omega$ the collection $\bigcup\{\mathcal{W}_n : m \leq n < \omega\}$ ($\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : m \leq n < \omega\}$, resp.) is an open cover of X. Furthermore, for each $m \in \omega$ the collection $\{\bigcup \mathcal{W}_n : m \leq n < \omega\}$ is an open cover of X. Thus, for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$ and the collection $\{\bigcup \mathcal{W}_n : n < \omega\}$ ($\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n < \omega\}$, resp.) is a large cover of X. Hence, $U_{fin}(\mathcal{O}, \mathcal{L})$ ($S^*_{fin}(\mathcal{O}, \mathcal{L})$, resp.) holds. □

Proposition 2.7. *If X is a Lindelöf space and X is the union of less than $\mathfrak{b}$ star-Menger spaces, then X is Menger.*

Proof. Let κ be a cardinal smaller than $\mathfrak{b}$ and put $X = \bigcup_{\alpha<\kappa} Y_\alpha$ with each Y_α being a star-Menger space. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is Lindelöf, we can assume that for each $n \in \omega$, $\mathcal{U}_n$ is countable and put $\mathcal{U}_n = \{U^i_n : i \in \omega\}$. Since for each $\alpha < \kappa$, Y_α is star-Menger, by Lemma 2.6, for each $\alpha < \kappa$ there is $\mathcal{V}^\alpha_n$ finite subset of $\mathcal{U}_n$ such that for every $m \in \omega$, $\{St(\bigcup \mathcal{V}^\alpha_n, \mathcal{U}_n) : m \leq n < \omega\}$ is an open cover of Y_α.

We define the family of functions f_α in the same fashion as we did in the proof of Theorem 2.3. Since it has size less than $\mathfrak{b}$ there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $f_\alpha \leq^* g$. For each $n \in \omega$, let $\mathcal{W}_n = \{U^i_n : i \leq g(n)\}$. With similar ideas, it can be shown that $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. □

Following [5] (see also [22]), we recall some modifications of the Menger and Hurewicz properties, called E^*_ω and E^{**}_ω properties, respectively. We say that a space X has the property E^*_ω (E^{**}_ω), if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of countable open covers of X, there exists a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is an open cover (γ-cover) of X. If we recall the definition of countably compact space (every countable cover has a finite subcover), properties E^*_ω and E^{**}_ω could be called countably Menger and countably Hurewicz, respectively. As pointed out in [22], in the class of Lindelöf spaces, the properties Menger and E^*_ω are the same. However, this fact is not true in general. The space ω_1 (with the order topology) has the property E^*_ω and it is not a Menger space.

Lemma 2.8. *If X can be written as a union of less than $\mathfrak{d}$ many Hurewicz spaces, then X is E^*_ω.*

Proof. Assume $X = \bigcup_{\alpha<\kappa} H_\alpha$ so that $\kappa < \mathfrak{d}$ and each H_α is Hurewicz. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of countable open covers of X. For each $n \in \omega$ let $\mathcal{U}_n = \{U^i_n : i \in \omega\}$. For each $\alpha < \kappa$ and $n \in \omega$, there exists $\mathcal{F}^\alpha_n \in [\mathcal{U}_n]^{<\omega}$ so that $\{\bigcup \mathcal{F}^\alpha_n : n \in \omega\}$ is a γ-cover of H_α. For each $\alpha < \kappa$ and $n \in \omega$, let $f_\alpha(n) = min\{m \in \omega : \mathcal{F}^\alpha_n \subseteq \{U^i_n : i \leq m\}\}$. Given that $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{d}$, there exists $g \in \omega^\omega$ so that for each $\alpha < \kappa$, $g \not\leq^* f_\alpha$. For each $n \in \omega$, let $G_n = \{U^i_n : i \leq g(n)\}$. Let us check that $\{\bigcup G_n : n \in \omega\}$ is an open cover of X. Let $x \in X$, then there is $\alpha < \kappa$ so that $x \in H_\alpha$. Thus, there is $n_0 \in \omega$ so that for each $n \geq n_0$, $x \in \bigcup \mathcal{F}^\alpha_n$. Pick $m \geq n_0$ with $f_\alpha(m) < g(m)$. Hence, $x \in \bigcup \mathcal{F}^\alpha_m \subseteq \bigcup G_m$. Therefore, X is E^*_ω. □

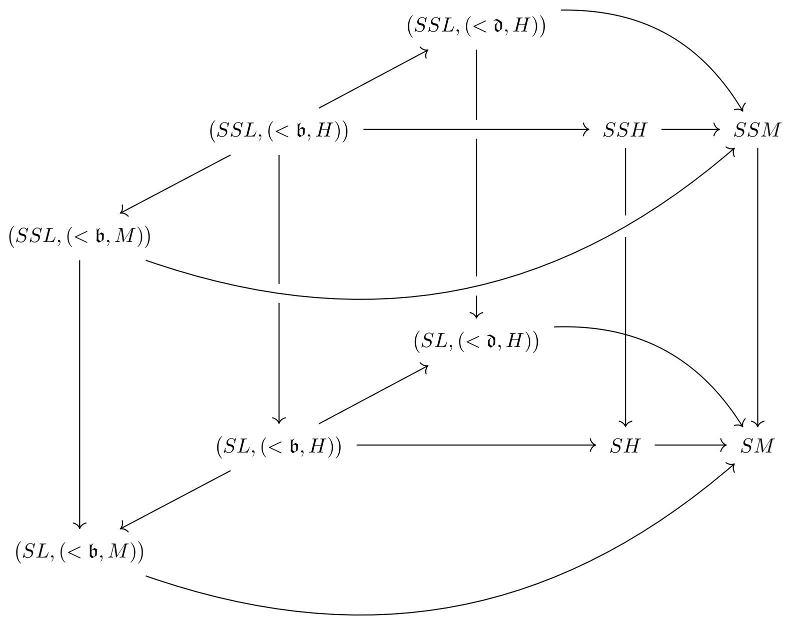
Analogous to Lemma 2.8, the following also holds:

Lemma 2.9. (1) *If X can be written as a union of less than $\mathfrak{b}$ many Menger spaces, then X is E^*_ω.*
(2) *If X can be written as a union of less than $\mathfrak{b}$ many Hurewicz spaces, then X is E^{**}_ω.*

Proof. (1) Assume $X = \bigcup_{\alpha<\kappa} M_\alpha$ so that $\kappa < \mathfrak{b}$ and each M_α is Menger. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of countable open covers of X. For each $n \in \omega$ let $\mathcal{U}_n = \{U^i_n : i \in \omega\}$. By Lemma 2.6, for each $\alpha < \kappa$ and $n \in \omega$, there exists $\mathcal{F}^\alpha_n \in [\mathcal{U}_n]^{<\omega}$ so that $\{\bigcup \mathcal{F}^\alpha_n : n \in \omega\}$ is a large cover of M_α. For each $\alpha < \kappa$ and $n \in \omega$, let $f_\alpha(n) = min\{m \in \omega : \mathcal{F}^\alpha_n \subseteq \{U^i_n : i \leq m\}\}$. Given that $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{b}$, there exists $g \in \omega^\omega$ so that for each $\alpha < \kappa$, $f_\alpha \leq^* g$. For each $n \in \omega$, let $G_n = \{U^i_n : i \leq g(n)\}$. Let us check that $\{\bigcup G_n : n \in \omega\}$ is an open cover of X. Let $x \in X$, then there is $\alpha < \kappa$ so that $x \in M_\alpha$. Thus, there is $n_0 \in \omega$ so that for each $n \geq n_0$, $f_\alpha(n) \leq g(n)$. Since $\{\bigcup \mathcal{F}^\alpha_n : n \in \omega\}$ is a large cover of M_α, there is $n \geq n_0$ such that $x \in \bigcup \mathcal{F}^\alpha_n \subseteq \bigcup_{i \leq f_\alpha(n)} U^i_n \subseteq \bigcup_{i \leq g(n)} U^i_n = \bigcup G_n$. Therefore, X is E^*_ω.

(2) Assume $X = \bigcup_{\alpha<\kappa} H_\alpha$ so that $\kappa < \mathfrak{b}$ and each H_α is Hurewicz. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of countable open covers of X. For each $n \in \omega$ let $\mathcal{U}_n = \{U^i_n : i \in \omega\}$. For each $\alpha < \kappa$ and $n \in \omega$, there exists $\mathcal{F}^\alpha_n \in [\mathcal{U}_n]^{<\omega}$ so that $\{\bigcup \mathcal{F}^\alpha_n : n \in \omega\}$ is γ-cover of H_α. For each $\alpha < \kappa$ and $n \in \omega$, let $f_\alpha(n) = min\{m \in \omega : \mathcal{F}^\alpha_n \subseteq \{U^i_n : i \leq m\}\}$. Given that $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{b}$, there exists $g \in \omega^\omega$ so that for each $\alpha < \kappa$, $f_\alpha \leq^* g$. For each $n \in \omega$, let $G_n = \{U^i_n : i \leq g(n)\}$. Let us check that $\{\bigcup G_n : n \in \omega\}$ is a γ-cover of X. Let $x \in X$, then there is $\alpha < \kappa$ so that $x \in H_\alpha$. Thus, there is $n_0 \in \omega$ so that for each $n \geq n_0$, $f_\alpha(n) \leq g(n)$. Since $\{\bigcup \mathcal{F}^\alpha_n : n \in \omega\}$ is a γ-cover of H_α, we can fix $n_1 \geq n_0$ such that for each $n \geq n_1$, $x \in \bigcup \mathcal{F}^\alpha_n \subseteq \bigcup_{i \leq f_\alpha(n)} U^i_n \subseteq \bigcup_{i \leq g(n)} U^i_n = \bigcup G_n$. Therefore, X is E^{**}_ω. □

Theorem 2.10. *For any space X the following holds:*



The proof is divided as Propositions 2.11, 2.12, 2.13, 2.14, 2.15 and 2.16.

Proposition 2.11. *If X is a strongly star-Lindelöf space and X is the union of less than $\mathfrak{d}$ Hurewicz spaces, then X is strongly star-Menger.*

Proof. Let κ be any cardinal smaller than $\mathfrak{d}$ and put $X=\bigcup_{\alpha<\kappa} H_\alpha$ with each H_α being a Hurewicz space. Let $\{\mathcal{U}_n : n\in\omega\}$ be a sequence of open covers of X. Since X is strongly star-Lindelöf, for each $n\in\omega$ there exists a countable set $C_n\subseteq X$ such that $St(C_n,\mathcal{U}_n)=X$. For each $n\in\omega$, put $C_n=\{x_n^i : i\in\omega\}$. Note that $St(C_n,\mathcal{U}_n)=\bigcup_{i\in\omega} St(x_n^i,\mathcal{U}_n)$ for each $n\in\omega$. So, for each $n\in\omega$, the collection $\mathcal{W}_n=\{St(x_n^i,\mathcal{U}_n) : i\in\omega\}$ is a countable open cover of X. By Lemma 2.8, X is E_ω^* and then we can get finite subcollections $\mathcal{F}_n$ of $\mathcal{W}_n$ so that $\{\bigcup\mathcal{F}_n : n\in\omega\}$ is a cover of X. Equivalently, we get finite subsets F_n of X such that the collection $\{St(F_n,\mathcal{U}_n) : n\in\omega\}$ is an open cover of X. Thus, X is strongly star-Menger. $\square$

Proposition 2.12. *If X is a strongly star-Lindelöf space and X is the union of less than $\mathfrak{b}$ Hurewicz spaces, then X is strongly star-Hurewicz.*

Proof. Following the first lines of the proof of Proposition 2.11 and applying Lemma 2.9 (2) to the collections of countable open covers $\mathcal{W}_n=\{St(x_n^i,\mathcal{U}_n) : i\in\omega\}$, we obtain that X is E_ω^{**} and thus, the result follows. $\square$

Using Lemma 2.9 (1) and same ideas as Propositions 2.11 y 2.12, the following result holds:

Proposition 2.13. *If X is a strongly star-Lindelöf space and X is the union of less than $\mathfrak{b}$ Menger spaces, then X is strongly star-Menger.*

Proposition 2.14. *If X is a star-Lindelöf space and X is the union of less than $\mathfrak{d}$ Hurewicz spaces, then X is star-Menger.*

Proof. Let κ be any cardinal smaller than $\mathfrak{d}$ and put $X=\bigcup_{\alpha<\kappa} H_\alpha$ with each H_α being a Hurewicz space. Let $\{\mathcal{U}_n : n\in\omega\}$ be a sequence of open covers of X. Since X is star-Lindelöf, for each $n\in\omega$ there exists a countable subset $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $St(\bigcup\mathcal{V}_n,\mathcal{U}_n)=X$. For each $n\in\omega$, put $\mathcal{V}_n=\{V_n^i : i\in\omega\}$. Note that $St(\bigcup\mathcal{V}_n,\mathcal{U}_n)=\bigcup_{i\in\omega} St(V_n^i,\mathcal{U}_n)$ for each $n \in \omega$. So, for each $n \in \omega$, the collection $\mathcal{W}_n = \{St(V_n^i, \mathcal{U}_n) : i \in \omega\}$ is a countable open cover of X. By Lemma 2.8, X is E_ω^* and then we can get finite subcollections $\mathcal{H}_n$ of $\mathcal{W}_n$ so that $\{\bigcup \mathcal{H}_n : n \in \omega\}$ is an open cover of X. That is, we get finite sets $\mathcal{F}_n$ of $\mathcal{U}_n$ such that the collection $\{St(\bigcup \mathcal{F}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Thus, X is star-Menger. □

Proposition 2.15. *If X is a star-Lindelöf space and X is the union of less than $\mathfrak{b}$ Hurewicz spaces, then X is star-Hurewicz.*

Proof. Arguing as in the first lines of the proof of Proposition 2.14 and applying Lemma 2.9 (2) to the collections of countable open covers $\mathcal{W}_n = \{St(V_n^i, \mathcal{U}_n) : i \in \omega\}$, we obtain that X is E_ω^{**} and thus, the result follows. □

Again, by using Lemma 2.9 (1) and same ideas as Propositions 2.14 y 2.15, the following result holds:

Proposition 2.16. *If X is a star-Lindelöf space and X is the union of less than $\mathfrak{b}$ Menger spaces, then X is star-Menger.*

Another interesting fact is that we also have theorems of same structure for the selective versions of the star selection principles. We recall the necessary definitions to state the analogous theorem for these selective versions.

The following version of the strongly star-Lindelöf property was introduced and studied by Bonanzinga in [1]:

Definition 2.17. A space X is absolutely strongly star-Lindelöf ($aSSL$) if for any open cover $\mathcal{U}$ of X and any dense subset D of X, there is a countable set $C \subseteq D$ such that $St(C, \mathcal{U}) = X$.

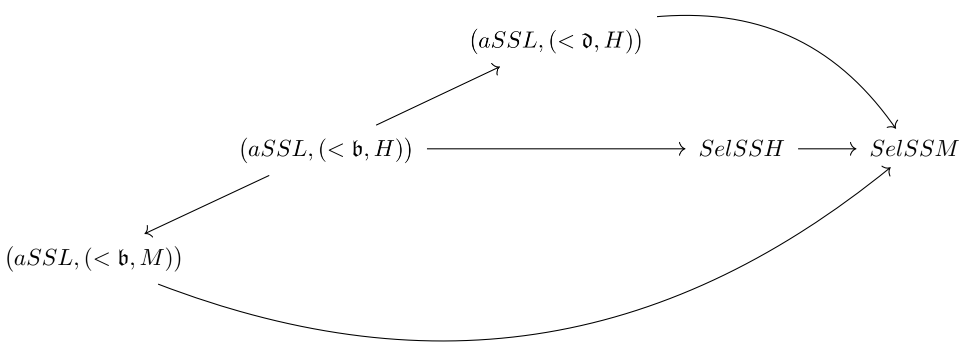
The selective versions below are stronger properties than the classical star selection principles (see for instance [7] and [3] for more information of these properties[§]).

Definition 2.18. We say that a space X is:

(1) selectively strongly star-Menger ($selSSM$) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X and each sequence $\{D_n : n \in \omega\}$ of dense sets of X, there exists a sequence $\{F_n : n \in \omega\}$ of finite sets such that $F_n \subseteq D_n$, $n \in \omega$, and $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X (defined in [10]).

(2) selectively strongly star-Hurewicz ($selSSH$) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X and each sequence $\{D_n : n \in \omega\}$ of dense sets of X, there exists a sequence $\{F_n : n \in \omega\}$ of finite sets such that $F_n \subseteq D_n$, $n \in \omega$, and $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X.

Theorem 2.19. *For any space X the following holds:*



[§]The Rothberger case and some other interesting properties are also given in [7]

The proof is divided as Propositions 2.20, 2.21, 2.22.

Proposition 2.20. *If X is an absolutely strongly star-Lindelöf space and X is the union of less than $\mathfrak{d}$ Hurewicz spaces, then X is selectively strongly star-Menger.*

Proof. Assume X is $aSSL$ and let $\kappa < \mathfrak{d}$ such that $X = \bigcup_{\alpha<\kappa} H_\alpha$ with each H_α being a Hurewicz space. Let $\{\mathcal{U}_n : n \in \omega\}$ be any sequence of open covers and let $\{D_n : n \in \omega\}$ be any sequence of dense subsets of X. For each $n \in \omega$ fix $E_n = \{d^n_q : q \in \omega\} \in [D_n]^{\leq\omega}$ such that $St(E_n, \mathcal{U}_n) = X$. Thus, for each $n \in \omega$, the collection $\{St(d^n_q, \mathcal{U}_n) : q \in \omega\}$ is a countable open cover of X. In a similar fashion as Proposition 2.11, we use Lemma 2.8 to conclude the proof. □

Proposition 2.21. *If X is an absolutely strongly star-Lindelöf space and X is the union of less than $\mathfrak{b}$ Hurewicz spaces, then X is selectively strongly star-Hurewicz.*

Proof. Assume X is $aSSL$ and let $\kappa < \mathfrak{b}$ such that $X = \bigcup_{\alpha<\kappa} H_\alpha$ with each H_α being a Hurewicz space. Let $\{\mathcal{U}_n : n \in \omega\}$ be any sequence of open covers and let $\{D_n : n \in \omega\}$ be any sequence of dense subsets of X. For each $n \in \omega$ fix $E_n = \{d^n_q : q \in \omega\} \in [D_n]^{\leq\omega}$ such that $St(E_n, \mathcal{U}_n) = X$. Thus, for each $n \in \omega$, the collection $\{St(d^n_q, \mathcal{U}_n) : q \in \omega\}$ is a countable open cover of X. In a similar fashion as Proposition 2.12, we use Lemma 2.9 (2) to complete the proof. □

Proposition 2.22. *If X is an absolutely strongly star-Lindelöf space and X is the union of less than $\mathfrak{b}$ Menger spaces, then X is selectively strongly star-Menger.*

Proof. Assume X is $aSSL$ and let $\kappa < \mathfrak{b}$ such that $X = \bigcup_{\alpha<\kappa} M_\alpha$ with each M_α being a Menger space. Let $\{\mathcal{U}_n : n \in \omega\}$ be any sequence of open covers and let $\{D_n : n \in \omega\}$ be any sequence of dense subsets of X. For each $n \in \omega$ fix $E_n = \{d^n_q : q \in \omega\} \in [D_n]^{\leq\omega}$ such that $St(E_n, \mathcal{U}_n) = X$. Thus, for each $n \in \omega$, the collection $\{St(d^n_q, \mathcal{U}_n) : q \in \omega\}$ is a countable open cover of X. In a similar way as Proposition 2.13, we use Lemma 2.9 (1) to conclude the proof. □

3. Iterated stars

A well-known study about iterations of star versions of Lindelöf properties was made in [11] (see also [20]). The previous section motivated a similar study for star selection properties. We start giving some results that involve refinements of open covers.

Let n be a positive integer. In [11] the properties n-star Lindelöf and strongly n-star Lindelöf are defined and the authors show that every n-star Lindelöf space is strongly $n+1$-star-Lindelöf (Theorem 3.1.1 (3) in [11]). In the class of metaLindelöf spaces the converse holds:

Proposition 3.1. *If X is metaLindelöf and strongly $n+1$-star Lindelöf then X is n-star Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. Since X is metaLindelöf we can assume that $\mathcal{U}$ is point-countable. Let C be a countable subset of X so that $St^n(St(C, \mathcal{U}), \mathcal{U}) = X$. Since $\mathcal{U}$ is point-countable, there exists a countable subset W of $\mathcal{U}$ so that $St(C, \mathcal{U}) \subseteq \bigcup W$. Hence, $St^{n+1}(C, \mathcal{U}) \subseteq St^n(\bigcup W, \mathcal{U})$ i.e., X is n-star-Lindelöf. □

Lemma 3.2 (Folklore)**.** *Let $\mathcal{U}$ be an open cover of a topological space X. If $A \subseteq B \subseteq X$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$, then for each $n \in \omega$, $St^n(A, \mathcal{V}) \subseteq St^n(B, \mathcal{U})$.*

Proposition 3.3. *Every paraLindelöf and n-star-Lindelöf space X is strongly n-star-Lindelöf for each $n \in \omega$.*

Proof. Assume X is a paraLindelöf n-star-Lindelöf space for some $n \in \omega$. Let $\mathcal{U}$ be an open cover of X. Without loss of generality we can assume that $\mathcal{U}$ is locally countable.

For each $x \in X$, let U_x be an open set so that $|\{U \in \mathcal{U} : U_x \cap U \neq \emptyset\}| \leq \omega$ and $\mathcal{V} := \{U_x : x \in X\}$ refines $\mathcal{U}$. Since X is n-star-Lindelöf, there exists a countable set $C \subseteq X$ so that $St^n(\bigcup_{x\in C} U_x, \mathcal{V}) = X$.
For $x \in C$, let $\mathcal{W}_x = \{U \in \mathcal{U} : U_x \cap U \neq \emptyset\}$. Then $|\mathcal{W}_x| \leq \omega$. If $\mathcal{W} = \bigcup_{x\in C} \mathcal{W}_x$, then $|\mathcal{W}| \leq \omega$. Thus, for each $W \in \mathcal{W}$, fix $y_W \in W$.
Claim: $St(\bigcup_{x\in C} U_x, \mathcal{V}) \subseteq St(\{y_W : W \in \mathcal{W}\}, \mathcal{U})$.
Indeed, let $y \in St(\bigcup_{x\in C} U_x, \mathcal{V})$, hence, there exists $V \in \mathcal{V}$ so that $y \in V$ and $V \cap \bigcup_{x\in C} U_x \neq \emptyset$. Let $U \in \mathcal{U}$ so that $V \subseteq U$. Then, $U \in \mathcal{W}$. Thus, $y \in St(y_U, \mathcal{U}) \subseteq St(\{y_W : W \in \mathcal{W}\}, \mathcal{U})$.
Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, and $St(\bigcup_{x\in C} U_x, \mathcal{V}) \subseteq St(\{y_W : W \in \mathcal{W}\}, \mathcal{U})$, by Lemma 3.2, $X = St^n(\bigcup_{x\in C} U_x, \mathcal{V}) \subseteq St^n(\{y_W : W \in \mathcal{W}\}, \mathcal{U})$. That is, X is strongly n-star-Lindelöf. □

Observe that if we have a paraLindelöf space X that is either n-star-Lindelöf or strongly n-star-Lindelöf for some $n \in \omega$, we can use Proposition 3.3 and Proposition 3.1, as many times as needed to get that X is star-Lindelöf. Since paraLindelöf, star-Lindelöf spaces are Lindelöf (check Theorem 2.6 in [8] or Theorem 2.24 in [28]), we have the following:

Corollary 3.4. *In the class of paraLindelöf spaces, for each $n \in \omega$, the following properties are equivalent:*

(i) *Lindelöf;*
(ii) *strongly n-star-Lindelöf;*
(iii) *n-star-Lindelöf.*

In Proposition 53 of [20], Matveev shows that every metaLindelöf strongly 2-star Lindelöf is absolutely strongly 2-star Lindelöf (he calls star Lindelöf what we call strongly star Lindelöf). Thus, we can ask the general case:

Question 3.5. Is it true that every metaLindelöf strongly n-star-Lindelöf space is absolutely strongly n-star-Lindelöf?

Additionally, since both properties, absolutely strongly $n+1$-star Lindelöf and n-star Lindelöf are stronger than strongly $n+1$-star Lindelöf, it is worth to investigate the following:

Question 3.6. What is the relationship between the properties absolutely strongly $n+1$-star Lindelöf and n-star Lindelöf?

We introduce similar definitions for the star versions of the Menger property:

Definition 3.7. (1) A space X is called k-star-Menger if for every sequence of open covers $\{\mathcal{U}_n : n \in \omega\}$ there exists a sequence $\{\mathcal{V}_n : n \in \omega\}$ so that for each $n \in \omega$, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{St^k(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.
(2) A space X is called strongly k-star-Menger if for every sequence of open covers $\{\mathcal{U}_n : n \in \omega\}$ there exists a sequence $\{F_n : n \in \omega\}$ so that for each $n \in \omega$, $F_n \in [X]^{<\omega}$ and $\{St^k(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.

Observe that 1-star-Menger and strongly 1-star-Menger spaces are precisely star-Menger and strongly star-Menger spaces, respectively.

It follows immediately from Proposition 3.1 that if X is metaLindelöf and strongly $k+1$-star-Menger then X is k-star Lindelöf. In addition, the next proposition is the Menger version of Theorem 3.1.1(3) in [11].

Proposition 3.8. *If X is k-star-Menger, then X is strongly $k+1$-star-Menger.*

Proof. Assume X is k-star-Menger. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. For each $n \in \omega$, let $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ so that $\{St^k(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of

X. Put $\mathcal{V}_n = \{V_n^i : i \in I_n\}$. Take for each $n \in \omega$ and $i \in I_n$, $x_n^i \in V_n^i$ and define for each $n \in \omega$, $F_n = \{x_n^i : i \in I_n\}$. Observe that for each $n \in \omega$, $\bigcup \mathcal{V}_n \subseteq St(F_n, \mathcal{U}_n)$. Hence,

$$St^k(\bigcup \mathcal{V}_n, \mathcal{U}_n) \subseteq St^k(St(F_n, \mathcal{U}_n), \mathcal{U}_n) = St^{k+1}(F_n, \mathcal{U}_n).$$

Thus, $\{St^{k+1}(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. That is, X is strongly $k+1$-star-Menger. □

In the class of metacompact spaces the properties k-star-Menger and strongly $k+1$-star-Menger coincide:

Proposition 3.9. *If X is strongly $k+1$-star-Menger and metacompact, then X is k-star-Menger.*

Proof. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is metacompact we can assume that for each $n \in \omega$, $\mathcal{U}_n$ is point-finite.
Since X is strongly $k+1$-star-Menger, for each $n \in \omega$, there exists $F_n \in [X]^{<\omega}$ so that

$$\{St^{k+1}(F_n, \mathcal{U}_n) : n \in \omega\}$$

is an open cover of X. For each $n \in \omega$, let $\mathcal{W}_n = \{U \in \mathcal{U}_n : F_n \cap U \neq \emptyset\}$. Note $|\mathcal{W}_n| < \omega$ and $St(F_n, \mathcal{U}_n) = \bigcup \mathcal{W}_n$. Then, for each $n \in \omega$ $St^k(St(F_n, \mathcal{U}_n), \mathcal{U}_n) = St^k(\bigcup \mathcal{W}_n, \mathcal{U}_n)$. Thus, $\{St^k(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. That is, X is k-star-Menger. □

Corollary 3.10. *If X is strongly 2-star-Menger and paracompact, then X is Menger.*

Proof. Since paracompact implies metacompact, if X is strongly 2-star-Menger and paracompact, using the previous result, X is star-Menger. In addition, paracompact star-Menger spaces are Menger (see [17]). □

Actually something stronger holds. In [8], the authors showed that for paraLindelöf spaces, the three Menger-type properties, strongly star-Menger, star-Menger and Menger, are equivalent. Theorem 3.11 shows that this fact is still true when taking iterations of those properties. Recall that a cover $\mathcal{B}$ is said to be a *star refinement* of a cover $\mathcal{U}$, which is denoted by $\mathcal{B} \prec^* \mathcal{U}$, if for each $B \in \mathcal{B}$, there is some $U \in \mathcal{U}$ so that $St(B, \mathcal{B}) \subseteq U$. A space is paracompact if every open cover has an open star refinement (see Theorem 5.1.12 in [12]).

Theorem 3.11. *In the class of paraLindelöf spaces, given any $k \in \omega$ the properties k-star-Menger, strongly k-star-Menger and Menger, are equivalent.*

Proof. Fix $m \in \omega$ and let X be a paraLindelöf, m-star-Menger space. By Corollary 3.4, X is Lindelöf and, in particular, paracompact.

Now let $\{\mathcal{U}_n : n \in \omega\}$ be any sequence of open covers of X. For each $i \leq m$ define $\mathcal{B}_n^i$ open cover of X so that $\mathcal{B}_n^m \prec^* \mathcal{B}_n^{m-1} \prec^* \cdots \prec^* \mathcal{B}_n^0 \prec^* \mathcal{U}_n$.
Claim: For each $n \in \omega$ and for each $W \in \mathcal{B}_n^m$, there is $U_W \in \mathcal{U}_n$ so that $St^m(W, \mathcal{B}_n^m) \subseteq U_W$.
Indeed, fix $n \in \omega$ and $W_m \in \mathcal{B}_n^m$, since $\mathcal{B}_n^m \prec^* \mathcal{B}_n^{m-1}$, there is $W_{m-1} \in \mathcal{B}_n^{m-1}$ so that $St(W_m, \mathcal{B}_n^m) \subseteq W_{m-1}$. By Lemma 3.2,

$$St^2(W_m, \mathcal{B}_n^m) = St(St(W_m, \mathcal{B}_n^m), \mathcal{B}_n^m) \subseteq St(W_{m-1}, \mathcal{B}_n^{m-1}).$$

Now, since $\mathcal{B}_n^{m-1} \prec^* \mathcal{B}_n^{m-2}$, there is $W_{m-2} \in \mathcal{B}_n^{m-2}$ so that $St(W_{m-1}, \mathcal{B}_n^{m-1}) \subseteq W_{m-2}$. Then, $St^2(W_m, \mathcal{B}_n^m) \subseteq W_{m-2}$. It is possible to repeat this process $m-2$ more times to get $U_{W_m} \in \mathcal{U}_n$ with $St^m(W_m, \mathcal{B}_n^m) \subseteq U_{W_m}$.
Since X is m-star-Menger, let $\mathcal{W}_n \in [\mathcal{B}_n^m]^{<\omega}$ so that $\{St^m(\bigcup \mathcal{W}_n, \mathcal{B}_n^m) : n \in \omega\}$ is an open cover of X.
For each $n \in \omega$ and each $W \in \mathcal{W}_n$ fix $U_W \in \mathcal{U}_n$ such that $St^m(W, \mathcal{B}_n^m) \subseteq U_W$. For all $n \in \omega$, let $\mathcal{V}_n = \{U_W : W \in \mathcal{W}_n\}$. Each $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is an open cover of X. Thus, X is Menger. □

4. Ψ-spaces

A natural question in this context is whether every 2-star-Menger is star-Menger. This is not the case, in fact, Tree ([31]) built a 2-starcompact (hence 2-star-Menger), space which is not strongly 2-star Lindelöf (in particular, not star Lindelöf and therefore, not star-Menger). Since, every star-Menger space is both 2-star-Menger and star-Lindelöf, it is worth asking whether the converse holds true, i.e., Is it true that every 2-star-Menger, star-Lindelöf space is star-Menger?. The answer is no, at least consistently (see Example 4.4 below). For this, we use a Luzin family $\mathcal{A}$. Recall that an almost disjoint family $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ is called a *Luzin family* (see [18] and [15]) if it satisfies that for each $\alpha < \omega_1$ and each $n \in \omega$ the set $\{\beta < \alpha : a_\beta \cap a_\alpha \subseteq n\}$ is finite. First of all, we introduce the analogous definitions to Definition 3.7 for the Rothberger property:

Definition 4.1. (1) A space X is called k-star-Rothberger if for every sequence of open covers $\{\mathfrak{U}_n : n \in \omega\}$ there exists a sequence $\{U_n : n \in \omega\}$ so that for each $n \in \omega$, $U_n \in \mathfrak{U}_n$ and $\{St^k(U_n, \mathfrak{U}_n) : n \in \omega\}$ is an open cover of X.

(2) A space X is called strongly k-star-Rothberger if for every sequence of open covers $\{\mathfrak{U}_n : n \in \omega\}$ there exists a sequence $\{x_n : n \in \omega\}$ so that for each $n \in \omega$, $x_n \in X$ and $\{St^k(x_n, \mathfrak{U}_n) : n \in \omega\}$ is an open cover of X.

Observe that 1-star-Rothberger and strongly 1-star-Rothberger spaces are precisely star-Rothberger and strongly star-Rothberger spaces, respectively.

It is surprising that, regardless of the size of the continuum, Ψ-spaces induced by a Luzin family, are always strongly 2-star-Rothberger.

Proposition 4.2. *If $\mathcal{A}$ is Luzin, then $\Psi(\mathcal{A})$ is strongly 2-star-Rothberger.*

Proof. Let $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ be a Luzin family. Let $\{\mathfrak{U}_n : n \in \omega\}$ be any sequence of open covers of $\Psi(\mathcal{A})$. For each $\alpha < \omega_1$, fix $U^0_\alpha \in \mathfrak{U}_0$ such that $a_\alpha \in U^0_\alpha$. For $\alpha < \omega_1$, let $f(\alpha) \in a_\alpha \cap \omega$ so that $a_\alpha \setminus f(\alpha) \subseteq U^0_\alpha$.
For each $k \in \omega$, let $W_k = \{\alpha < \omega_1 : f(\alpha) = k\}$. Observe $\bigcup_{k\in\omega} W_k = \omega_1$. Since $f : \omega_1 \setminus \omega \to \omega_1$ is a regressive function, by Fodor's Lemma, there is $A_0 \subseteq \omega_1$ stationary and $n_0 \in \omega$ such that for each $\alpha \in A_0$, $f(\alpha) = n_0$. Now, defining a function $g : A_0 \to \omega_1$ as $g(\alpha) = min\ U^0_\alpha$, we can apply again Fodor's Lemma to get $A_1 \subseteq A_0$ and $n_1 \in \omega$ so that $A_1 = \{\alpha < \omega_1 : min\ U^0_\alpha = n_1\}$ is stationary. For each $k \in \omega$ and each $\alpha \in A_1$, let $B_{\alpha,k} = \{\beta < \alpha : a_\beta \cap a_\alpha \subseteq max\{k, n_0, n_1\}\}$. Since $\mathcal{A}$ is Luzin, each $B_{\alpha,k}$ is finite.
Claim: For each $k \in \omega$, $G_k = \bigcup_{\alpha\in A_1}(\alpha \setminus B_{\alpha,k})$ is cofinite.
Indeed, if for some $k \in \omega$, $\omega_1 \setminus G_k = J$ is infinite, let $\alpha < \omega_1$ such that $|J \cap \alpha| = \omega$. Pick $\beta \in A_1$ with $\beta > \alpha$. Observe that since $B_{\beta,k}$ is finite, we get $J \cap \alpha$ is almost contained in $\beta \setminus B_{\beta,k} \subseteq G_k$, which is a contradiction.
Since for each $k \in \omega$, G_k is cofinite, then $W_k \setminus G_k$ is finite. Hence, $\bigcup_{k\in\omega} W_k \cap G_k$ is cocountable. Now we show that $\{a_\beta : \beta \in \bigcup_{k\in\omega} W_k \cap G_k\} \subseteq St^2(n_1, \mathfrak{U}_0)$. Fix $k \in \omega$ and let $\beta \in W_k \cap G_k$, then $f(\beta) = k$ and there is $\alpha \in A_1$ such that $\beta \in \alpha \setminus B_{\alpha,k}$. Thus, $a_\beta \cap a_\alpha \nsubseteq max\{k, n_0, n_1\}$. Fix $m \in a_\beta \cap a_\alpha$ with $m > max\{k, n_0, n_1\}$. Hence, $m \in U^0_\alpha$, $m \in U^0_\beta$ and $m > n_1 = minU^0_\alpha$. Therefore, $m \in St(n_1, \mathfrak{U}_0)$ and $a_\beta \in U^0_\beta \subseteq St(m, \mathfrak{U}_0) \subseteq St^2(n_1, \mathfrak{U}_0)$. Therefore, $St^2(n_1, \mathfrak{U}_0)$ contains all but countably many members of $\Psi(\mathcal{A})$. Thus, the countable set of points $\{y_n : n \in \mathbb{N}\}$ of $\Psi(\mathcal{A})$ that are not contained in $St^2(n_1, \mathfrak{U}_0)$ are recollected at stage $n > 0$ with $St^2(y_n, \mathfrak{U}_n)$, hence $\Psi(\mathcal{A})$ is strongly 2-star-Rothberger. □

Another fact, which is also interesting, is that Ψ-spaces induced by a maximal almost-disjoint family can be characterized in terms of the *strongly* 2-*starcompact* property (see [11] for information about iterated (strongly) starcompact property). It is worth to mention that the equivalences (3) – (6) in Proposition 4.3 were showed in [11] for spaces in general (by using pseudocompactness instead of a maximal almost disjoint family) and the proof for (6) ⇒ (1) is contained in the proof of Example 2.2.5 in same article. For convenience of the reader, we outline the proof of these equivalences for Ψ-spaces.

Proposition 4.3 ([11])**.** *For an a.d. family $\mathcal{A}$, the following are equivalent:*

(1) *$\Psi(\mathcal{A})$ is strongly 2-starcompact.*
(2) *$\Psi(\mathcal{A})$ is strongly k-starcompact for every $k \geq 2$.*
(3) *$\Psi(\mathcal{A})$ is 2-starcompact.*
(4) *$\Psi(\mathcal{A})$ is k-starcompact for every $k \geq 2$.*
(5) *$\Psi(\mathcal{A})$ is k-starcompact for some $k \geq 2$.*
(6) *$\mathcal{A}$ is maximal.*

Proof. Since the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) always hold, we just show (5) ⇒ (6) and (6) ⇒ (1).

Assume that $\Psi(\mathcal{A})$ is k-starcompact for some $k \geq 2$. If $\mathcal{A}$ is not maximal, there is some $B \subseteq \omega$ which is almost disjoint from every member of $\mathcal{A}$. Consider the open cover $\mathcal{U}$ consisting of singletons from ω together with sets of the form $A \setminus B$ for $A \in \mathcal{A}$. Then for any $\mathcal{W} \subseteq \mathcal{U}$, $St^k(\bigcup \mathcal{W}, \mathcal{U})$ intersects B just in those natural numbers whose singletons were chosen in $\mathcal{W}$, as all other members of $\mathcal{U}$ are disjoint from B. Therefore $\Psi(A)$ is not k-starcompact.

If $\Psi(\mathcal{A})$ is not 2-strongly starcompact, then there is an open cover $\mathcal{U}$ so that the star of every finite subcollection misses some element of ω, otherwise taking another star would give all of $\Psi(\mathcal{A})$. Now we can choose an infinite subset B with increasing enumeration $\{b_i : i < \omega\}$ of ω recursively so that for every $j < \omega$, $b_j \notin St(\{b_i : i < j\}, \mathcal{U})$. Now we claim that B is almost disjoint from every member of $\mathcal{A}$, and so $\mathcal{A}$ cannot be maximal. To see this, suppose that there is $A \in \mathcal{A}$ which intersects B infinitely. Then the neighborhood of A in $\mathcal{U}$ contains two points b_{j_0}, b_{j_1} with $j_0 < j_1$. So this neighborhood is contained in $St(b_{j_0}, \mathcal{U})$, contradicting the choice of j_1. □

Example 4.4. ($\mathfrak{d} = \omega_1$) There is a Tychonoff, strongly 2-star-Rothberger, strongly star-Lindelöf, not star-Menger space.

Proof. Let $\mathcal{A}$ be a Luzin family. Since $\Psi(\mathcal{A})$ is separable, it is strongly star-Lindelöf and, by Proposition 4.2, it is strongly 2-star-Rothberger. Since $\mathfrak{d} = \omega_1$, by Proposition 9 in [4], $\Psi(\mathcal{A})$ is not star-Menger. □

Corollary 4.5. *If $\mathfrak{d} = \omega_1$ then there is a Tychonoff, 2-star-Menger, star-Lindelöf, not star-Menger space.*

In contrast to Ψ-spaces induced from Luzin families, we also have examples of Ψ-spaces that do not hold any iteration of star-Menger property.

Example 4.6. There is a Ψ-space which is not m-star-Menger for any $m < \omega$.

Proof. Identify $\omega^{<\omega}$ with ω and let $\mathcal{A}$ be the branches of the tree $\omega^{<\omega}$. Now, for each n, let $\mathcal{U}_n$ be the open cover consisting of the sets $N_s = \{a \in \Psi(\mathcal{A}) : s \sqsubseteq a\}$, where $s \in \omega^{n+1}$, and the sets $\{x\}$ for all $x \notin \bigcup\{N_s : s \in \omega^{n+1}\}$. Then, for each $n \in \omega$, $\mathcal{U}_n$ consists of pairwise disjoint clopen sets and therefore, no new elements are picked up when taking iterations of stars.

Suppose $\mathcal{V}_n \subseteq \mathcal{U}_n$ is finite. By induction, define $x \in \mathcal{A}$ so that $N_{x\restriction(n+1)} \notin \mathcal{V}_n$. It is clear that $x \notin St^m(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for any $m, n < \omega$. □

We can also define Ψ-spaces associated to maximal almost disjoint (MAD for short) families that do not satisfy any iteration of the star-Rothberger property.

Example 4.7. There is a MAD family $\mathcal{A}$ so that $\Psi(\mathcal{A})$ is not k-star-Rothberger for any k.

Proof. Enumerate all infinite subsets of ω as $\langle b_\alpha : \alpha < \mathfrak{c}, \alpha \text{ even}\rangle$ and all functions in $\prod_n {}^n3$ as $\langle s_\alpha : \alpha < \mathfrak{c}, \alpha \text{ odd}\rangle$. Note that any ordinal can be written as $\alpha + n$ for some $n < \omega$, and "odd" and "even" above refer to the parity of this integer n.

Let $\{B_t : t \in {}^{<\omega}3\}$ be a sequence of infinite subsets of ω so that $B_\emptyset = \omega$ and for all $t \in {}^{<\omega}2$, $\{B_{t^\frown 0}, B_{t^\frown 1}, B_{t^\frown 2}\}$ partition B_t.

We will define $\mathcal{A} = \{a_\alpha : \alpha < \mathfrak{c}\}$. Suppose a_β, $\beta < \alpha$, have already been defined.

We will construct a_α infinite subsets of ω and x_α branches of the tree so that

(1) For $\beta < \alpha$, a_α is almost disjoint from a_β.
(2) For each n, $\{z_m : m \geq n\} \subseteq B_{x_\alpha \restriction n}$, where $\{z_m : m < \omega\}$ is the increasing enumeration of a_α (so that in particular, $a_\alpha \subseteq^* B_{x_\alpha \restriction n}$ for any n).

If α is even, then we will construct a_α so that

3'. If b_α is almost disjoint from each a_β, $\beta < \alpha$, then $a_\alpha \subseteq b_\alpha$.

Let us assume the case where b_α is almost disjoint from each a_β, $\beta < \alpha$, and hence the third condition applies. Note that if $3'$ is satisfied, then so is the first condition. Let x_α be a branch through the tree so that $B_{x_\alpha \restriction n} \cap b_\beta$ is infinite for each n. Define $\{z_n : n < \omega\}$ to be an increasing sequence of natural numbers so that $z_n \in B_{x_\alpha \restriction n} \cap b_\alpha$.

If α is odd, then we will construct a_α so that

3". For each n, $\{z_m : m \geq n\} \cap B_{s_\alpha(n+1)} = \emptyset$.

This construction will proceed in ω steps. Let S be the subtree of ${}^{<\omega}3$ of all t so that for all $s \leq t$, $s \neq s_\alpha(n)$, where n is the length of s. The resulting subtree S still splits at every node.

Let x_α be a branch through this tree which is not x_β for any $\beta < \alpha$. This is possible since the tree has $\mathfrak{c}$ branches.

In step n, pick $z_n \in B_{x_\alpha \restriction (n+1)}$ greater than all previously chosen z_m, $m < n$. Let $a_\alpha = \{z_n : n < \omega\}$. As $B_{x_\alpha \restriction n}$ is $\subseteq$-decreasing along the branch, this ensures that for each n, $\{z_m : m \geq n\} \subseteq B_{x_\alpha \restriction n}$.

If $\beta < \alpha$, then let i be so that $x_\alpha(i) \neq x_\beta(i)$. We have that $a_\alpha \subseteq^* B_{x_\alpha \restriction (i+1)}$ and $a_\beta \subseteq^* B_{x_\beta \restriction (i+1)}$, yet $B_{x_\alpha \restriction (i+1)}$ is disjoint from $B_{x_\beta \restriction (i+1)}$. Therefore a_α is almost disjoint from a_β.

Since $x_\alpha \restriction (n+1) \neq s_\alpha(n+1)$ and $\{z_m : m \geq n\} \subseteq B_{x_\alpha \restriction n}$, we have that 3" holds.

For each n, let $a_\alpha^{(n)}$ be the set a_α with the least n elements removed and let $\mathcal{U}_n$ be the open cover of $\Psi(\mathcal{A})$ consisting of singletons from ω together with $\{a_\alpha\} \cup a_\alpha^{(n)}$. This sequence of covers shows that $\Psi(\mathcal{A})$ is not k-star-Rothberger. For any selection of sets $\{U_n : n < \omega\}$, for each n there is some $s(n) \in {}^n3$ so that $U_n \subseteq B_{s(n)}$. Now take α so that $s = s_\alpha$. Then a_α is disjoint from the k-iterated star of U_n in $\mathcal{U}_n$ for each n by construction. □

We finish this section giving a result on Ψ-spaces that combines the style of the results in Section 2 with iterations of a star selection property introduced in Section 3.

Theorem 4.8. *For any $k \in \omega$, if X is the union of less than $\mathfrak{b}$ strongly k-star-Menger Ψ-spaces on ω, then X is strongly k-star-Menger.*

Proof. Fix $k \in \omega$. Let κ be a cardinal less than $\mathfrak{b}$. For each $\alpha < \kappa$, let Ψ_α be a Ψ-space on ω defined by an a.d. family $\mathcal{A}_\alpha$ and put $X = \bigcup_{\alpha<\kappa} \Psi_\alpha$ where each Ψ_α is strongly k-star-Menger. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X consisting of basic open sets (for each $\alpha < \kappa$, $n \in \omega$ and $U \in \mathcal{U}_n$, $|U \cap \mathcal{A}_\alpha| \leq 1$). For each $\alpha < \kappa$, let $F_n^\alpha \in [\Psi_\alpha]^{<\omega}$ such that for every $m \in \omega$, $\{St^k(F_n^\alpha, \mathcal{U}_n) : m \leq n < \omega\}$ is an open cover of Ψ_α.

Fix $\alpha < \kappa$ and $n \in \omega$. For each $A \in F_n^\alpha \cap \mathcal{A}_\alpha$, let U_A be a member of $\mathcal{U}_n$ such that $A \in U_A$. So, $U_A = \{A\} \cup A \setminus F_A$ for some $F_A \in [A]^{<\omega}$. For each $A \in F_n^\alpha \cap \mathcal{A}_\alpha$, fix

$n_A \in A \setminus F_A$. Thus, for each $\alpha < \kappa$ and for each $n \in \omega$, let

$$G_n^\alpha = (F_n^\alpha \cap \omega) \cup \{n_A : A \in F_n^\alpha \cap \mathcal{A}_\alpha\} \cup \{\bigcup F_A : A \in F_n^\alpha \cap \mathcal{A}_\alpha\}.$$

Then, for each $\alpha < \kappa$ and each $n \in \omega$, we have $G_n^\alpha \in [\omega]^{<\omega}$ and $St(F_n^\alpha, \mathcal{U}_n) \subseteq St(G_n^\alpha, \mathcal{U}_n)$. Indeed, let $x \in St(F_n^\alpha, \mathcal{U}_n)$. Then there exists $U \in \mathcal{U}_n$ such that $x \in U$ and $U \cap F_n^\alpha \neq \emptyset$. We have two cases:
If $U \cap (F_n^\alpha \cap \omega) \neq \emptyset$, then $U \cap G_n^\alpha \neq \emptyset$ and therefore, $x \in U \subseteq St(G_n^\alpha, \mathcal{U}_n)$.
If $U \cap (F_n^\alpha \cap \mathcal{A}_\alpha) \neq \emptyset$, then $U = \{A\} \cup A \setminus F$ for some $A \in F_n^\alpha \cap \mathcal{A}_\alpha$ and for some $F \in [A]^{<\omega}$. Then $x \in St(F_A \cup \{n_A\}, \mathcal{U}_n) \subseteq St(G_n^\alpha, \mathcal{U}_n)$.

We conclude that $St(F_n^\alpha, \mathcal{U}_n) \subseteq St(G_n^\alpha, \mathcal{U}_n)$.

Now, we define, for each $\alpha < \kappa$, a function $f_\alpha : \omega \to \omega$ as $f_\alpha(n) = max(G_n^\alpha)$ for each $n \in \omega$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{b}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $f_\alpha \leq^* g$. For each $n \in \omega$, let $D_n = \{i \in \omega : 0 \leq i \leq g(n)\}$. Then each D_n is a finite subset of ω and it follows that $\{St^k(D_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Indeed, let $x \in X$. Then, there exists $\alpha < \kappa$ such that $x \in \Psi_\alpha$. Since $f_\alpha \leq^* g$, there is $m \in \omega$ so that for every $n \geq m$, $f_\alpha(n) \leq g(n)$. Furthermore, since the collection $\{St^k(F_n^\alpha, \mathcal{U}_n) : m \leq n < \omega\}$ is an open cover of Ψ_α, let $n \geq m$ such that $x \in St^k(F_n^\alpha, \mathcal{U}_n)$. We obtain that $St(F_n^\alpha, \mathcal{U}_n) \subseteq St(G_n^\alpha, \mathcal{U}_n) \subseteq St(\{1, \dots, f_\alpha(n)\}, \mathcal{U}_n) \subseteq St(\{1, \dots, g(n)\}, \mathcal{U}_n) = St(D_n, \mathcal{U}_n)$. Thus, $x \in St^k(F_n^\alpha, \mathcal{U}_n) \subseteq St^k(D_n, \mathcal{U}_n)$. Therefore, the collection $\{St^k(D_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Thus, X is strongly k-star-Menger. □

5. Normal star-Menger not strongly star-Menger not Dowker space

Recall that X is a Dowker space if and only if X is normal and its Cartesian product with the closed unit interval I is not normal. Equivalently, X is normal and not countably paracompact. In [8] the following questions were posed:

Question 5.1 ([8] Question 2.4)**.** Is there a normal star-Menger space which is not strongly star-Menger?

Question 5.2 ([8] Question 2.21)**.** Are normal, countably paracompact star-Menger spaces strongly star-Menger? I.e., if X is normal, star-Menger, not strongly star-Menger, is X a Dowker Space?

In this section we present a consistent example (Example 5.4 below), of a normal star-Menger not strongly star-Menger not Dowker space. This space answers consistently in the affirmative Question 5.1 and in the negative Question 5.2.
In [29] Tall presented an example of a separable normal space with an uncountable discrete subspace. Below we provide details of the construction of such example for sake of completeness:

Example 5.3 ([29] Example E)**.** Assuming $2^{\aleph_0} = 2^{\aleph_1}$ there exists a separable normal T_1 space with an uncountable closed subspace.

Construction: Let L be a set of cardinality $\aleph_1$ disjoint from ω. The existence of a strongly independent family $\mathcal{F}$[¶] of subsets of $\aleph_0$ of size $2^{\aleph_0} = \mathfrak{c}$ is guaranteed by the Fichtenholz-Kantorovitch-Hausdorff Theorem[‖].

Write $\mathcal{F} = \{A_\alpha : \alpha < \mathfrak{c}\}$. Since $|L| = \aleph_1$, $|\mathcal{P}(L)| = 2^{\aleph_1}$. Assuming $2^{\aleph_0} = 2^{\aleph_1}$ it is possible to build a function $f : \mathcal{P}(L) \to \{A_\alpha : \alpha < \mathfrak{c}\} \cup \{\omega \setminus A_\alpha : \alpha < \mathfrak{c}\}$ which is bijective

¶For an infinite cardinal κ, a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ is called *independent* if for all pairs of disjoint $F, G \in [\mathcal{F}]^{<\omega}$ we have:

$$C_{F,G} = \bigcap_{A \in F} A \cap \bigcap_{A \in G} (\kappa \setminus A) \neq \emptyset$$

(Assume $\bigcap \emptyset = \kappa$). If in addition, for each pair (F, G) as above, $|C_{F,G}| = \kappa$, $\mathcal{F}$ is called *strongly independent*.
‖For every infinite cardinal κ there exists a strongly independent family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{F}| = 2^\kappa$.

and complement-preserving (for each $B \subseteq L$, $f(L \setminus B) = \omega \setminus f(B)$).
Now let $X = L \cup \omega$ with a subbase φ for a topology defined by

(1) if $M \subseteq L$, then $M \cup f(M) \in \varphi$,
(2) if $n \in \omega$, then $\{n\} \in \varphi$,
(3) if $p \in X$, then $X \setminus \{p\} \in \varphi$.

Observe that by condition (3) X is T_1. By (2) ω is open, therefore $L = X \setminus \omega$ is closed and, by (1) for any $x \in L$, $\{x\} \cup f(\{x\})$ is open such that $[\{x\} \cup f(\{x\})] \cap L = \{x\}$, that is L is discrete. X is separable since ω is dense in X: let U be any nonempty basic open set, then

$$U = \bigcap_{U \in F} U \cap \bigcap_{U \in G} U \cap \bigcap_{U \in H} U$$

where F, G, H are finite (possibly empty), each $U \in F$ is a subbasic open set defined as in (1), each $U \in G$ is a subbasic open set defined as in (2), and each $U \in H$ is a subbasic open set defined as in (3). To show $U \cap \omega \neq \emptyset$ it is enough to observe that $|(\bigcap_{U \in F} U) \cap \omega| = \omega$. This is always the case since $\mathcal{F}$ is a strongly independent family. Now let Y, Z be disjoint closed subsets of X and observe:

$$\begin{aligned} U_Y &= ((Y \setminus L) \cup [(Y \cap L) \cup f(Y \cap L)]) \cap (X \setminus Z) \\ &= (Y \cup f(Y \cap L)) \cap (X \setminus Z) \\ U_Z &= ((Z \setminus L) \cup [(L \setminus Y) \cup f(L \setminus Y)]) \cap (X \setminus Y) \end{aligned}$$

are open sets and $Y \subseteq U_Y$, $Z \subseteq U_Z$. Assume $x \in U_Y \cap U_Z$, then $x \in X \setminus (Y \cup Z)$ and $x \in f(Y \cap L) \cap f(L \setminus Y)$. But this is a contradiction since f is complement preserving: $f(L \setminus Y) = f(L \setminus (Y \cap L)) = \omega \setminus (Y \cap L)$. Hence, X is normal. ■

The following example presented by Song in [26] and [27] is a modification of Example 5.3. Song proved, in particular, that this space is normal, star-Lindelöf and not strongly star-Lindelöf (actually he showed something stronger: there is $\mathcal{U} \in \mathcal{O}(X)$ such that for all $L \subseteq X$ Lindelöf subspace of X, $St(L, \mathcal{U}) \neq X$).

Example 5.4 ([26], [27])**.** Assuming $2^{\aleph_0} = 2^{\aleph_1}$ there exists a normal T_1 space which is star-Lindelöf and not strongly star Lindelöf.

Construction: Let $X_0 = L \cup \omega$ denote the space built in Example 5.3. Let $X = L \cup (\omega_1 \times \omega)$ and topologize it as follows, a basic open set of

- **(i):** $x \in L$ is a set of the form $V_\alpha^U(x) = (U \cap L) \cup ((\alpha, \omega_1) \times (U \cap \omega))$ where U is a neighbourhood of $x \in X_0$ and $\alpha < \omega_1$.
- **(ii):** $\langle \alpha, n \rangle \in (\omega_1 \times \omega)$ is a set of the form $V_W(\langle \alpha, n \rangle) = W \times \{n\}$ where W is a neighbourhood of α in ω_1 with the usual topology.

Condition (i) guarantees that X is T_1. Furthermore, $\omega_1 \times \omega$ is open in X and for $x \in L$, if we let $U = \{x\} \cup f(\{x\})$. then for any $\alpha < \omega_1$, $V_\alpha^U(x) \cap L = \{x\}$. That is, L is closed and discrete in X.

X is normal: Let $Y, Z \subseteq X$ closed and disjoint. Define $Y_L = Y \cap L$ and $Z_L = Z \cap L$ and for each $n \in \omega$, $Y_n = Y \cap (\omega_1 \times \{n\})$, $Z_n = Z \cap (\omega_1 \times \{n\})$. Since $Y \cap Z = \emptyset$ and $\omega_1 \times \{n\}$ is a copy of ω_1 with the usual topology (for each $n \in \omega$), then we can find clopen sets $Y'_n, Z'_n \subseteq \omega_1 \times \{n\}$ such that $Y'_n \cap Z'_n = \emptyset$, $Y_n \subseteq Y'_n$, $Z_n \subseteq Z'_n$ and so that for each $n \in \omega$, Y'_n is cofinal in $\omega_1 \times \{n\}$ if and only if Y_n is cofinal in $\omega_1 \times \{n\}$ and Z'_n is cofinal in $\omega_1 \times \{n\}$ if and only if Z_n is cofinal in $\omega_1 \times \{n\}$. This is possible since for each $n \in \omega$, Y_n and Z_n cannot be both cofinal (otherwise $Y_n \cap Z_n \neq \emptyset$). Let

$$\mathcal{Y} = Y_L \cup \bigcup_{n \in \omega} Y'_n, \qquad \mathcal{Z} = Z_L \cup \bigcup_{n \in \omega} Z'_n$$

Observe $Y \subseteq \mathcal{Y}$, $Z \subseteq \mathcal{Z}$ and $\mathcal{Y} \cap \mathcal{Z} = \emptyset$.
Claim: $\mathcal{Y}$ and $\mathcal{Z}$ are closed in X.
Indeed, if $\langle \alpha, m\rangle \in (\omega_1 \times \omega) \setminus \mathcal{Y}$, since Y'_m is clopen in $\omega_1 \times \{m\}$, then there is U open neighbourhood of $\langle \alpha, m\rangle$ in $\omega_1 \times \{m\}$ (and therefore open neighbourhood in X), such that $U \cap Y'_m = \emptyset$. Now, let $x \in L \setminus \mathcal{Y}$ and assume that for each U open neighbourhood of x in X_0 and each $\alpha < \omega_1$, $V^U_\alpha(x) \cap \mathcal{Y} \neq \emptyset$. This implies that for each U open neighbourhood of x in X_0 and each $\alpha < \omega_1$ there is some $n \in \omega$ such that $V^U_\alpha(x) Y'_n \neq \emptyset$ and Y'_n is cofinal in $\omega_1 \times \{n\}$. Then Y_n is cofinal in $\omega_1 \times \{n\}$ and $V^U_\alpha(x) Y_n \neq \emptyset$. Hence, $x \in \overline{Y} = Y$ which is a contradition. Thus, $\mathcal{Y}$ is closed. A similar argument shows that $\mathcal{Z}$ is closed.
Since Y_L and Z_L are disjoint closed subsets of X_0 and X_0 is normal (recall X_0 is the space constructed in Example 5.3), then there exist disjoint open sets U_Y, U_Z in X_0 such that $Y_L \subseteq U_Y$, $Z_L \subseteq U_Z$. Let

$$V_Y = (U_Y \cap Y) \cup \bigcup_{n \in U_Y \cap \omega} (\omega_1 \times \{n\}), \qquad V_Z = (U_Z \cap Z) \cup \bigcup_{n \in U_Z \cap \omega} (\omega_1 \times \{n\}).$$

Observe that V_Y and V_Z are disjoint open subsets in X and $Y_L \subseteq V_Y$, $Z_L \subseteq V_Z$. Let $W_Y = \mathcal{Y} \cup (V_Y \setminus \mathcal{Z})$, $W_Z = \mathcal{Z} \cup (V_Z \setminus \mathcal{Y})$. Hence, W_Y and W_Z are open sets in X, $W_Y \cap W_Z = \emptyset$, and $y \subseteq W_Y$, $Z \subseteq W_Z$.

X **is not strongly star-Lindelöf:** List $L = \{x_\alpha : \alpha < \omega_1\}$. Since L is a closed discrete subset of X_0, for $\alpha < \omega_1$ let D_α be an open neighbourhood of x_α in X_0 such that $D_\alpha \cap L = \{x_\alpha\}$. Hence,

$$\mathcal{U} = \{V^{D_\alpha}_\alpha(x_\alpha) : \alpha < \omega_1\} \cup \{\omega_1 \times \omega\} \in \mathcal{O}(X).$$

Assume $E \in [X]^\omega$, we show $St(E, \mathcal{U}) \neq X$. Since E is countable, fix $\beta_0, \beta_1 < \omega_1$ such that $sup\{\alpha : x_\alpha \in E \cap L\} < \beta_0$ and $sup\{\gamma : \langle \gamma, n\rangle \in E \textit{ for some } n \in \omega\} < \beta_1$. Let $\alpha = max\{\beta_0, \beta_1\}$ and observe $E \cap V^{D_\alpha}_\alpha(x_\alpha) = \emptyset$ Since $V^{D_\alpha}_\alpha(x_\alpha)$ is the only element of $\mathcal{U}$ that contains x_α, then $x_\alpha \notin St(E, \mathcal{U})$. Thus, X is not strongly star-Lindelöf.

X **is star-Lindelöf:** Let $\mathcal{U} \in \mathcal{O}(X)$ and define

$$M = \{n \in \omega : (\exists U \in \mathcal{U})(\exists \beta < \omega_1)[(\beta, \omega_1) \times \{n\} \subseteq U]\}.$$

For each $n \in M$ fix $U_n \in \mathcal{U}$ and $\beta_n < \omega_1$ such that $(\beta, \omega_1) \times \{n\} \subseteq U_n$. Put $\mathcal{V}' = \{U_n : n \in M\}$.
Claim: $L \subseteq St(\bigcup \mathcal{V}', \mathcal{U})$.
Indeed, let $x \in L$, there is $U^x \in \mathcal{U}_n$ such that $x \in U^x$ and therefore, there is U open neighbourhood of x in X_0 and $\alpha < \omega_1$ such that $V^U_\alpha(x) \subseteq U^x$. Since $V^U_\alpha(x) \cap (\omega_1 \times \omega) = (\alpha, \omega_1) \times (U \cap \omega)$ and $U = N \cup f(N)$ for some $N \subseteq L$, with $x \in N$, it holds true that $n \in f(N) \to n \in M$. Then, for $n \in f(N)$, $V^U_\alpha(x) \cap (\omega_1 \times \{n\}) \cap [(\beta_n, \omega_1) \times \{n\}] \neq \emptyset$. Thus, $V^U_\alpha(x) \cap U_n \neq \emptyset$. Hence, $U^x \cap U_n \neq \emptyset$. Therefore $x \in St(U_n, \mathcal{U}) \subseteq St(\bigcup \mathcal{V}', \mathcal{U})$.
Now, $\omega_1 \times \omega$ is a countable union of strongly starcompact spaces, then there is a countable $\mathcal{V}'' \subseteq \mathcal{U}$ such that $\omega_1 \times \omega \subseteq St(\bigcup \mathcal{V}'', \mathcal{U})$. If we let $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$, then $St(\bigcup \mathcal{V}, \mathcal{U}) = X$. ■

Proposition 5.5. *Assuming $2^{\aleph_0} = 2^{\aleph_1}$ and $\aleph_1 < \mathfrak{d}$ the space X built in Example 5.4 is normal, star-Menger, and is not either strongly star-Menger nor Dowker.*

Proof. It has been shown that X is normal and not strongly star-Lindelöf (in particular, X is not strongly star-Menger). It remains to show that it is star-Menger and is not a Dowker space.

X **is star-Menger**: let $(\mathcal{U}_n : n \in \omega)$ be any sequence of open covers of X. Write $L = \{x_\alpha : \alpha < \omega_1\}$ and for each $\alpha < \omega_1$ and each $n \in \omega$, let $f_\alpha(n) = min\{i \in \omega : (\exists U \in \mathcal{U}_n)(\exists \beta < \omega_1)[x_\alpha \in U \wedge (\beta, \omega_1) \times \{i\} \subseteq U]\}$. Observe that for each $\alpha < \omega_1$, $f_\alpha : \omega \to \omega$ is well defined. Since $\{f_\alpha : \alpha < \omega_1\}$ has size less than $\mathfrak{d}$, there is a funtion $g \in \omega^\omega$ such that for all $\alpha < \omega_1 : g \not\leq^* f_\alpha$. For $n \in \omega$ let

$$M_n = \{i \in \omega : (\exists U \in \mathcal{U}_n)(\exists \beta < \omega_1)[(\beta, \omega_1) \times \{i\} \subseteq U]\}.$$

Now, for each $n \in \omega$ and each $i \in M_n$, fix $U_n^i \in \mathcal{U}_n$ and $\beta_n^i < \omega_1$ such that $(\beta_n^i, \omega_1) \times \{i\} \subseteq U_n^i$ and let $\mathcal{V}_n = \{U_n^i : i \in M_n \cap g(n)\}$.
Claim: $L \subseteq \bigcup\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$.
Indeed, fix $x_\alpha \in L$. There is $n \in \omega$ such that $f_\alpha(n) < g(n)$. Hence, there are $U \in \mathcal{U}_n$ and $\beta < \omega_1$ such that $x_\alpha \in U$ and $(\beta, \omega_1) \times \{f_\alpha(n)\} \subseteq U$. Thus, $f_\alpha(n) \in M_n$ and $U_n^{f_\alpha(n)} \in \mathcal{V}_n$. In addition, $U_n^{f_\alpha(n)} \cap U \neq \emptyset$. Hence, $x \in St(\bigcup \mathcal{V}_n, \mathcal{U}_n) \subseteq \bigcup\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$.

X **it is not a Dowker space**: Let us recall the following characterization: A normal space D is a Dowker space (see [24]) if, and only if, D has a countable increasing open cover $\{U_n : n \in \omega\}$ such that there is no closed cover $\{F_n : n \in \omega\}$ of D with $F_n \subseteq U_n$ for each $n \in \omega$. Hence, let $\{U_n : n \in \omega\}$ be any countable increasing open cover ($U_0 \subseteq U_1 \subseteq \cdots$) of X, we must find a countable cover of closed sets $\{F_n : n \in \omega\}$, such that for each $n \in \omega$, $F_n \subseteq U_n$.

For each $i \in \omega$ define $n_i = min\{n \in \omega : i \leq n \wedge (\exists \gamma < \omega_1)[[\gamma, \omega_1) \times \{i\} \subseteq U_n]\}$. Observe that since $\{U_n : n \in \omega\}$ is a countable cover of X, n_i is well defined for each $i \in \omega$. In addition, for each $n \in \omega$ and $i \in \omega$ with $i \leq n_i \leq n$ let

$$\gamma_i^n = min\{\gamma < \omega_1 : [\gamma, \omega_1) \times \{i\} \subseteq U_n\} \qquad (*)$$

Since for each $n \in \omega$, $U_n \subseteq U_{n+1}$, then γ_i^n is well defined. Now, for $n \in \omega$ let

$$F_n = (\bigcup_{i \leq n} \{[\gamma_i^n, \omega_1) \times \{i\} : i \leq n_i \leq n\}) \cup (U_n \cap L).$$

Claim:

- **(1):** For each $n \in \omega$, F_n is closed,
- **(2):** For each $n \in \omega$, $F_n \subseteq U_n$,
- **(3):** $\bigcup_{n \in \omega} F_n = X$.

Indeed, to show (1), fix $n \in \omega$. First assume $x \in (X \setminus F_n) \cap (\omega_1 \times \omega)$. Hence $x = \langle \alpha, m \rangle$ for some $\alpha < \omega_1$ and $m \in \omega$. If $F_n \cap (\omega_1 \times \{m\}) = \emptyset$, any $U \subseteq \omega_1 \times \{m\}$ open neighbourhood of x is disjoint from F_n. If $F_n \cap (\omega_1 \times \{m\}) \neq \emptyset$, then $\alpha < \gamma_m^n$ and for each $\beta < \alpha$, $(\beta, \alpha] \times \{m\}$ is an open neighbourhood of x disjoint from F_n. Now, assume $x \in (X \setminus F_n) \cap L$, let $N \subseteq L$ such that $N \cap F_n = \emptyset$ and $x \in N$. Observe that $U = N \cup f(N) \setminus (n+1) = (N \cup f(N)) \cap (\bigcap_{j \leq n+1}(X_0 \setminus \{j\}))$ is an open neighbourhood of x in X_0 (see condition (1) and (3) of Example 5.3). Hence, for any $\alpha < \omega_1$, $V_\alpha^U(x)$ ($= [U \cap L] \cup [(\alpha, \omega_1) \times (U \cap \omega)]$) is an open neighbourhood of x in X such that $V_\alpha^U(x) \cap F_n = \emptyset$ since $F_n \subseteq \omega_1 \times [0, n]$ and $V_\alpha^U(x) \cap (\omega_1 \times [0, n]) = \emptyset$. Thus, F_n is closed.
To show (2), fix $n \in \omega$. If $x \in F_n \cap L$, then $x \in U_n$. If $x = \langle \alpha, m \rangle \in F_n \cap (\omega_1 \times \omega)$, then there is some $i \leq n_i \leq n$ such that $\langle \alpha, m \rangle \in [\gamma_i^n, \omega_1) \times \{i\}$. Thus , $m = i$ and $[\gamma_i^n, \omega_1) \times \{i\} \subseteq U_n$. Hence $F_n \subseteq U_n$.
Let us show (3). If $x \in X \cap L$, then there is some $n \in \omega$ such that $x \in U_n$. Hence, $x \in U_n \cap L \subseteq F_n$. If $x \in X \setminus L$, there is some $i \in \omega$ such that $x \in \omega_1 \times \{i\}$. By $(*)$ and the fact that $U_n \subseteq U_{n+1}$, $\{\gamma_i^n : n \in \omega\}$ is a decreasing sequence of ordinals. Since $U_n : n \in \omega)$ covers X, there is some $m \in \omega$ such that $\gamma_i^m = 0$. Thus, $x \in F_m$. □

Acknowledgment. The authors are grateful to the referees for their valuable comments and suggestions the helped to improve the writing and notation of this paper. The first-listed author was supported for this research by Consejo Nacional de Ciencia y Tecnología (CONACYT, México), Scholarship 769010.

References

[1] M. Bonanzinga, *Star-Lindelöf and absolutely star-Lindelöf spaces*, Quest. Answ. Gen. Topol. **16**, 79104, 1998.

[2] M. Bonanzinga, F. Cammaroto, Lj.D.R. Kočinac, *Star- Hurewicz and related properties*, Appl. Gen.Topol. **5**, 79-89, 2004.

[3] M. Bonanzinga, F. Maesano, *Selectively strongly star-Menger spaces and related properties*, Atti della Accademia Peloritana dei Pericolanti, **99** (2), A2 2021.

[4] M. Bonanzinga, M. Matveev, *Some covering properties for Ψ-spaces*, Mat. Vesn. **61**, 3-11, 2009.

[5] L. Bukovský, J. Haleš, *On Hurewicz properties*, Topology Appl. **132**, 71-79, 2003.

[6] D.K. Burke, *Covering properties*, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 347-422, 1984.

[7] J. Casas-de la Rosa, S. A. Garcia-Balan, *Variations of star selection principles on small spaces*, Filomat **36** (14), 4903-4917, 2022.

[8] J. Casas-de la Rosa, S. A. Garcia-Balan, P. J. Szeptycki, *Some star and strongly star selection principles*, Topology Appl. **258**, 572-587, 2019.

[9] D. Chandra, N. Alam, *On certain star versions of the Scheepers property*, arXiv: 2207.08595 [math.GN]

[10] M. V. Cuzzupè, Some selective and monotone versions of covering properties and some results on the cardinality of a topological space, PhD thesis, University of Catania, 2017.

[11] E.K. van Douwen, G.M. Reed, A.W. Roscoe, I.J. Tree, *Star covering properties*, Topology Appl. **39**, 71-103, 1991.

[12] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, Sigma Series in Pure Mathematics **6**, 1989.

[13] S. A. Garcia-Balan, Results on star selection principles and weakenings of normality in Ψ-spaces, PhD thesis, York University, 2020.

[14] F. Hernández-Hernández, M. Hrušák, *Topology of Mrówka-Isbell spaces.* In: Hrušák, Tamariz, Tkachenko (Eds.), Pseudocompact Topological Spaces, Springer International Publishing AG, 2018.

[15] M. Hruák, O. Guzmán, *n-Luzin gaps*, Topology Appl. **160**, 1364-1374, 2013.

[16] W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, Math. Z. **24** (1), 401-421, 1925.

[17] Lj.D.R. Kočinac, *Star-Menger and related spaces*, Publ. Math. (Debr.) **55**, 421-431, 1999.

[18] N.N. Luzin, *On subsets of the series of natural numbers*, Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 403410, 1947.

[19] Lj.D.R. Kočinac, *Star selection principles: A survey*, Khayyam J. Math. **1**, 82-106, 2015.

[20] M.V. Matveev, *A survey on star covering properties*, Topology Atlas, Preprint No. **330**, 1998.

[21] K.Menger, *Einige überdeckungssätze der Punltmengen-lehre*, Sitzungberichte Abt.2a, Mathematik, Astronomie, Physik, Meteorologie and Mechanik (Wiener Akademie, Wien) **133**, 421-444, 1924.

[22] D. Repovš, L. Zdomskyy, *On the Menger covering property and D-spaces*, Proc. Amer. Math. Soc. **140** (3), 10691074, 2012.

[23] F. Rothberger, *Eine Verschärfung der Eigenschaft C*, Fund. Math. **30**, 50-55, 1938.
[24] M. E. Rudin, *Dowker Spaces*, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 761-780, 1984.
[25] M. Scheepers, *Combinatorics of open covers I: Ramsey Theory*, Topology Appl. **69**, 31-62, 1996.
[26] Y.-K. Song, *Remarks on countability and star covering properties*, Topology Appl. **158**, 1121-1123, 2011.
[27] Y.-K. Song, *Remarks on neighborhood star-Lindelöf spaces II*, Filomat **27** (5), 875-880, 2013.
[28] Y.-K. Song, X. Wei-Feng, *Remarks on new star-selection principles in topology*, Topology Appl. **268**, 106921, 2019.
[29] F. D. Tall, *Normality versus collectionwise normality*, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 685-732, 1984.
[30] F. D. Tall, *Lindelöf spaces which are Menger, Hurewicz, Alster, productive, or D*, Topology Appl. **158** (18), 2556-2563, 2011.
[31] Ian J. Tree, *Constructing regular 2-starcompact spaces that are not strongly 2-star-Lindelöf*, Topology Appl. **47**, 129-132, 1992.