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U_NApproximation for the Eigenvalue Spectrum of Time-Dependent, One-Speed and One-Dimensional Neutron Transport Problem with Anisotropic Scattering

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Abstract: The eigenvalue spectrum is studied in one-speed time-dependent neutron transport theory in a uniform finite slab. The linear anisotropic scattering kernel together with the combination of forward and backward scattering is used and time-dependent neutron transport equation is reduced to a stationary one. Numerical results of eigenvalues for various combinations of the scattering parameters and the selected values of the time decay constant are performed by using the Chebyshev polynomials approximation method.

Key words: Neutron transport equation, anisotropic scattering, eigenvalues, U_N approximation

Zamana Bağlı, Tek-Hızlı ve Tek-Boyutlu Anizotropik Saçılmalı Nötron Transport Probleminin Özdeğer Spektrumu için U_NYaklaşımı

Özet: Düzgün sonlu bir dilimde tek-hızlı zamana bağlı nötron transport teorisinde özdeğer spektrumu çalışılmıştır. İleri ve geri saçılmayla birlikte lineer anizotropik saçılma foksniyonu kullanılmış ve zamana bağlı nötron transport denklemi kararlı denkleme indirgenmiştir. Nümerik özdeğerler, Chebyshev polinomları yaklaşımı yöntemi kullanılarak, zaman bozunum sabitinin seçilmiş bazı değerleri ve saçılma parametrelerinin farklı değerleri için hesaplanmıştır.

Anahtar kelimeler: Nötron transport denklemi, anizotropik saçılma, özdeğerler, $U_N$yaklaşımı

1. Introduction

In a nuclear reactor, the number of neutrons decreases with time after a short neutron pulse. Then, basic and higher order eigenvalues of time (decay constants) can be calculated by taking into account behavior of the neutrons migrated in the system. Fundamental time eigenvalue is defined as the smallest number of λ's for the time-dependent system which have a neutron distribution with a term exp(λt). Therefore, when the neutron flux density in a pulsed neutron experiment decays exponentially it is possible to remove the time dependence from the neutron flux and to get a stationary one in transport theory. The time eigenvalues of many systems can be determined by calculating the critical size of those systems in stationary condition [1,2].

One of the important problems in the solution of neutron transport equation is the calculation of eigenvalues. In nuclear reactor physics, the important coefficients such as diffusion length, diffusion coefficient and buckling depend on the parameter *c*, the number of secondary neutrons per collision. Since the eigenvalues depend on the values of *c*, it is known as the fundamental eigenvalue in the transport theory.

In this study, a new theoretical scheme is described for solving the time-dependent neutron transport equation in plane geometry. First, the neutron angular flux is expanded in terms of the second kind of Chebyshev polynomials. It is then used in transport equation together with the linear anisotropic scattering kernel including backward and forward scattering. Then, time-dependent transport equation reduced to a stationary transport equation by applying the procedure described above. Finally, the eigenvalue spectrum is obtained and numerical results are given in the tables for various values of the anisotropic scattering parameters and time decay constants.

2. The Second Kind of Chebyshev Polynomials (U_N) Method

The linear transport equation for a time-dependent neutron population without source can be written as [1],

$$\frac{1}{v}\frac{\partial\psi(r,\mathbf{\Omega},t)}{\partial t}=-\mathbf{\Omega}\cdot\nabla\psi(r,\mathbf{\Omega},t)-\sigma_T^*\psi(r,\mathbf{\Omega},t)+\sigma_S^*\int f(\mathbf{\Omega}'\cdot\mathbf{\Omega})\psi(r,\mathbf{\Omega}',t)\,\mathrm{d}\mathbf{\Omega}' \tag{1}$$

where $\mathbf{\Omega}'$ is the direction of neutron velocity before (and $\mathbf{\Omega}$ after) a collision and $\psi(r,\mathbf{\Omega},t)$ is the time-dependent angular flux of neutrons at position r in direction $\mathbf{\Omega}$. σ_T^* and σ_S^* denote the macroscopic total and scattering differential cross-sections of the time-dependent system, respectively; v is the average neutron velocity. $f(\mathbf{\Omega}'\cdot\mathbf{\Omega})$ is the scattering kernel and it is assumed to be of the form of a combination of linearly anisotropic scattering and backward-forward-isotropic scattering. The time dependence of the angular flux of neutrons is assumed to be exponential with time and given by [1],

$$\psi(\mathbf{r},\mathbf{\Omega},t)=\psi(\mathbf{r},\mathbf{\Omega})\exp(-\lambda t)\,. \tag{2}$$

The time-independent stationary transport equation for one-dimensional case can be written as,

$$\begin{aligned}&\mu\frac{\partial\psi(x,\mu)}{\partial x}+\sigma_T(1-\alpha c)\psi(x,\mu)\\&=\frac{c\sigma_T}{2}(1-\alpha-\beta)\int_{-1}^{1}\psi(x,\mu')(1+3b_1\mu\mu')\,\mathrm{d}\mu'+\beta c\sigma_T\psi(x,-\mu)\end{aligned} \tag{3}$$

subject to free space boundary and symmetry conditions:

$$\psi(a,\mu)=0\,, \tag{4a}$$

$$\psi(x,\mu)=\psi(-x,\mu),\qquad \mu>0\,. \tag{4b}$$

Here α and β parameters denote the forward and backward scattering probabilities in a collision, respectively and they vary over the range of $0\le\alpha,\beta\le 1$, $\alpha+\beta\le 1$ and. The parameter b_1 is the average cosine of the scattering angle and is restricted to the range $|b_1| \le 1/3$ to ensure the positivity of the scattering function for all angles [1]. σ_T is the total macroscopic cross-section, c is the mean number of secondary neutrons per collision in the time-independent critical system and a is the critical half thickness of the finite homogeneous slab in units of mean free path. By following the same general notation as those of Sahni et al. [1], the time decay constant Λ can be related with the c as,

$$\Lambda = 1 - \frac{1}{c} \tag{5}$$

In this study, the neutron angular flux is expanded in terms of the Chebyshev polynomials of second kind as in the previous works [3],

$$\psi(x,\mu) = \frac{2}{\pi}\sum_{n=0}^{\infty}\Phi_n(x)U_n(\mu),\; -a \le x \le a, -1 \le \mu \le 1. \tag{6}$$

By inserting Eq.(6) into Eq.(1) and using the recurrence and orthogonality relations of the Chebyshev polynomials of second kind, one can obtain the U_N moments of the angular flux for n = 0 and n = 1, respectively;

$$\frac{d\Phi_1(x)}{dx} + 2\sigma_T\left(1 - c\left(\alpha+\beta\right)\right)\Phi_0(x) = 2c\sigma_T\left(1-\alpha-\beta\right)\sum_{n=0}^{\infty}\frac{\Phi_{2n}(x)}{2n+1}, \tag{7a}$$

$$\frac{d\Phi_2(x)}{dx} + \frac{d\Phi_0(x)}{dx} + 2\sigma_T\left[1 - c\left(\alpha-\beta\right)\right]\Phi_1(x) = 6b_1c\sigma_T\left(1-\alpha-\beta\right)\sum_{n=0}^{\infty}\frac{n\Phi_{2n-1}(x)}{4n^2-1}, \tag{7b}$$

and in general,

$$\frac{d\Phi_{n+1}(x)}{dx} + \frac{d\Phi_{n-1}(x)}{dx} + 2\sigma_T\left[1 - c\left(\alpha+\left(-1\right)^n\beta\right)\right]\Phi_n(x) = 0, n \ge 2. \tag{7c}$$

A well-known procedure for the eigenvalue spectrum is applied in the form of [4],

$$\Phi_n(x) = G_n\left(\upsilon\right)\exp\left(\sigma_T x/\upsilon\right) \tag{8}$$

and by inserting Eq. (8) in Eqs.(7) one can obtain analytical expressions for n = 0, 1 and in general, respectively;

$$G_1(\nu) + 2\nu\left[1 - c(\alpha+\beta)\right]G_0(\nu) = 2\nu c(1-\alpha-\beta)\sum_{n=0}^{\infty}\frac{G_{2n}(\nu)}{2n+1}, \tag{9a}$$

$$G_2(\nu) + G_0(\nu) + 2\nu\left[1 - c(\alpha-\beta)\right]G_1(\nu) = 6b_1\nu\, c(1-\alpha-\beta)\sum_{n=1}^{\infty}\frac{nG_{2n-1}(\nu)}{4n^2-1}, \tag{9b}$$

$$G_{n+1}(\nu)+G_{n-1}(\nu)+2\nu\left\{1-c\left[\alpha+(-1)^{n}\beta\right]\right\}G_{n}(\nu)=0\,, \qquad n\geq 2. \tag{9c}$$

Here, the normalizations $G_{-1}(\upsilon)=0$ and $G_{0}(\upsilon)=1$ are used. Setting $G_{N+1}(\upsilon)=0$ is identical with $\Phi_{N+1}(x)=0$ in U_{N} approximation and other polynomial expansion based techniques; therefore the discrete and continuum eingenvales (υ) may be calculated for various values of c, α, β and b_1. The other way to compute eigenvalues is to use the determinant of the coefficients matrix of $G_n(\upsilon)$,

$$\left[\mathbf{M}(\upsilon)\right]\mathbf{G}(\upsilon)=\mathbf{0}, \tag{10}$$

where $\mathbf{M}(\upsilon)$ is the $(N+1)\times(N+1)$ coefficient matrix, $\mathbf{0}$ is a null vector and $\mathbf{G}(\upsilon)=\left[G_0, G_1, ..., G_N\right]^T$.

Table 1. Eigenvalue spectrum for forward scattering (α = 0.3 β = 0.0)

Λ	b_1	$N = 4$	$N = 6$	$N = 8$	$N = 10$
		6.027434i	6.021161i	6.021224i	6.021223i
		0.624152	0.414057	0.301062	0.238856
	-0.3	-	1.082479	0.859564	0.686140
		-	-	1.242446	1.070303
		-	-	-	1.314906
		6.881926i	6.876434i	6.876476i	6.876476i
		0.624313	0.414021	0.301074	0.238851
0.0099	0.0	-	1.082589	0.859513	0.686173
		-	-	1.242494	1.070274
		-	-	-	1.314928
		8.248743i	8.244161i	8.244186i	8.244186i
		0.624475	0.413985	0.301087	0.238846
	0.3	-	1.082700	0.859462	0.686207
		-	-	1.242541	1.070245
		-	-	-	1.314950
		1.256274i	1.227171i	1.231290i	1.230467i
		0.736310	0.488752	0.347064	0.273272
	-0.3	-	1.204820	0.967334	0.769808
		-	-	1.365650	1.184053
		-	-	-	1.438921
		1.464334i	1.440329i	1.443178i	1.442725i
		0.745744	0.488844	0.347847	0.273263
0.1666	0.0	-	1.211624	0.969623	0.772573
		-	-	1.369176	1.186327
		-	-	-	1.440897
		1.852166i	1.834921i	1.836443i	1.836276i
		0.757193	0.488944	0.348667	0.273254
	0.3	-	1.221182	0.972540	0.775854
		-	-	1.374456	1.189484
		-	-	-	1.443962

		0.539726i	0.464582i	0.456200i	0.434115i
		1.278792	0.956408	0.675996	0.551185
	-0.3	-	1.941681	1.597050	1.277870
		-	-	2.189654	1.913075
		-	-	-	2.304528
		0.664403i	0.578930i	0.554821i	0.535271i
		1.487368	1.050162	0.746806	0.583846
0.5000	0.0	-	2.031785	1.669034	1.354252
		-	-	2.232771	1.958734
		-	-	-	2.328269
		1.165834i	1.091129i	0.983094i	0.988384i
		3.790778i	3.651811i	3.662755i	3.661979i
	0.3	-	1.386406	0.961407	0.654621
		-	-	1.994687	1.647346
		-	-	-	2.211459

Table 2. Eigenvalue spectrum for isotropic scattering (α = 0.0 β = 0.0)

Λ	b_1	N = 4	N = 6	N = 8	N = 10
		5.041691i	5.038026i	5.038051i	5.038051i
		0.434424	0.288244	0.209662	0.166363
	-0.3	-	0.754118	0.598735	0.477987
		-	-	0.865790	0.745745
		-	-	-	0.916378
		5.754027i	5.750816i	5.750833i	5.750833i
		0.434500	0.288226	0.209668	0.166360
0.0099	0.0	-	0.754169	0.598709	0.478003
		-	-	0.865812	0.745730
		-	-	-	0.916388
		6.890942i	6.888261i	6.888271i	6.888271i
		0.434577	0.288209	0.209674	0.166358
	0.3	-	0.754221	0.598683	0.478019
		-	-	0.865833	0.745714
		-	-	-	0.916398
		1.044406i	1.029511i	1.031180i	1.030961i
		0.459087	0.303786	0.217502	0.171686
	-0.3	-	0.766541	0.612922	0.487912
		-	-	0.871969	0.754483
		-	-	-	0.919826
		1.210088i	1.197559i	1.198700i	1.198581i
		0.462075	0.303635	0.217734	0.171662
0.1666	0.0	-	0.768865	0.613381	0.488749
		-	-	0.873177	0.755073
		-	-	-	0.920499
		1.501750i	1.492094i	1.492727i	1.492681i
		0.465406	0.303478	0.217972	0.171637
	0.3	-	0.771748	0.613913	0.489674
		-	-	0.874746	0.755801
		-	-	-	0.921395
		0.388283i	0.356252i	0.357995i	0.354266i
		0.509016	0.355490	0.247097	0.194197
	-0.3	-	0.780305	0.636335	0.508269

		-	-	0.877790	0.765559
		-	-	-	0.922880
		0.4601975i	0.430653i	0.430950i	0.428767i
		0.5432450	0.363842	0.252612	0.195500
0.5000	0.0	-	0.797756	0.647670	0.519441
		-	-	0.886432	0.773642
		-	-	-	0.927692
		0.635379i	0.616934i	0.615418i	0.615126i
		0.622125	0.377095	0.260039	0.197036
	0.3	-	0.849553	0.673315	0.540865
		-	-	0.917114	0.795849
		-	-	-	0.947090

Table 3. Eigenvalue spectrum for isotropic scattering (α = 0.0 β = 0.3)

Λ	b_1	$N = 4$	$N = 6$	$N = 8$	$N = 10$
		4.668376i	4.664044i	4.664083i	4.664082i
		0.456549	0.302822	0.220196	0.174693
	-0.3	-	0.791743	0.628653	0.501842
		-	-	0.908719	0.782791
		-	-	-	0.961706
		5.033318i	5.029301i	5.029331i	5.029331i
		0.456611	0.302808	0.220200	0.174691
0.0099	0.0	-	0.791786	0.628633	0.501855
		-	-	0.908738	0.782779
		-	-	-	0.961715
		5.500143i	5.496468i	5.496491i	5.496491i
		0.456675	0.302794	0.220205	0.174690
	0.3	-	0.791829	0.628613	0.501868
		-	-	0.908756	0.782768
		-	-	-	0.961723
		0.928498i	0.910266i	0.912639i	0.912214i
		0.508403	0.335325	0.238373	0.187466
	-0.3	-	0.828819	0.664394	0.529067
		-	-	0.938031	0.813048
		-	-	-	0.987776
		1.004558i	0.988090i	0.990045i	0.989734i
		0.511593	0.335356	0.238629	0.187463
0.1666	0.0	-	0.831195	0.665178	0.529998
		-	-	0.939278	0.813841
		-	-	-	0.988480
		1.105392i	1.090939i	1.092465i	1.092256i
		0.515087	0.335387	0.238891	0.187460
	0.3	-	0.833979	0.666056	0.531007
		-	-	0.940784	0.814765
		-	-	-	0.989343
		0.311974i	0.270716i	0.261299i	0.251000i
		0.704783	0.508911	0.361744	0.286860
	-0.3	-	0.997466	0.820737	0.663175
		-	-	1.107114	0.970073
		-	-	-	1.158884

		0.332201i	0.289465i	0.277410i	0.267636i
		0.743684	0.525081	0.373403	0.291923
0.5000	0.0	-	1.015893	0.834517	0.677126
		-	-	1.116385	0.979367
		-	-	-	1.164135
		0.358164i	0.313829i	0.298299i	0.289339i
		0.803207	0.546876	0.388371	0.297993
	0.3	-	1.047629	0.855463	0.697003
		-	-	1.133885	0.994896
		-	-	-	1.174690

3. Results and Discussion

In this study, we investigated the applicability of the second kind of Chebyshev polynomial approximation method for the general eigenvalue spectrum of the time-dependent neutron transport equation in slab geometry. Numerically calculated eigenvalues for $c>1$ are given in Tables 1, 2 and 3 for forward, isotropic and backward scattering, respectively.

The analytical expressions for all $G_n(\nu)$ are present in Eqs.(9) and by using these the discrete and continuum ν eigenvalues can be calculated by setting $G_{N+1}(\nu)=0$ for various values of c, b_1, α and β. This comes from essential idea of the U_N method for which $\Phi_{N+1}(x)=0$ as in the case of spherical harmonics (P_N) method. As can be seen in Table 1 when $c > 1$ then at least one pair of the roots is purely imaginary and the others lie as pairs in the interval $[-1, 1]$ as expected since it is possible to see in literature [5, 6] that when $c > 1$ asymptotic roots or discrete eigenvalues are purely imaginary.

As a result, the Chebyshev polynomials of second kind other than the traditional Legendre polynomials are proved to be applied to solve the time-dependent neutron transport equation. For this purpose, the eigenvalue spectrum of the neutrons is investigated as a first step successfully. Therefore, one can use this spectrum to solve other problems such as criticality, albedo, scalar flux, etc in transport theory.

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