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Rough statistical convergence of double sequences in intuitionistic fuzzy normed spaces

Ahmet Özcan[1], **Aykut Or**[2]

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Rough convergence, Statistical convergence, Rough statistical convergence, Intuitionistic fuzzy normed spaces

Abstract — This paper proposes rough convergence and rough statistical convergence of a double sequence in intuitionistic fuzzy normed spaces. It then defines the rough statistical limit points and rough statistical cluster points of a double sequence in these spaces. Afterwards, this paper examines some of their basic properties. Finally, it discusses the need for further research.

Subject Classification (2020): 40A35, 40G15.

1. Introduction

Based on the concept of density of positive natural numbers, statistical convergence was independently defined by Fast [1] and Steinhaus [2] in 1951. Moreover, Zygmund [3] studied the concept of statistical convergence under the name of almost all convergence in 1939. Afterward, in 2003, Mursaleen and Edely [4] investigated the concept of statistical convergence in double sequence space.

Phu [5] has defined the concept of rough convergence in finite dimensional normed spaces as a natural generalization of ordinary convergence. He has also shown that the set LIM_x^r, the set of all the rough limit points, is bounded, closed, and convex. Using the concept of natural density, Aytar [6] has defined the concept of rough statistical convergence. Furthermore, Malik and Maity [7, 8] have studied the concepts of rough convergence and rough statistical convergence of double sequences in normed linear spaces, respectively. Besides, many studies on these concepts have been conducted [9–11].

The theory of fuzzy sets was introduced by Zadeh [12] in 1965. Then, the concept of fuzzy norms on a linear space was proposed by Cheng and Mordeson [13], and some properties of the fuzzy norm have been studied [14]. Atanassov [15, 16] has proposed the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Park [17] has suggested the concept of intuitionistic fuzzy metric space. After, Saadati and Park

[1]ozcannaahmet@gmail.com; [2]aykutor@comu.edu.tr (Corresponding Author)
[1,2]Department of Mathematics, Faculty of Sciences, Çanakkale Onsekiz Mart University, Çanakkale, Türkiye
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[18] defined the concept of intuitionistic fuzzy normed space (IFNS). Moreover, they have also introduced the concepts of convergence and the Cauchy sequences in an IFNS. Afterward, Karakuş et al. [19] proposed and investigated the concept of statistical convergence in an IFNS. Savaş and Gürdal [20] have suggested a generalization of statistical convergence in an IFNS. Mursaleen [21] has defined the concept of statistical convergence of double sequences in an IFNS. Moreover, Antal et al. [22] have proposed the concept of rough statistical convergence in an IFNS.

The present paper can be summarized as follows: In the second part of the present study, some basic definitions and properties to be required for the next section are provided. Section 3 proposes the concepts of rough convergence and rough statistical convergence of double sequences in an IFNS and studies some of their basic properties. Moreover, it defines the concepts of rough statistical (r-st) limit points and r-st cluster points of a double sequence in an IFNS and investigates some of their basic properties. The final section discusses the need for further research.

2. Preliminary

This section presents some of the basic definitions to be required in the next sections.

Definition 2.1. [7] Let (x_{jk}) be a double sequence in a normed space $(\mathbb{X}, \|\cdot\|)$ and $r \geq 0$. Then, the double sequence (x_{jk}) is said to be rough convergent (r-convergent) to $x_0 \in \mathbb{X}$, if for all $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $j, k \geq N_\varepsilon$,

$$\left\|x_{jk} - x_0\right\| < r + \varepsilon$$

It is denoted by $x_{jk} \xrightarrow{r} x_0$. The element x_0 is called an r-limit point of the double sequence (x_{jk}).

Definition 2.2. [4] The double natural density of the set $A \subseteq \mathbb{N} \times \mathbb{N}$ is defined by

$$\delta_2(A) = \lim_{m,n\to\infty} \frac{|\{(j,k) \in A : j \leq m \text{ and } k \leq n\}|}{mn}$$

where $|\{(j,k) \in A : j \leq m \text{ and } k \leq n\}|$ denotes the number of elements of A not exceeding m and n, respectively. It can be observed that if the set A is finite, then $\delta_2(A) = 0$.

Definition 2.3. [8] Let (x_{jk}) be a double sequence in a normed space $(\mathbb{X}, \|\cdot\|)$ and $r \geq 0$. Then, (x_{jk}) is referred to as r-statistically convergent to $x_0 \in \mathbb{X}$, if for all $\varepsilon > 0$,

$$\delta_2\left(\left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \left\|x_{jk} - x_0\right\| \geq r + \varepsilon\right\}\right) = 0$$

In this case we write $x_{jk} \xrightarrow{r-st_2} x_0$. The element x_0 is called an r-statistical limit point of the double sequence (x_{jk}).

Definition 2.4. [22] An intuitionistic fuzzy normed space (IFNS) is the triplet $(\mathbb{X}, \mu, \nu)$ with linear space $\mathbb{X}$ and fuzzy sets μ, ν on $\mathbb{X} \times \mathbb{R}$, if the following conditions for all $x, y \in \mathbb{X}$ and $s, u \in \mathbb{R}$ are valid:

i. $\mu(x;u) = 0$ and $\nu(x;u) = 1$, for $u \notin \mathbb{R}^+$

ii. $\mu(x;u) = 1$ and $\nu(x;u) = 0$, for $u \in \mathbb{R}^+ \Leftrightarrow x = 0$

iii. $\mu(\alpha x;u) = \mu\left(x; \frac{u}{|\alpha|}\right)$ and $\nu(\alpha x;u) = \nu\left(x; \frac{u}{|\alpha|}\right)$, for $\alpha \neq 0$

iv. $\min\{\mu(x;s), \mu(y;u)\} \leq \mu(x+y; s+u)$ and $\max\{\nu(x;s), \nu(y;u)\} \geq \nu(x+y; s+u)$

v. $\lim_{u\to\infty}\mu(x;u)=1$, $\lim_{u\to 0}\mu(x;u)=0$, $\lim_{u\to\infty}\nu(x;u)=0$, and $\lim_{u\to 0}\nu(x;u)=1$

Moreover, the ordered pair (μ,ν) is called an intuitionistic fuzzy norm.

Example 2.5. [18] Let $(\mathbb{X},\|\cdot\|)$ be a normed space and for all $u>0$ and $x\in\mathbb{X}$,

$$\mu(x;u)=\frac{u}{u+\|x\|} \text{ and } \nu(x;u)=\frac{\|x\|}{u+\|x\|}$$

Since the conditions in Definition 2.4 are valid, the triplet $(\mathbb{X},\mu,\nu)$ is an IFNS.

Definition 2.6. [18] Let $(\mathbb{X},\mu,\nu)$ be an IFNS, $x\in\mathbb{X}$, $\varepsilon\in(0,1)$, and $u>0$. Then, the open ball with center x and radius ε is the set $B(x,\varepsilon,u)=\{y\in\mathbb{X}:\mu(x-y;u)>1-\varepsilon \text{ and } \nu(x-y;u)<\varepsilon\}$.

Definition 2.7. [21] Let $(\mathbb{X},\mu,\nu)$ be an IFNS. Then, a double sequence (x_{jk}) in $\mathbb{X}$ is called convergent to $x_0\in\mathbb{X}$ with respect to the intuitionistic fuzzy norm (μ,ν), if there exists $N_\varepsilon\in\mathbb{N}$ for all $u>0$ and $\varepsilon\in(0,1)$,

$$\mu\left(x_{jk}-x_0;u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-x_0;u\right)<\varepsilon, \text{ for all } j,k\geq N_\varepsilon$$

and denoted by $(\mu,\nu)-\lim_{j,k\to\infty}x_{jk}=x_0$ or $x_{jk}\xrightarrow{(\mu,\nu)}x_0$.

Definition 2.8. [21] Let $(\mathbb{X},\mu,\nu)$ be an IFNS. Then, a double sequence $\left(x_{jk}\right)$ is said to be statistically convergent to $x_0\in\mathbb{X}$ with respect to the intuitionistic fuzzy norm (μ,ν), for all $u>0$ and $\varepsilon\in(0,1)$, if

$$\delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-x_0;u\right)\leq 1-\varepsilon \text{ or } \nu\left(x_{jk}-x_0;u\right)\geq\varepsilon\right\}\right)=0$$

and denoted by $st_2^{(\mu,\nu)}-\lim_{j,k\to\infty}x_{jk}=x_0$.

3. Rough Statistical Convergence

This section defines the concepts of rough convergence and rough statistical convergence of double sequences in an IFNS and examines some of their basic properties.

Definition 3.1. Let $(\mathbb{X},\mu,\nu)$ be an IFNS and $r\geq 0$. Then, a double sequence (x_{jk}) in $\mathbb{X}$ is said to be r-convergent to $x_0\in\mathbb{X}$, with respect to the norm (μ,ν), if there exists $N_\varepsilon\in\mathbb{N}$ for all $u>0$ and $\varepsilon\in(0,1)$ such that

$$\mu(x_{jk}-x_0;r+u)>1-\varepsilon \text{ and } \nu(x_{jk}-x_0;r+u)<\varepsilon, \text{ for all } j,k\geq N_\varepsilon$$

In this case, we write $r_{(\mu,\nu)}-\lim_{j,k\to\infty}x_{jk}=x_0$ or $x_{jk}\xrightarrow{r_{(\mu,\nu)}}x_0$, where x_0 is called an $r_{(\mu,\nu)}-$limit point of the double sequence (x_{jk}).

Note 3.2. For $r=0$, the concept of rough convergence in IFNSs becomes the concept of ordinary convergence in IFNSs.

The $r_{(\mu,\nu)}-$limit point of a double sequence may not be unique. Therefore, the set of all the $r_{(\mu,\nu)}$-limit points for a double sequence (x_{jk}) is as follows:

$${}_{\nu}^{\mu}LIM_{x_{jk}}^{r}:=\left\{x_0\in\mathbb{X}:x_{jk}\xrightarrow{r_{(\mu,\nu)}}x_0\right\}$$

If ${}_{\nu}^{\mu}LIM_{x_{jk}}^{r}\neq\emptyset$, the double sequence (x_{jk}) is rough convergent.

Definition 3.3. Let $(\mathbb{X},\mu,\nu)$ be an IFNS and $r \geq 0$. Then, a double sequence (x_{jk}) in $\mathbb{X}$ is referred to as rough statistically convergent to $x_0 \in \mathbb{X}$ with respect to the norm (μ,ν), for all $u > 0$ and $\varepsilon \in (0,1)$, if

$$\delta_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk}-x_0; r+u) \leq 1-\varepsilon \text{ or } \nu(x_{jk}-x_0; r+u) \geq \varepsilon\}) = 0$$

and denoted by $r -_\nu^\mu st_2 - \lim_{j,k\to\infty} x_{jk} = x_0$ or $x_{jk} \xrightarrow{r-_\nu^\mu st_2} x_0$.

Note 3.4. If $r = 0$, then rough statistical convergence coincide with statistical convergence in IFNSs

The rough statistical limit of a double sequence may not be unique. Hence, the set of rough statistical limit points is denoted as follows:

$$st_2 -_\nu^\mu LIM_{x_{jk}}^r := \left\{x_0 \in \mathbb{X} : x_{jk} \xrightarrow{r-_\nu^\mu st_2} x_0\right\}$$

Let (x_{jk}) be a unbounded sequence. Then, ${}_\nu^\mu LIM_{x_{jk}}^r$ is empty set. However, this is not achived in the case of rough statistical convergence. Hence, $st_2 -_\nu^\mu LIM_{x_{jk}}^r$ may not be empty set.

Example 3.5. Let us consider a real normed space $(\mathbb{X},\|\cdot\|)$ and, for all $u > 0$ and $x \in \mathbb{X}$,

$$\mu(x,u) = \frac{u}{u+\|x\|} \text{ and } \nu(x,u) = \frac{\|x\|}{u+\|x\|}$$

Then, the triplet $(\mathbb{X},\mu,\nu)$ is an IFNS. For all $j,k \in \mathbb{N}$, define

$$x_{jk} = \begin{cases} (-1)^{j+k}, & j \text{ and } k \text{ are non-squares} \\ jk, & \text{otherwise} \end{cases}$$

Then,

$$st_2 -_\nu^\mu LIM_{x_{jk}}^r = \begin{cases} \emptyset, & r < 1 \\ [1-r, r-1], & r \geq 1 \end{cases}$$

and $LIM_{x_{jk}}^r = \emptyset$, for all $r \geq 0$.

Afterward, the concept of rough statistically bounded sequence in an IFNS is as follows:

Definition 3.6. Let $(\mathbb{X},\mu,\nu)$ be an IFNS and $r \geq 0$. Then, a double sequence (x_{jk}) in $\mathbb{X}$ is said to be statistical bounded with respect to the norm (μ,ν), if there exists a real number $M > 0$ for all $u > 0$ and $\varepsilon \in (0,1)$ such that

$$\delta_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk}; M) \leq 1-\varepsilon \text{ or } \nu(x_{jk}; M) \geq \varepsilon\}) = 0$$

Definition 3.7. [8] A double subsequence $x' = \left(x_{j_p k_q}\right)$ of a double sequence $x = \left(x_{jk}\right)$ is called a dense subsequence, if $\delta_2\left(\left\{\left(j_p, k_q\right) \in \mathbb{N}\times\mathbb{N} : p,q \in \mathbb{N}\right\}\right) = 1$.

Example 3.8. Let us consider the IFNS in Example 3.5 and, for all $j,k \in \mathbb{N}$, define

$$x_{jk} = \begin{cases} jk, & j \text{ and } k \text{ are squares} \\ 0, & \text{otherwise} \end{cases}$$

Thus, $st_2 -_\nu^\mu LIM_{x_{jk}}^r = [-r,r]$. Moreover, for the subsequence $x' = (x_{j_t k_t})$ of (x_{jk}) such that j_t and k_t are squares, $st_2 -_\nu^\mu LIM_{x'}^r = \emptyset$. It can be seen that $st_2 -_\nu^\mu LIM_{x_{jk}}^r \nsubseteq st_2 -_\nu^\mu LIM_{x'}^r$. However, this inclusion for the rough statistical convergent sequences and their dense subsequences in an IFNS is valid. The following theorem explains this state.

Theorem 3.9. Let $(\mathbb{X},\mu,\nu)$ be an IFNS. If $x'=\left(x_{j_pk_q}\right)$ is a dense subsequence of $x=\left(x_{jk}\right)$, then

$$st_2-^{\mu}_{\nu}LIM^r_x \subseteq st_2-^{\mu}_{\nu}LIM^r_{x'}$$

Proof.

The proof is obvious.

Theorem 3.10. Let $(\mathbb{X},\mu,\nu)$ be an IFNS. A double sequence (x_{jk}) in $\mathbb{X}$ is statistically bounded if and only if $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}\neq\emptyset$, for all $r>0$.

Proof.

Let $(\mathbb{X},\mu,\nu)$ be an IFNS.

$(\Rightarrow)$: Let a double sequence (x_{jk}) be statistically bounded in the IFNS. Then, for all $u>0$, $\varepsilon\in(0,1)$, and $r>0$, there exists a real number $M>0$ such that

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk};M)\leq 1-\varepsilon \text{ or } \nu(x_{jk};M)\geq\varepsilon\})=0$$

Let $K=\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk};M)\leq 1-\varepsilon \text{ or } \nu(x_{jk};M)\geq\varepsilon\}$. For $k\in K^c$, $\mu(x_{jk};M)>1-\varepsilon$ and $\nu(x_{jk};M)<\varepsilon$. Moreover,

$$\mu(x_{jk};r+M)\geq\ \min\{\mu(0;r),\mu(x_{jk};M)\}=\min\{1,\mu(x_{jk};M)\}>1-\varepsilon$$

and

$$\nu(x_{jk};r+M)\leq\ \max\{\nu(0;r),\nu(x_{jk};M)\}=\max\{0,\nu(x_{jk};M)\}<\varepsilon$$

Hence, $0\in st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}$. Consequently, $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}\neq\emptyset$.

$(\Leftarrow)$: Let $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}\neq\emptyset$, for all $r>0$. Then, there exists $x_0\in\mathbb{X}$ such that $x_0\in st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}$. For all $u>0$ and $\varepsilon\in(0,1)$,

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-x_0;r+u)\leq 1-\varepsilon \text{ or } \nu(x_{jk}-x_0;r+u)\geq\varepsilon\})=0$$

Therefore, almost all x_{jk} are contained in some ball with center x_0 which implies that double sequence (x_{jk}) is statistically bounded in an IFNS. Theorem 3.11 shows that rough statistical convergence of a double sequences in an IFNS has many arithmetic properties similar to those of ordinary convergence.

Theorem 3.11. Let (x_{jk}) and (y_{jk}) be two double sequences in an IFNS. Then, for all $r\geq 0$, the following holds:

i. If $x_{jk}\xrightarrow{r-^{\mu}_{\nu}st_2}x_0$ and $\alpha\in\mathbb{F}$, then $\alpha x_{jk}\xrightarrow{r-^{\mu}_{\nu}st_2}\alpha x_0$.

ii. If $x_{jk}\xrightarrow{r-^{\mu}_{\nu}st_2}x_0$ and $y_{jk}\xrightarrow{r-^{\mu}_{\nu}st_2}y_0$, then $x_{jk}+y_{jk}\xrightarrow{r-^{\mu}_{\nu}st_2}x_0+y_0$.

Proof.

Let (x_{jk}) and (y_{jk}) be two double sequences in an IFNS and $r\geq 0$.

i. Let $x_{jk}\xrightarrow{r-^{\mu}_{\nu}st_2}x_0$ and $\alpha\in\mathbb{F}$. Therefore, if for all $u>0$ and $\varepsilon\in(0,1)$,

$$K=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-x_0;\frac{r+u}{|\alpha|}\right)\leq 1-\varepsilon \text{ or } \nu\left(x_{jk}-x_0;\frac{r+u}{|\alpha|}\right)\geq\varepsilon\right\}$$

then $\delta_2(K)=0$. Let $(t,s)\in K^c$. Then, $\mu\left(x_{ts}-x_0;\frac{r+u}{|\alpha|}\right)>1-\varepsilon$ and $\nu\left(x_{ts}-x_0;\frac{r+u}{|\alpha|}\right)<\varepsilon$. Hence,

$$\mu(\alpha x_{ts}-\alpha x_0;r+u)=\mu\left(x_{ts}-x_0;\frac{r+u}{|\alpha|}\right)>1-\varepsilon \tag{3.1}$$

and

$$\nu(\alpha x_{ts}-\alpha x_0;r+u)=\nu\left(x_{ts}-x_0;\frac{r+u}{|\alpha|}\right)<\varepsilon \tag{3.2}$$

Let $H=\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(\alpha x_{jk}-\alpha x_0;r+u)>1-\varepsilon$ and $\nu(\alpha x_{jk}-\alpha x_0;r+u)<\varepsilon\}$. From Equations (3.1) and (3.2), $(t,s)\in H$. Therefore, $K^c\subseteq H$. Consequently, $\alpha x_{jk}\xrightarrow{r-^{\mu}_{\nu}st_2}\alpha x_0$.

ii. Let $x_{jk}\xrightarrow{r-^{\mu}_{\nu}st_2}x_0$ and $y_{jk}\xrightarrow{r-^{\mu}_{\nu}st_2}y_0$. Therefore, if for all $u>0$ and $\varepsilon\in(0,1)$,

$$A=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-x_0;\frac{r+u}{2}\right)\le 1-\varepsilon \text{ or } \nu\left(x_{jk}-x_0;\frac{r+u}{2}\right)\ge\varepsilon\right\}$$

and

$$B=\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(y_{jk}-y_0;r+u)\le 1-\varepsilon \text{ or } \nu(y_{jk}-y_0;r+u)\ge\varepsilon\}$$

Then, $\delta_2(A)=0$ and $\delta_2(B)=0$. Let $(t,s)\in A^c\cap B^c$. Then,

$$\mu\left(x_{ts}-x_0;\frac{r+u}{2}\right)>1-\varepsilon \text{ and } \nu\left(x_{ts}-x_0;\frac{r+u}{2}\right)<\varepsilon$$

and

$$\mu\left(y_{ts}-y_0;\frac{r+u}{2}\right)>1-\varepsilon \text{ and } \nu\left(y_{ts}-y_0;\frac{r+u}{2}\right)<\varepsilon$$

Hence,

$$\mu(x_{ts}+y_{ts}-(y_0+x_0);r+u)\ge\min\left\{\mu\left(x_{ts}-x_0;\frac{r+u}{2}\right),\mu\left(y_{ts}-y_0;r+\frac{r+u}{2}\right)\right\}>1-\varepsilon \tag{3.3}$$

and

$$\nu(x_{ts}+y_{ts}-(y_0+x_0);r+u)\ge\max\left\{\nu\left(x_{ts}-x_0;\frac{r+u}{2}\right),\nu\left(y_{ts}-y_0;\frac{r+u}{2}\right)\right\}<\varepsilon \tag{3.4}$$

Let $C=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}+y_{jk}-(x_0+y_0);r+u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}+y_{jk}-(x_0+y_0);r+u\right)<\varepsilon\right\}$. From the Equations (3.3) and (3.4), $(t,s)\in C$. Therefore, $C\subseteq A^c\cap B^c$. Consequently, $x_{jk}+y_{jk}\xrightarrow{r-^{\mu}_{\nu}st_2}x_0+y_0$.

Theorem 3.12 and Theorem 3.13 prove some topological properties of the r-statistical limit set of a double sequence in IFNSs.

Theorem 3.12. Let (x_{jk}) be a double sequence in an IFNS $(\mathbb{X},\mu,\nu)$ and $r\ge 0$. Then, the set $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}$ is closed.

Proof.

Let (x_{jk}) be a double sequence in an IFNS and $r\ge 0$. If $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}=\emptyset$, then the theorem is valid. Therefore, let $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}\ne\emptyset$, for all $r\ge 0$ and $y_0\in\overline{st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}}$. Then, $y_{jk}\in st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}$ such that $y_{jk}\xrightarrow{(\mu,\nu)}y_0$. Then, for all $u>0$ and $\varepsilon\in(0,1)$, there exists a $k_1\in\mathbb{N}$ such that

$$\mu\left(y_{jk}-y_0;\frac{u}{2}\right)>1-\varepsilon \text{ and } \nu\left(y_{jk}-y_0;\frac{u}{2}\right)<\varepsilon, \text{ for all } j,k\ge k_1$$

Let $y_{mn} \in st_2 -_\nu^\mu LIM_{x_{jk}}^r$ for $m, n > k_1$ such that

$$\delta_2\left((j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - y_{mn}; r + \frac{u}{2}\right) \le 1-\varepsilon \text{ or } \nu\left(x_{jk} - y_{mn}; r + \frac{u}{2}\right) \ge \varepsilon\right) = 0$$

For $(t,s) \in A = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - y_{mn}, r + \frac{u}{2}\right) > 1-\varepsilon \text{ and } \nu\left(x_{jk} - y_{mn}, r + \frac{u}{2}\right) < \varepsilon\}$

$$\mu\left(x_{ts} - y_{mn}, r + \frac{u}{2}\right) > 1-\varepsilon \text{ and } \nu\left(x_{ts} - y_{mn}, r + \frac{u}{2}\right) < \varepsilon$$

Then,

$$\mu(x_{ts} - y_0, r+u) \ge \min\left\{\mu\left(x_{jk} - y_{mn}, r + \frac{u}{2}\right), \mu\left(y_{mn} - y_0, \frac{u}{2}\right)\right\} > 1-\varepsilon \tag{3.5}$$

and

$$\nu(x_{ts} - y_0, r+u) \le \max\left\{\mu\left(x_{jk} - y_{mn}, r + \frac{u}{2}\right), \mu\left(y_{mn} - y_0, \frac{u}{2}\right)\right\} < \varepsilon \tag{3.6}$$

Let $B = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - y_0, r+u\right) > 1-\varepsilon \text{ and } \nu\left(x_{jk} - y_0, r+u\right) < \varepsilon\}$. From the Equations (3.5) and (3.6), $(t,s) \in B$. Since $A \subseteq B$, then $\delta_2(A) \le \delta_2(B)$. Consequently,

$$y_0 \in st_2 -_\nu^\mu LIM_{x_{jk}}^r$$

Theorem 3.13. Let (x_{jk}) be a double sequence in an IFNS $(\mathbb{X}, \mu, \nu)$ and $r \ge 0$. Then, the set $st_2 -_\nu^\mu LIM_{x_{jk}}^r$ is convex.

Proof.
Let (x_{jk}) be a double sequence in an IFNS, $r \ge 0$, and $x_1, x_2 \in st_2 -_\nu^\mu LIM_{x_{jk}}^r$. For the convexity of the set $st_2 -_\nu^\mu LIM_{x_{jk}}^r$, we should show that $[(1-\lambda)x_1 + \lambda x_2] \in st_2 -_\nu^\mu LIM_{x_{jk}}^r$, for all $\lambda \in (0,1)$. For all $u > 0$ and $\varepsilon \in (0,1)$, let

$$M_1 = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - x_1; \frac{r+u}{2(1-\lambda)}\right) \le 1-\varepsilon \text{ or } \nu\left(x_{jk} - x_1; \frac{r+u}{2(1-\lambda)}\right) \ge \varepsilon\right\}$$

and

$$M_2 = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk} - x_2; \frac{r+u}{2\lambda}\right) \le 1-\varepsilon \text{ or } \nu\left(x_{jk} - x_2; \frac{r+u}{2\lambda}\right) \ge \varepsilon\right\}$$

By assumption we have $\delta_2(M_1) = 0$ and $\delta_2(M_2) = 0$. For $k \in M_1^c \cap M_2^c$,

$$\begin{aligned}
\mu(x_{jk} - [(1-\lambda)x_1 + \lambda x_2]; r+u) &= \mu((1-\lambda)(x_{jk} - x_1) + \lambda(x_{jk} - x_2); r+u) \\
&\ge \min\left\{\mu\left((1-\lambda)(x_{jk} - x_1); \frac{r+u}{2}\right)\mu\left(\lambda(x_{jk} - x_2); \frac{r+u}{2}\right)\right\} \\
&= \min\left\{\mu\left(x_{jk} - x_1; \frac{r+u}{2(1-\lambda)}\right), \mu\left(x_{jk} - x_2; \frac{r+u}{2\lambda}\right)\right\} \\
&> 1-\varepsilon
\end{aligned}$$

and

$$\begin{aligned}\nu(x_{jk}-[(1-\lambda)x_1+\lambda x_2];r+u) &= \nu((1-\lambda)(x_{jk}-x_1)+\lambda(x_{jk}-x_2);r+u)\\ &\geq \max\left\{\nu\left((1-\lambda)(x_{jk}-x_1);\frac{r+u}{2}\right),\nu\left(\lambda(x_{jk}-x_2);\frac{r+u}{2}\right)\right\}\\ &= \max\left\{\nu\left(x_{jk}-x_1;\frac{r+u}{2(1-\lambda)}\right),\nu\left(x_{jk}-x_2;\frac{r+u}{2\lambda}\right)\right\}\\ &> \varepsilon\end{aligned}$$

Thus,

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-[(1-\lambda)x_1+\lambda x_2];r+u)\leq 1-\varepsilon \text{ or } \nu(x_k-[(1-\lambda)x_1+\lambda x_2];r+u)\geq 1-\varepsilon\})=0$$

Consequently, $[(1-\lambda)x_1+\lambda x_2]\in st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}$ and so $st_2-^{\mu}_{\nu}LIM^r_{x_{jk}}$ is a convex set.

Theorem 3.14. Let (x_{jk}) be a double sequence in an IFNS $(\mathbb{X},\mu,\nu)$ and $r\geq 0$. If there exists a double sequence (y_{jk}) in $\mathbb{X}$, statistically convergent to $x_0\in\mathbb{X}$ with respect to the norm (μ,ν) and, for all $\varepsilon\in(0,1)$ and $j,k\in\mathbb{N}$, $\mu(x_{jk}-y_{jk};r)>1-\varepsilon$ and $\nu(x_{jk}-y_{jk};r)<\varepsilon$, then (x_{jk}) is rough statistically convergent to $x_0\in\mathbb{X}$ with respect to the norm (μ,ν).

Proof.

Let (x_{jk}) be a double sequence in an IFNS, $r\geq 0$, $u>0$ and there exists a double sequence (y_{jk}) in $\mathbb{X}$ such that $y_{jk}\xrightarrow{st_2^{(\mu,\nu)}}x_0$ and $\mu(x_{jk}-y_{jk};r)>1-\varepsilon$ and $\nu(x_{jk}-y_{jk};r)<\varepsilon$, for all $j,k\in\mathbb{N}$. For given $\varepsilon\in(0,1)$, let

$$A=\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(y_{jk}-x_0;u)\leq 1-\varepsilon \text{ or } \nu(y_{jk}-x_0;u)\geq\varepsilon\}$$

and

$$B=\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-y_{jk};r)\leq 1-\varepsilon \text{ or } \nu(x_{jk}-y_{jk};r)\geq\varepsilon\}$$

Clearly, $\delta_2(A)=0$ and $\delta_2(B)=0$. For $(j,k)\in A^c\cap B^c$,

$$\mu(x_{jk}-x_0;r+u)\geq\min\{\mu(x_{jk}-y_{jk};r),\mu(y_{jk}-x_0;u)\}>1-\varepsilon$$

and

$$\nu(x_{jk}-x_0;r+u)\leq\max\{\nu(x_{jk}-y_{jk};r),\nu(y_{jk}-x_0;u)\}<\varepsilon$$

Then, for all $(j,k)\in A^c\cap B^c$,

$$\mu(x_{jk}-x_0;r+u)>1-\varepsilon \text{ and } \nu(x_{jk}-x_0;r+u)<\varepsilon$$

This implies that

$$\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-x_0;r+u)\leq 1-\varepsilon \text{ or } \nu(x_{jk}-x_0;r+u)\geq\varepsilon\}\subseteq A\cup B$$

Then,

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-x_0;r+u)\leq 1-\varepsilon \text{ or } \nu(x_{jk}-x_0;r+u)\geq\varepsilon\})\leq\delta_2(A)+\delta_2(B)$$

Hence, $\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-x_0;r+u)\le 1-\varepsilon \text{ or } \nu(x_{jk}-x_0;r+u)\ge\varepsilon\})=0$. Consequently, $x_{jk}\xrightarrow{r-_{\nu}^{\mu}st_2} x_0$.

Theorem 3.15. Let $x=(x_{jk})$ be a sequence in an IFNS and $r>0$. Then, there does not exist $y,z\in st_2-_{\nu}^{\mu} LIM^r_{x_{jk}}$ such that $\mu(y-z;mr)\le 1-\varepsilon$ or $\nu(y-z;mr)\ge\varepsilon$, for $\varepsilon\in(0,1)$ and $m>2$.

Proof.

Let (x_{jk}) be a sequence in an IFNS and $r>0$. Assume that there exists $y,z\in st_2-_{\nu}^{\mu} LIM^r_{x_{jk}}$ such that for $m>2$,

$$\mu(y-z;mr)\le 1-\varepsilon \text{ or } \nu(y-z;mr)\ge\varepsilon$$

For given $\varepsilon\in(0,1)$ and $u>0$. K_1 and K_2 are denoted as follows:

$$K_1=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-y;r+\frac{u}{2}\right)\le 1-\varepsilon \text{ or } \nu\left(x_{jk}-y;r+\frac{u}{2}\right)\ge\varepsilon\right\}$$

and

$$K_2=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-z;r+\frac{u}{2}\right)\le 1-\varepsilon \text{ or } \nu\left(x_{jk}-z;r+\frac{u}{2}\right)\ge\varepsilon\right\}$$

Hence, $\delta(K_1)=0$ and $\delta(K_2)=0$. For $(j,k)\in K_1^c\cap K_2^c$,

$$\mu(y-z;2r+u)\ge\min\left\{\mu\left(x_{jk}-z;r+\frac{u}{2}\right),\mu\left(x_{jk}-y;r+\frac{u}{2}\right)\right\}>1-\varepsilon$$

and

$$\nu(y-z;2r+u)\le\max\left\{\nu\left(x_{jk}-z;r+\frac{u}{2}\right),\nu\left(x_{jk}-y;r+\frac{u}{2}\right)\right\}<\varepsilon$$

Hence, $\mu(y-z;2r+u)>1-\varepsilon$ and $\nu(y-z;2r+u)<\varepsilon$. Then,

$$\mu(y-z;mr)>1-\varepsilon \text{ or } \nu(y-z;mr)<\varepsilon, \text{ for } m>2$$

which is contradiction to the hypothesis. Therefore, there does not exists $y,z\in st_2-_{\nu}^{\mu} LIM^r_{x_{jk}}$ such that $\mu(y-z;mr)\le 1-\varepsilon$ or $\nu(y-z;mr)\ge\varepsilon$, for $\varepsilon\in(0,1)$ and $m>2$.

Next, the concept of rough statistical cluster points of a double sequence in an IFNS is defined, and some related results are proposed.

Definition 3.16. Let $(\mathbb{X},\mu,\nu)$ be an IFNS, $\gamma\in\mathbb{X}$, and $r\ge 0$. Then, γ is called rough statistical cluster point of the double sequence (x_{jk}) in $\mathbb{X}$ with respect to the norm (μ,ν) (briefly, $r-_{\nu}^{\mu} st_2$-cluster point of(x_{jk})) if for all $u>0$ and $\varepsilon\in(0,1)$

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-\gamma;r+u)>1-\varepsilon \text{ and } \nu(x_{jk}-\gamma;r+u)<\varepsilon\})>0$$

The set of all the $r-_{\nu}^{\mu} st_2$-cluster points of $x=(x_{jk})$ in an IFNS is denoted by $\Gamma^r_{(\mu,\nu)_2}(x)$. If $r=0$, then $\Gamma^r_{(\mu,\nu)_2}(x)=\Gamma_{(\mu,\nu)_2}(x)$.

Theorem 3.17. Let $x=(x_{jk})$ be a double sequence in an IFNS and $r\ge 0$. Then, $\Gamma^r_{(\mu,\nu)_2}(x)$ is a closed set.

Proof.

Let (x_{jk}) be a double sequence in an IFNS and $r\ge 0$.

i. If $\Gamma^r_{(\mu,\nu)_2}(x)=\emptyset$, then the theorem is valid.

ii. Let $\Gamma^r_{(\mu,\nu)_2}(x) \neq \emptyset$ and $y_0 \in \overline{\Gamma^r_{(\mu,\nu)_2}(x)}$. Then, there is a double sequence (y_{jk}) in $\Gamma^r_{(\mu,\nu)_2}(x)$ such that $y_{jk} \xrightarrow{(\mu,\nu)} y_0$, for all $j,k \in \mathbb{N}$. It is sufficient to show that $y_0 \in \Gamma^r_{(\mu,\nu)_2}(x)$. As $y_{jk} \xrightarrow{(\mu,\nu)} y_0$, for all $u>0$ and $\varepsilon \in (0,1)$, there exists $k_\varepsilon \in \mathbb{N}$ such that $\mu\left(y_{jk}-y_0;\frac{u}{2}\right)>1-\varepsilon$ and $\nu\left(y_{jk}-y_0;\frac{u}{2}\right)<\varepsilon$, for all $j,k \geq k_\varepsilon$. Let $j_0,k_0 \in \mathbb{N}$ such that $j_0,k_0 \geq k_\varepsilon$. Then,

$$\mu\left(y_{j_0k_0}-y_0;\frac{u}{2}\right)>1-\varepsilon \text{ and } \nu\left(y_{j_0k_0}-y_0;\frac{u}{2}\right)<\varepsilon$$

Since $y_{jk} \in \Gamma^r_{(\mu,\nu)_2}(x)$, $y_{j_0k_0} \in \Gamma^r_{(\mu,\nu)_2}(x)$. Thus,

$$\delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(y_{jk}-y_{j_0k_0};r+\frac{u}{2}\right)>1-\varepsilon \text{ and } \nu\left(y_{jk}-y_{j_0k_0};r+\frac{u}{2}\right)<\varepsilon\right\}\right)>0$$

Let

$$A=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-y_{j_0k_0};r+\frac{u}{2}\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-y_{j_0k_0};r+\frac{u}{2}\right)<\varepsilon\right\}$$

Choose $(t,s)\in A$. Then, $\mu\left(x_{ts}-y_{j_0k_0};r+\frac{u}{2}\right)>1-\varepsilon$ and $\nu\left(x_{jk}-y_{j_0k_0};r+\frac{u}{2}\right)<\varepsilon$. Therefore,

$$\mu\left(x_{ts}-y_0;r+u\right)\geq\min\left\{\mu\left(x_{ts}-y_{j_0k_0};r+\frac{u}{2}\right),\mu\left(y_{j_0k_0}-y_0;\frac{u}{2}\right)\right\}>1-\varepsilon \tag{3.7}$$

and

$$\nu\left(x_{ts}-y_0;r+u\right)\leq\max\left\{\nu\left(x_{ts}-y_{j_0k_0};r+\frac{u}{2}\right),\nu\left(y_{j_0k_0}-y_0;\frac{u}{2}\right)\right\}<\varepsilon \tag{3.8}$$

Let $B=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-y_0;r+u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-y_0;r+\frac{u}{2}\right)<\varepsilon\right\}$. From the Equations (3.7) and (3.8), $(t,s)\in B$. Thereby, $A\subseteq B$ and so $\delta_2(A)\leq\delta_2(B)$. Therefore,

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-y_0;r+u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-y_0;r+u\right)<\varepsilon\})>0$$

Consequently, $y_0\in\Gamma^r_{(\mu,\nu)_2}(x)$.

Theorem 3.18. Let $x=(x_{jk})$ be a double sequence in an IFNS. Then, for an arbitrary $\gamma\in\Gamma_{(\mu,\nu)_2}(x)$ and $\varepsilon\in(0,1)$, $\mu(\xi-\gamma;r)>1-\varepsilon$ and $\nu(\xi-\gamma;r)<\varepsilon$, for all $\xi\in\Gamma^r_{(\mu,\nu)_2}(x)$.

Proof.

Let $x=(x_{jk})$ be a double sequence in an IFNS and $\gamma\in\Gamma_{(\mu,\nu)_2}(x)$. Then, for all $u>0$ and $\varepsilon\in(0,1)$,

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-\gamma;u)>1-\varepsilon \text{ and } \nu(x_{jk}-\gamma;u)<\varepsilon\})>0$$

Let $A=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-\gamma;u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-\gamma;u\right)<\varepsilon\right\}$. Choose $(t,s)\in A$.

Then, $\mu\left(x_{ts}-\gamma;u\right)>1-\varepsilon$ and $\nu\left(x_{ts}-\gamma;u\right)<\varepsilon$. Thus,

$$\mu\left(x_{ts}-\xi;r+u\right)\geq\min\left\{\mu\left(x_{ts}-\gamma;u\right),\mu(\xi-\gamma;r)\right\}>1-\varepsilon \tag{3.9}$$

and

$$\nu\left(x_{ts}-\xi;r+u\right)\leq\max\left\{\nu\left(x_{ts}-\gamma;u\right),\nu(\xi-\gamma;r)\right\}<\varepsilon \tag{3.10}$$

Let $B=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-\xi;r+u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-\xi;r+u\right)<\varepsilon\right\}$. From the Equations (3.9) and (3.10),

$(t,s) \in B$. Thereby $A \subseteq B$, $\delta_2(A) \leq \delta_2(B)$. Therefore,

$$\delta_2\left(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk}-\xi; r+u) > 1-\varepsilon \text{ and } \nu(x_{jk}-\xi; r+u) < \varepsilon\}\right) > 0$$

Consequently, $\xi \in \Gamma^r_{(\mu,\nu)_2}(x)$.

Theorem 3.19. Let $x = (x_{jk})$ be a double sequence in an IFNS, $r > 0$ and $c \in \mathbb{X}$. Then,

$$\Gamma^r_{(\mu,\nu)_2}(x) = \bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)} \overline{B(c,\varepsilon,r)}$$

Proof.

Let $x = (x_{jk})$ be a double sequence in an IFNS and $r > 0$. Let $\gamma \in \bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)} \overline{B(c,\varepsilon,r)}$, then there exists $c \in \Gamma_{(\mu,\nu)_2}(x)$ such that for all $r > 0$ and given $\varepsilon \in (0,1)$, $\mu(c-\gamma; r) > 1-\varepsilon$ and $\nu(c-\gamma; r) < \varepsilon$. Fix $u > 0$. Since $c \in \Gamma_{(\mu,\nu)_2}(x)$, there exists a set

$$K = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk}-c; u\right) > 1-\varepsilon \text{ and } \nu\left(x_{jk}-c; u\right) < \varepsilon\}$$

such that $\delta_2(K) > 0$. For $(j,k) \in K$,

$$\mu\left(x_{jk}-\gamma; r+u\right) \geq \min\{\mu\left(x_{jk}-c; u\right), \mu(c-\gamma; r)\} > 1-\varepsilon$$

and

$$\nu\left(x_{jk}-\gamma; r+u\right) \leq \max\{\nu\left(x_{jk}-c; u\right), \nu(c-\gamma; r)\} < \varepsilon$$

This implies that $\delta_2\left(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk}-\gamma; r+u\right) > 1-\varepsilon \text{ and } \nu\left(x_{jk}-\gamma; r+u\right) < \varepsilon\}\right) > 0$. Hence, $\gamma \in \Gamma^r_{(\mu,\nu)_2}(x)$. Therefore, $\bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)} \overline{B(c,\lambda,r)} \subseteq \Gamma^r_{(\mu,\nu)_2}(x)$.

Conversely, let $\gamma \in \Gamma^r_{(\mu,\nu)_2}(x)$ and $\gamma \notin \bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)} \overline{B(c,\varepsilon,r)}$ and so $\gamma \notin \overline{B(c,\varepsilon,r)}$, for all $c \in \Gamma_{(\mu,\nu)_2}(x)$. Then,

$$\mu(\gamma-c; r) \leq 1-\varepsilon \text{ or } \nu(\gamma-c; r) \geq \varepsilon, \text{ for all } c \in \Gamma_{(\mu,\nu)_2}(x)$$

By Theorem 3.18, for $\gamma \in \Gamma_{(\mu,\nu)_2}(x)$, $\mu(\gamma-c; r) > 1-\varepsilon$ and $\nu(\gamma-c; r) < \varepsilon$, for all $c \in \Gamma^r_{(\mu,\nu)_2}(x)$ which is a contradiction to the assumption. Therefore, $\gamma \in \bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)} \overline{B(c,\varepsilon,r)}$. Hence, $\Gamma^r_{(\mu,\nu)_2}(x) \subseteq \bigcup_{c\in\Gamma_{(\mu,\nu)_2}(x)} \overline{B(c,\varepsilon,r)}$.

Theorem 3.20. Let $x = (x_{jk})$ be a double sequence in an IFNS and $r > 0$. Then, for all $\varepsilon \in (0,1)$,

i. If $c \in \Gamma_{(\mu,\nu)_2}(x)$, then $st_2 -^{\mu}_{\nu} LIM^r_{x_{jk}} \subseteq \overline{B(c,\varepsilon,r)}$.

ii. $st_2 -^{\mu}_{\nu} LIM^r_{x_{jk}} = \bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)} \overline{B(c,\varepsilon,r)} = \left\{\xi \in \mathbb{X} : \Gamma_{(\mu,\nu)_2}(x) \subseteq \overline{B(\xi,\varepsilon,r)}\right\}$

Proof.

Let $x = (x_{jk})$ be a double sequence in an IFNS.

i. Consider $\xi \in st_2 -^{\mu}_{\nu} LIM^r_{x_{jk}}$ and $c \in \Gamma_{(\mu,\nu)_2}(x)$. For all $u > 0$ and $\varepsilon \in (0,1)$, let

$$A = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu\left(x_{jk}-\xi : r+u\right) > 1-\varepsilon \text{ and } \nu\left(x_{jk}-\xi; r+u\right) < \varepsilon\}$$

and

$$B=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-c;u\right)>1-\varepsilon \text{ and } \nu\left(x_{jk}-c;u\right)<\varepsilon\right\}$$

Thus, $\delta_2\left(A^c\right)=0$ and $\delta_2(B)\neq 0$. For $(j,k)\in A\cap B$,

$$\mu(\xi-c;r)\geq\min\left\{\mu\left(x_{jk}-c;u\right),\mu\left(x_{jk}-\xi;r+u\right)\right\}>1-\varepsilon$$

and

$$\nu(\xi-c;r)\leq\max\left\{\nu\left(x_{jk}-c;u\right),\nu\left(x_{jk}-\xi;r+u\right)\right\}<\varepsilon$$

Therefore, $\xi\in\overline{B(c,\varepsilon,r)}$. Hence, $st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}\subseteq\overline{B(c,\varepsilon,r)}$.

ii. From the statement *i.*, $st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}\subseteq\bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}$. Let $y\in\bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}$. Then, $\mu(y-c;r)\geq 1-\varepsilon$ and $\nu(y-c;r)\leq\varepsilon$, for all $c\in\Gamma_{(\mu,\nu)_2}(x)$. This implies that

$$\Gamma_{(\mu,\nu)_2}(x)\subseteq\overline{B(y,\varepsilon,r)}$$

and so

$$\bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}\subseteq\left\{\xi\in\mathbb{X}:\Gamma_{(\mu,\nu)_2}(x)\subseteq\overline{B(\xi,\varepsilon,r)}\right\}$$

Further, let $y\notin st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}$. Then, for $u>0$,

$$\delta_2(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-y;r+u)\leq 1-\varepsilon \text{ or } \nu(x_{jk}-y;r+u)\geq\varepsilon\})\neq 0$$

which implies that a statistical cluster point c exists for the sequence x such that

$$\mu(y-c;r+u)\leq 1-\varepsilon \text{ or } \nu(y-c;r+u)\geq\varepsilon$$

Thus, $\Gamma_{(\mu,\nu)_2}(x)\nsubseteq\overline{B(y,\varepsilon,r)}$ and $y\notin\left\{\xi\in\mathbb{X}:\Gamma_{(\mu,\nu)_2}(x)\subseteq\overline{B(\xi,\varepsilon,r)}\right\}$. Therefore,

$$\left\{\xi\in\mathbb{X}:\Gamma_{(\mu,\nu)_2}(x)\subseteq\overline{B(\xi,\varepsilon,r)}\right\}\subseteq st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}$$

and so $\bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}\subseteq st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}$. Consequently,

$$st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}=\bigcap_{c\in\Gamma_{(\mu,\nu)_2}(x)}\overline{B(c,\varepsilon,r)}=\left\{\xi\in\mathbb{X}:\Gamma_{(\mu,\nu)_2}(x)\subseteq\overline{B(\xi,\varepsilon,r)}\right\}$$

Theorem 3.21. Let (x_{jk}) be a double sequence in an IFNS. If (x_{jk}) is statistically convergent to $\xi\in\mathbb{X}$ with respect to the norm (μ,ν), then for all $\varepsilon\in(0,1)$ and $r>0$ $st_2-_{\nu}^{\mu}LIM_{x_{jk}}^{r}=\overline{B(\xi,\varepsilon,r)}$ is hold.

Proof.

Let (x_{jk}) be a double sequence in an IFNS and (x_{jk}) be statistically convergent to $\xi\in\mathbb{X}$ with respect to the norm (μ,ν) and $u>0$. Since $x_{jk}\xrightarrow{st_2^{(\mu,\nu)}}\xi$, then there exists a set

$$A=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-\xi;u\right)\leq 1-\varepsilon \;\text{ or }\; \nu\left(x_{jk}-\xi;u\right)\geq\varepsilon\right\}$$

such that $\delta_2(A) = 0$. Let $y \in \overline{B(\xi,\varepsilon,r)} = \{y \in \mathbb{X} : \mu(y-\xi;r) \geq 1-\varepsilon, \nu(y-\xi;r) \leq \varepsilon\}$. For $(j,k) \in A^c$,

$$\mu\left(x_{jk} - y; r+u\right) \geq \min\left\{\mu\left(x_{jk}-\xi;u\right), \mu(y-\xi;r)\right\} > 1-\varepsilon$$

and

$$\nu\left(x_{jk} - y; r+u\right) \leq \max\left\{\nu\left(x_{jk}-\xi;u\right), \nu(y-\xi;r)\right\} < \varepsilon$$

This implies that $y \in st_2\text{-}_{\nu}^{\mu}LIM_{x_{jk}}^r$, i.e., $\overline{B(\xi,\varepsilon,r)} \subseteq st_2\text{-}_{\nu}^{\mu}LIM_{x_{jk}}^r$. On the other hand, $st_2\text{-}_{\nu}^{\mu}LIM_{x_{jk}}^r \subseteq \overline{B(\xi,\varepsilon,r)}$. Hence, $st_2\text{-}_{\nu}^{\mu}LIM_{x_{jk}}^r = \overline{B(\xi,\varepsilon,r)}$.

Theorem 3.22. Let $x = (x_{jk})$ be a double sequence in an IFNS. If x is statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm (μ,ν), then $\Gamma_{(\mu,\nu)_2}^r(x) = st_2\text{-}_{\nu}^{\mu}LIM_{x_{jk}}^r$ for some $r > 0$.

Proof.

Let $x = (x_{jk})$ be a double sequence in an IFNS and x be statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm (μ,ν). Then, $\Gamma_{(\mu,\nu)_2}(x) = \{\xi\}$. By Theorem 3.19, for some $r > 0$ and $\varepsilon \in (0,1)$, $\Gamma_{(\mu,\nu)_2}^r(x) = \overline{B(\xi,\varepsilon,r)}$. Moreover, by Theorem 3.21, $\overline{B(\xi,\varepsilon,r)} = st_2\text{-}_{\nu}^{\mu}LIM_{x_{jk}}^r$. Hence, $\Gamma_{(\mu,\nu)_2}^r(x) = st_2\text{-}_{\nu}^{\mu}LIM_{x_{jk}}^r$.

4. Conclusion

This paper studies the concept of rough statistical convergence, a generalization of rough convergence, and statistical convergence in an IFNS. Then, it defines the concepts of r-st limit and r-st cluster points' sets and investigates some of their basic properties.

In the future studies, researchers can study the concepts proposed herein for triple sequences. Moreover, they can define the concept of rough ideal convergence of a double sequence in an IFNS and examines its basic properties.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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