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SUBORDINATION THEOREMS FOR A CLASS RELATED TO Q-FRACTIONAL DIFFERENTIAL OPERATOR

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ABSTRACT. By using the definition of q-difference operator, we defined the new q-Al-Oboudi-Al-Amoudi operator, which generalize modified Al-Oboudi-Al-Amoudi operator. Using the new operator, we defined a new class of uniformly functions and obtained subordination result for functions in it. Our results not only generalize previous results but also modified some previous results.

1. Introduction

The class of univalent analytic functions

$$F(z) = z + \sum_{k=2}^{\infty} a_k z^k, (a_k \geq 0),\ z \in \mathcal{D} = \{z \in \mathbb{C} :| z |< 1\}, \tag{1}$$

is denoted by $\mathcal{S}$.

The class of convex functions $\mathcal{K}$ satisfies

$$\text{Re}\left\{1 + \frac{zF^{''}(z)}{F^{'}(z)}\right\} > 0.$$

If F, g are analytic in $\mathcal{D}$, then F is subordinate to g, written F $\prec g$ if there exists a Schwarz function $w(z)$ analytic in $\mathcal{D}$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathcal{D}$, such that $\text{F}(z) = g(w(z))$. (*see* [14, 16])

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For F given by (1) and g given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{2}$$

the Hadamard product (or convolution) is

$$(F * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * F)(z).$$

For $F \in \mathcal{S},\ 0 < q < 1$, the $q-$derivative operator ∇_q is given by (Jackson [15]) and many authors studied it for example see ([1], [4 − 6], [9], [16, 17] and [22, 23]).

$$\nabla_q F(z) = \begin{cases} \frac{F(z)-F(qz)}{(1-q)z} & ,z\neq 0 \\ F'(0) & ,z=0 \end{cases}$$

that is

$$\nabla_q F(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{3}$$

where

$$[k]_q = \frac{1-q^k}{1-q},\ [0]_q = 0. \tag{4}$$

The fractional $q-$derivative operator of order α for analytic function F defined in a simply connected domain, contains zero is defined by [5],

$$D_{q,z}^{\alpha} F(z) = \frac{1}{\Gamma_q(1-\alpha)} \int_0^z \frac{F(t)}{(z-t)^{\alpha}} d_q t\ ,\ 0 \le \alpha < 1,$$

$$\begin{aligned} \Omega_q^{\alpha} F(z) &= \Gamma_q(2-\alpha)\, z^{\alpha} D_{q,z}^{\alpha} F(z), \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(k+1-\alpha)} a_k z^k \ \ (0 < q < 1,\ 0 \le \alpha < 1), \end{aligned} \tag{5}$$

where multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $\log(z-t)$, to be real when $z - t > 0$ (for $q \to 1^-$ see [19], [20]).

Definition 1. *For $\lambda \ge 0,\ 0 \le \alpha < 1,\ 0 < q < 1,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \mathbb{N} = \{1, 2, ...\}$ and F is given by (1) we defined new $q-$fractional derivative operator as follows,*

$$\begin{aligned} D_{\lambda,q}^{0,0} F(z) &= F(z), \\ D_{\lambda,q}^{1,\alpha} F(z) &= (1-\lambda)\, \Omega_q^{\alpha} F(z) + \lambda z D_q \left(\Omega_q^{\alpha} F(z)\right) = D_{\lambda,q}^{\alpha} F(z), \\ D_{\lambda,q}^{2,\alpha} F(z) &= D_{\lambda,q}^{\alpha} \left(D_{\lambda,q}^{\alpha} F(z)\right), \end{aligned} \tag{6}$$

$$\begin{aligned} D_{\lambda,q}^{n,\alpha}F(z) &= D_{\lambda,q}^{\alpha}\left(D_{\lambda,q}^{n-1,\alpha}F(z)\right), \\ &= z+\sum_{k=2}^{\infty}\Psi_{k,n,q}(\alpha,\lambda)a_k z^k, \end{aligned}$$

where

$$\Psi_{k,n,q}(\alpha,\lambda)=\frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(k+1-\alpha)}\left[1+\lambda([k]_q-1)\right]^n. \tag{7}$$

We note that:

(i) $D_{1,q}^{n,0}F(z)=D_q^n\ F(z)$ [8, 18].

(ii) $\lim_{q\to 1^-} D_{\lambda,q}^{n,\alpha}F(z)=D_{\lambda}^{n,\alpha}F(z)$, where this operator modified the operator of [3, 7],

(iii) $\lim_{q\to 1^-} D_{\lambda,q}^{0,\alpha}F(z)=D_z^{\alpha}\ F(z)$ (see [19, 20]),

(iv) $\lim_{q\to 1^-} D_{1,q}^{n,0}F(z)=D^nF(z)$ (see [21]),

(v) $\lim_{q\to 1^-} D_{\lambda,q}^{n,0}F(z)=D_{\lambda}^nF(z)$ (see [2]),

Definition 2. *For $\lambda,\mu\geq 0$, $\gamma\geq 1, 0\leq\alpha,\beta<1, 0\leq\delta\leq 1, n\in\mathbb{N}_0$, a function $F\in\mathcal{S}$ is in the class $S_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, if*

$$Re\left\{\frac{\gamma z\nabla_q G(z)}{G(z)}-(\gamma-1)\right\}>\mu\left|\frac{\gamma z\nabla_q G(z)}{G(z)}-\gamma\right|+\beta, \tag{8}$$

where

$$G(z)=(1-\delta)D_{\lambda,q}^{n,\alpha}F(z)+\delta z\left(\nabla_q D_{\lambda,q}^{n,\alpha}F(z)\right). \tag{9}$$

We note that as $q\to 1^-$: $S_{\lambda,q}^{n,\alpha}(0,1,\mu,\beta)=SP_{\alpha,\lambda}^n(\mu,\beta)$ and $S_{\lambda,q}^{n,\alpha}(1,1,\mu,\beta)=UCV_{\alpha,\lambda}^n(\mu,\beta)$ [3, 7, with $\Psi_{k,n,q}(\alpha,\lambda)$ of the form (1.7)]. For different values of n, α, $\lambda,\delta,\gamma,\mu$ and β, we get the classes defined by [3], [8], [10 − 13], and [17].

2. Main Results

Unless indicated, let $0\leq\alpha,\beta<1$, $\lambda,\mu\geq 0$, $\gamma\geq 1$, $0\leq\delta\leq 1$, $n\in\mathbb{N}_0$, $0<q<1$ and $\Psi_{k,n,q}(\alpha,\lambda)$ as (7). The following definition and lemma are needed.

Definition 3. [24]. *A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $F(z)\in\mathcal{K}$ then,*

$$\sum_{k=1}^{\infty}a_kb_kz^k\prec F(z)\qquad(z\in\mathcal{D};a_1=1).$$

Lemma 1. [24]. *The sequence* $\{b_k\}_{k=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$\Re\left\{1+2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \qquad (z \in \mathcal{D}).$$

Theorem 1. *If* $F \in \mathcal{S}$*, satisfies*

$$\sum_{k=2}^{\infty}\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)\left|a_k\right| \leq 1-\beta, \tag{10}$$

then, $F \in S_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$.

Proof. Assume that (10) holds. Since for real β and complex number w, □

$$\operatorname{Re}(w) \geq \beta \Leftrightarrow |w+(1-\beta)|-|w-(1+\beta)| \ \geq 0, \tag{11}$$

then by Definition 2 it is sufficient to show that

$$\begin{aligned} &\left|\frac{\gamma z \nabla_q G(z)}{G(z)}-(\gamma-1)-\mu\left|\frac{\gamma z \nabla_q G(z)}{G(z)}-\gamma\right|-(1+\beta)\right| \leq \\ &\left|\frac{\gamma z \nabla_q G(z)}{G(z)}-(\gamma-1)-\mu\left|\frac{\gamma z \nabla_q G(z)}{G(z)}-\gamma\right|+(1-\beta)\right|. \end{aligned} \tag{12}$$

For the right-hand side of (12)

$$\begin{aligned} R \ :&=\left|\frac{\gamma z \nabla_q G(z)}{G(z)}-(\gamma-1)-\mu\left|\frac{\gamma z \nabla_q G(z)}{G(z)}-\gamma\right|+(1-\beta)\right| \\ &= \frac{1}{|G(z)|}\left|\gamma z \nabla_q G(z)+(2-\beta-\gamma)G(z)-\mu e^{i\theta}\left|\gamma z \nabla_q G(z)-\gamma G(z)\right|\right| \\ &> \frac{|z|}{|G(z)|}\{2-\beta-\sum_{k=2}^{\infty}\left[2-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right] \\ &\times \left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)\left|a_k\right|\}. \end{aligned}$$

Similarly, the left

$$\begin{aligned} L \ :&=\left|\frac{\gamma z \nabla_q G(z)}{G(z)}-(\gamma-1)-\mu\left|\frac{\gamma z \nabla_q G(z)}{G(z)}-\gamma\right|-(1+\beta)\right| \\ &= \frac{1}{|G(z)|}\left|\gamma z \nabla_q G(z)-(\gamma-1)G(z)-\mu e^{i\theta}\left|\gamma z \nabla_q G(z)-\gamma G(z)\right|-(1+\beta)G(z)\right| \end{aligned}$$

$$< \frac{|z|}{|G(z)|}\{\beta+\sum_{k=2}^{\infty}\left[\gamma\left([k]_q-1\right)(1+\mu)-\beta\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)\,|a_k|\}.$$

Since

$$\begin{aligned} R-L &> \frac{|z|}{|G(z)|}\{2(1-\beta)-2\sum_{k=2}^{\infty}\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right] \\ &\times \left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)\,|a_k|\} \\ &\geq 0, \end{aligned}$$

then (12) is satisfied, so $F \in S_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$.

Let $\acute{S}_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$ be the class of functions satisfy (10) so $\acute{S}_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta) \subset$

$S_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$.

Theorem 2. *Let $F \in \acute{S}_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$ and $g \in \mathcal{K}$, then*

$$\left(\frac{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{2\{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta\}}\right)(F*g)(z) \prec g(z) \quad (13)$$

and

$$\Re\{F(z)\} > -\frac{\{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta\}}{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}. \quad (14)$$

The constant factor $\frac{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{2\{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta\}}$ *in* (13) *cannot be replaced by a larger one.*

Proof. Let $F \in \acute{S}_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$ and $g(z)=z+\sum_{k=2}^{\infty}b_k z^k \in \mathcal{K}$, then □

$$\left(\frac{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{2\{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta\}}\right)(F*g)(z)$$

$$= \left(\frac{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{2\{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta\}}\right)\left(z+\sum_{k=2}^{\infty}a_k b_k z^k\right). \quad (15)$$

Thus, by Definition 3, (13) will be true if

$$\left\{\frac{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{2\left\{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta\right\}}a_k\right\}_{k=1}^{\infty} \tag{16}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to

$$\Re\left\{1+\sum_{k=1}^{\infty}\frac{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{\left\{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta\right\}}a_k z^k\right\}>0, \tag{17}$$

where

$$\Theta(k)=\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)\quad (k\geq 2),$$

is an increasing function of k $(k \geq 2)$, when $|z| = r < 1$, we have,

$$\begin{aligned}
&\Re\left\{1+\sum_{k=1}^{\infty}\frac{\Theta(2)}{\Theta(2)+1-\beta}a_k z^k\right\}\\
=\;&\Re\left\{1+\frac{\Theta(2)}{\Theta(2)+1-\beta}z+\frac{\sum_{k=2}^{\infty}\Theta(2)}{\Theta(2)+1-\beta}a_k z^k\right\}\\
\geq\;&1-\frac{\Theta(2)}{\Theta(2)+1-\beta}r-\frac{\sum\limits_{k=2}^{\infty}\Theta(k)|a_k|}{\Theta(2)+1-\beta}r^k\\
>\;&1-\frac{\Theta(2)}{\Theta(2)+1-\beta}r-\frac{1-\beta}{\Theta(2)+1-\beta}r\\
=\;&1-r>0\qquad(|z|=r<1).
\end{aligned}$$

By taking the convex function $g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$. To prove the sharpness of $\frac{\Theta(2)}{2[\Theta(2)+1-\beta]}$, the function $F_0(z)\in \acute{S}_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$ given by

$$F_0(z)=z-\frac{1-\beta}{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}z^2. \tag{18}$$

Thus from (14), we have

$$\frac{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{2\left\{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta\right\}}F_0(z)\prec\frac{z}{1-z}.$$

Moreover, it can easily to verify for $F_0(z)$ given by (18) that

$$\min_{|z|\leq r}\left\{\Re\frac{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{2\left\{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta\right\}}F_0(z)\right\}=-\frac{1}{2} \tag{19}$$

This shows that the $\frac{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{2\{[1-\beta+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta\}}$ is the best possible .

Taking $\lim_{q\to 1^-}$in Theorem 2, we have

Corollary 1. *Let $F \in \acute{S}_{\lambda}^{n,\alpha}(\delta, \gamma, \mu, \beta)$ whose coefficients satisfy* (10) *when $q \to 1^-$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{K}$, then*

$$\left(\frac{[1-\beta+\gamma(1+\mu)](1+\delta)\Psi_{2,n}(\alpha,\lambda)}{2\{[1-\beta+\gamma(1+\mu)](1+\delta)\Psi_{2,n}(\alpha,\lambda)+1-\beta\}} \right)(F*g)(z) \prec g(z) \tag{20}$$

and

$$\Re\{F(z)\} > -\frac{\{[1-\beta+\gamma(1+\mu)](1+\delta)\Psi_{2,n}(\alpha,\lambda)+1-\beta\}}{[1-\beta+\gamma(1+\mu)](1+\delta)\Psi_{2,n}(\alpha,\lambda)}.$$

The factor $\frac{[1-\beta+\gamma(1+\mu)](1+\delta)\Psi_{2,n}(\alpha,\lambda)}{2\{[1-\beta+\gamma(1+\mu)](1+\delta)\Psi_{2,n}(\alpha,\lambda)+1-\beta\}}$ in (2.11) cannot be replaced by a larger one.

Remark 1. *Note that for $\gamma = 1$ and $\delta = 0, 1$ respectively in Corollary 1 modified Theorems 2.4 and 2.8 of* [7].

Taking $\gamma = 0$ in Theorem 2, we have

Corollary 2. *Let $F \in \acute{S}_{\lambda,q}^{n,\alpha}(\delta, 0, \mu, \beta)$ whose coefficients satisfy* (10) *when $\gamma = 0$ and $g \in \mathcal{K}$, then*

$$\left(\frac{(1-\beta)(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{2[(1-\beta)(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta]} \right)(F*g)(z) \prec g(z) \tag{21}$$

and

$$\Re\{F(z)\} > -\frac{[(1-\beta)(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta]}{(1-\beta)(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}.$$

The factor $\frac{(1-\beta)(1+\delta)\Psi_{2,n}(\alpha,\lambda)}{2[(1-\beta)(1+\delta)\Psi_{2,n}(\alpha,\lambda)+1-\beta]}$ in (2.12) *cannot be replaced by a larger one.*

Taking $\mu = 0$ in Theorem 2, we have

Corollary 3. *Let $F \in \acute{S}_{\lambda,q}^{n,\alpha}(\delta, \gamma, 0, \beta)$ whose coefficients satisfy* (10) *with $\mu = 0$ and $g \in \mathcal{K}$. Then*

$$\left(\frac{(1-\beta+\gamma q)(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{2[(1-\beta+\gamma q)(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta]} \right)(F*g)(z) \prec g(z) \tag{22}$$

and

$$\Re\{F(z)\} > -\frac{[(1-\beta+\gamma q)(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta]}{(1-\beta+\gamma q)(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}.$$

The factor $\frac{(1-\beta+\gamma q)(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)}{2[(1-\beta+\gamma q)(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)+1-\beta]}$ in (22) *cannot be replaced by a larger one.*

3. Conclusions

Throughout the paper, first by using the definition of $q-$difference operator we defined new q- Al-Oboudi - Al-Amoudi operator and which modified Al-Oboudi - Al-Amoudi operator. After that, we used the new operator to introduce new class $S_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$ which generalized a class of uniformly univalent functions. Finally, we obtained some subordination factor sequence results for this class and its subclasses. Our results modified previous results.

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