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Research Article

Weakly (k, n)-absorbing (primary) hyperideals of a Krasner (m, n)-hyperring

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Abstract

In this paper, we introduce new expansion classes, namely weakly (k, n)-absorbing hyperideals and weakly (k, n)-absorbing primary hyperideals of a Krasner (m, n)-hyperring, including (k, n)-absorbing hyperideal and (k, n)-absorbing primary hyperideal. In summary, we give generalizations of (k, n)-absorbing hyperideal and (k, n)-absorbing primary hyperideal. Also, we examine the relationships between classical hyperideals and the new hyperideals and explore some ways to connect them. Additionally, some main results and examples are given to explain the structures of these concepts. Finally, we study a version of Nakayama's lemma on a commutative Krasner (m, n)-hyperring.

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1. Introduction

Hyperstructures, which are an extension of classical algebraic structures, have an important role in mathematics due to its applications to other fields ranging from automata, cryptography, coding theory, articial intelligence and probabilities. Firstly, these structures (particularly, hypergroups) were introduced by F.Marty in (1934). Afterwards, many authors have been introduced and studied various hyperstructures, for examples, hypergroups, hyperrings, Krasner hyperring, multiplicative hyperring, hypermodules, i.e. (See [7]- [11]). Basic illustrations and results of Krasner hyperrings can be seen in [8] and [9]. Recently, many authors have turned to this topic because the generalizations of various hyperideal of Krasner (m, n)-hyperring have an important place in this theory [1–13].

Let H be an algebraic hyperstructure with m-ary hyperoperation f and n-ary hyperoperation g. Then, (H, f, g) is defined as a Krasner (m, n)-hyperring if the following holds: **i.** (H, f) is a canonical m-ary hypergroup, **ii.** (H, g) is an n-ary semigroup, **iii.** g has distributive property with regarding to f such that $g(x_1^{i-1}, f(y_1^m), x_{i+1}^n) = f(g(x_1^{i-1}, y_1, x_{i+1}^n), ..., g(x_1^{i-1}, y_m, x_{i+1}^n))$ for each $x_1^{i-1}, x_{i+1}^n, y_1^m \in H$ for $i \in \{1, ..., n\}$, **iv.** 0

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is a zero element of the n-ary hyperoperation g for each $x_2^n \in H$, $g(0, x_2^n) = g(x_2, 0, x_3^n) = ... = g(x_2^n, 0)$ (For more information, see [11]). The above notation x_i^j will denote the sequence of $x_i, x_{i+1}, ..., x_j$ if $j \geq i$ and it is the empty symbol if $j < i$. A Krasner (m, n)-hyperring is called commutative if (H, g) is a commutative n-ary semigroup and also it has a scalar identity if there is an element 1_g with $x = g(x, 1_g^{n-1})$ for every $x \in H$. Let (H, f, g) be a Krasner (m, n)-hyperring with scalar identity 1_g. Let I be a non empty subset of (H, f, g). Then I is called an i-hyperideal of (H, f, g) if the following holds: **i.** I is a subhypergroup of the canonical m-ary hypergroup (H, f); that is, (I, f) is a canonical n-ary hypergoup, **ii.** $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$ for every $x_1^n \in H$. Then I is called a hyperideal of H if it is an i-hyperideal for every $i \in \{1, ..., n\}$. Let A be a subset of H. Then $< A >$ is the hyperideal generated by elements of A. The radical of I, $rad(I)$, is the intersection of all n-ary prime hyperideals P containing I. By [2, Theorem 4.23], $rad(I) = \{h \in H | g(h^u, 1_H^{n-u}) \in I\}$ for $u \leq n$ and $rad(I) = \{h \in H | g_{(l)}(h^u) \in I\}$ for $u > n,\ u = l(n-1)+1$. As a result of the definition of rad, we have that if $I = 0$, then $rad(0) = \{h \in H | g(h^u, 1_H^{n-u}) = 0\}$ for $u \leq n$ and $rad(0) = \{h \in H | g_{(l)}(h^u) = 0\}$ for $u > n,\ u = l(n-1)+1$. The $rad(0)$ is called the nil radical of H. In [2], the notions of n-ary prime and n-ary primary hyperideals of Krasner (m, n)-hyperrings were introduced and their basic properties were given. A hyperideal $H \neq I$ is called an n-ary prime if $g(U_1^n) \subseteq I$ for hyperideals U_i of H for each $i \in \{1, ..., n\}$ implies $U_1 \subseteq I$ or ...or $U_n \subseteq I$. By [2, Lemma 4.5], it is obtained that any hyperideal $H \neq I$ is an n-ary prime if and only if $g(x_1^n) \in I$ for each $x_i \in H$ for $i \in \{1, ..., n\}$ implies $x_1 \in I$ or ... or $x_n \in I$. A hyperideal $H \neq I$ is called an n-ary primary if $g(x_1^n) \in I$ for each $x_i \in H$ for $i \in \{1, ..., n\}$ implies $x_i \in I$ or $g(x_1^{i-1}, 1_H, x_{i+1}^n) \in rad(I)$ for $i \in \{1, ..., n\}$. A commutative Krasner (m, n)-hyperring H is defined as n-ary hyperdomain if $g(x_1^n) = 0$ implies $x_1 = 0$ or ... or $x_n = 0$ for each $x_1^n \in H$. In [9], the authors introduced the concepts of (k, n)-absorbing hyperideals and (k, n)-absorbing primary hyperideals of a Krasner (m, n)-hyperring. A proper hyperideal P of (H, f, g) is called an (k, n)-absorbing hyperideal if $g(x_1^{kn-k+1}) \in P$ for each $x_1^{kn-k+1} \in H$ implies that there exist $(k-1)n-k+2$ of x_i's whose g-product is an element of P. A proper hyperideal P of (H, f, g) is defined as (k, n)-absorbing primary if $g(x_1^{kn-k+1}) \in P$ for each $x_1^{kn-k+1} \in H$ implies that $g(x_1, ..., x_{(k-1)n-k+2}) \in P$ or a g-product of $(k-1)n-k+2$ of x_i's (other than $g(x_1, ..., x_{(k-1)n-k+2})$) is in $rad(P)$.

In this paper, we introduce the concepts of weakly (k, n)-absorbing hyperideals and weakly (k, n)-absorbing primary hyperideals of a Krasner (m, n)-hyperring. Among many results in this paper, we show that every (k, n)-absorbing hyperideal of a Krasner (m, n)-hyperring is a weakly (k, n)-absorbing hyperideal but the converse need not to be hold in Example 2.2. We obtain that if P is a weakly (k, n)-absorbing hyperideal of H, $rad(P)$ may not be weakly (k, n)-absorbing by Example 2.3. It is shown that if P is a weakly n-ary primary hyperideal of H and $rad(P) = P$, then P is a weakly (k, n)-absorbing in Proposition 3.5. We conclude that if $rad(P)$ is a weakly $(k+1, n)$-absorbing primary, then P is a weakly $(k+1, n)$-absorbing primary for all $k \geq 2$ in Theorem 3.6. In Theorem 4.4, we give a version of Nakayama's Lemma for a commutative Krasner (k, n)-hyperring. Then, in Theorem 4.10, we give another version of Nakayama's Lemma for strongly weakly (k, n)-absorbing primary hyperideal of a commutative Krasner (k, n)-hyperring. Finally, we give some characterizations of these concepts on cartesian product of commutative Krasner (m, n)-hyperrings with scalar identities in Theorem 5.1-Theorem 5.2.

Throughout this paper, H denotes commutative Krasner (m, n)-hyperring with scalar identity element.

2. On weakly (k,n)-absorbing hyperideals

Definition 2.1. Let $k \in \mathbb{Z}^+$. A proper hyperideal P of H is called weakly (k,n)-absorbing if $0 \neq g(x_1^{kn-k+1}) \in P$ for each $x_1^{kn-k+1} \in H$ implies that there exist $(k-1)n-k+2$ of x_i's whose g-product is an element of P.

Note that if $k=1$, then P is called weakly n-ary prime hyperideal of H and if $n=2$ and $k=1$, then P is defined as weakly prime hyperideal of a Krasner hyperring.
In the next examples, we give the relationship between weakly (k,n)-absorbing hyperideal and (k,n)-absorbing hyperideal of a Krasner (m,n)-hyperring.

Example 2.2. Every (k,n)-absorbing hyperideal of a Krasner (m,n)-hyperring is a weakly (k,n)-absorbing hyperideal. But the converse of the explanation may not be generally true. Consider the set $H=\{0,1,2,3\}$. In [2, Example 4.7], it is verified that H is a Krasner $(2,4)$-hyperring with the addition hyperoperation f, determined by [2] and the hyperoperation g, defined as $g(x_1,x_2,x_3,x_4)=2$ if one of $x_1^4 \in \{2,3\}$ or 0 if otherwise. Then it is easily seen that $I=<0>$ is a weakly $(1,4)$-absorbing hyperideal of H. However, since $g(1,1,2,3) \in I$ but $g(1) \notin I$, $g(2) \notin I$ and $g(3) \notin I$ then I is not a $(1,4)$-absorbing hyperideal of R.

By [9, Theorem 3.4], it is given that $rad(P)$ is a (k,n)-absorbing hyperideal of H when P is a (k,n)-absorbing hyperideal of H. But if P is a weakly (k,n)-absorbing hyperideal, $rad(P)$ may not be a weakly (k,n)-absorbing hyperideal.

Example 2.3. Let $R=\mathbb{Z}_3[X,Y]$ and $I=<X^3Y^3>$. Note that $H=R/I$ is a (m,n)-Krasner hyperring with ordinary addition and ordinary multiplication. Let $k=1$ and $n=2$. It is clear that $I/I=0_H$ is a weakly $(1,2)$-absorbing hyperideal of H. But $rad(0_H)$ is not weakly $(1,2)$-absorbing hyperideal of H since $0 \neq 2XY+I=(2X+I)(Y+I) \in rad(0_H)$ but $2X+I \notin rad(0_H)$ and $Y+I \notin rad(0_H)$.

Theorem 2.4. *If P is a weakly (k,n)-absorbing hyperideal of H, then P is a weakly (u,n)-absorbing hyperideal for every $u \geq k$.*

Proof. It can be easily seen that the idea is true in a similar manner to the proof of [9, Theorem 3.7]. □

Let I be a hyperideal of a commutative Krasner (m,n)-hyperring H. Recall from [2] that the set $H/I=\{f(x_1^{i-1},I,x_{i+1}^m) \mid x_1^{i-1},x_{i+1}^m \in H\}$ is a commutative Krasner (m,n)-hyperring with m-ary hyperoperation f and n-hyperoperation g. Let (H,f,g) and (H',f',g') be two Krasner (m,n)-hyperrings. Then a map $\psi : H \to H'$ is said to be a homomorphism if $\psi(f(x_1^m))=f'(\psi(x_1),...,\psi(x_m))$, $\psi(g(y_1^n))=g'(\psi(y_1),...,\psi(y_n))$ for all $x_1^m, y_1^n \in H$ and $\psi(0_H)=0'_H$ (For more information, see [11]). Note that the map $\pi : H \to H/I$, is given with $\pi(x)=f(x,I,0^{m-2})$, is a homomorphism in [11].

Theorem 2.5. *Let (H,f,g) and (H',f',g') be two commutative Krasner (m,n)-hyperrings with scalar identities and $\psi : H \to H'$ be a homomorphism. The following holds:*

(1) *If Q is a weakly (k,n)-absorbing hyperideal of H', then $\psi^{-1}(Q)$ is a weakly (k,n)-absorbing hyperideal of H.*
(2) *Let $\psi : H \to H'$ be an epimorphism and P a weakly (k,n)-absorbing hyperideal of H with $kerf(\psi) \subseteq P$. Then $\psi(P)$ is a weakly (k,n)-absorbing hyperideal of H'.*

Proof. (1) Let $0 \neq g(x_1^{kn-k+1}) \in \psi^{-1}(Q)$ for any $x_1^{kn-k+1} \in H$. Then $0 \neq \psi(g(x_1^{kn-k+1}))= g'(\psi(x_1),...,\psi(x_{kn-k+1})) \in Q$. By the assumption, we get that there exist $(k-1)n-k+2$ of $\psi(x_i)$'s whose g'-product is in Q. By the homomorphism ψ, we have that the image ψ of $(k-1)n-k+2$ of x_i's whose g'-product is an element of Q and therefore, there exist $(k-1)n-k+2$ of x_i's whose g-product is an element of $\psi^{-1}(Q)$.

(2) Let $0 \neq g'(y_1^{kn-k+1}) \in \psi(P)$ for any $y_1^{kn-k+1} \in H'$. Then there are $x_i \in H$ for each $i \in \{1, 2, ..., kn-k+1\}$ with $\psi(x_i) = y_i$ since ψ is an epimorphism. We get $0 \neq g'(y_1^{kn-k+1}) = g'(\psi(x_1), ..., \psi(x_{kn-k+1})) = \psi(g(x_1^{kn-k+1})) \in \psi(P)$ by the assumption. We obtain $0 \neq g(x_1^{kn-k+1}) \in P$ as P contains $kerf(\psi)$. Thus, there exist $(k-1)n-k+2$ of x_i's whose g-product is an element of P. The proof is completed since ψ is a homomorphism. □

Let us give the following theorem without proof as a result of Theorem 2.5.

Theorem 2.6. *Let P and Q be two proper hyperideals of H such that $Q \subseteq P$. If P is a weakly (k,n)-absorbing hyperideal, then P/Q is a weakly (k,n)-absorbing hyperideal of H/Q.*

3. On weakly (k, n)-absorbing primary hyperideals

Definition 3.1. Let $k \in \mathbb{Z}^+$. A proper hyperideal P of H is called a weakly (k,n)-absorbing primary if $0 \neq g(x_1^{kn-k+1}) \in P$ for each $x_1^{kn-k+1} \in H$ implies $g(x_1^{(k-1)n-k+2}) \in P$ or a g-product of $(k-1)n-k+2$ of x_i's, other than $g(x_1^{(k-1)n-k+2})$, is in $rad(P)$.

Let $k = 1$. Then P is called weakly n-ary primary hyperideal of H and also, if $n = 2$ and $k = 1$, then P is defined as weakly primary hyperideal of H.

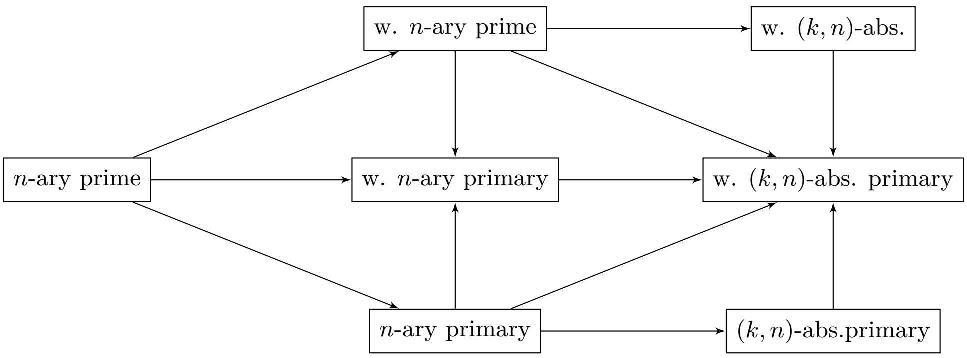

Figure 1. Relationships between n-ary prime (primary) hyperideal and other classical hyperideals of a commutative Krasner (m,n)-hyperring for $k \geq 2$

In the above figure, "weakly" is denoted by "w." and "absorbing" is denoted by "abs.", shortly. Actually, we obtain the above diagram which gives the relationships between n-ary prime (primary), weakly n-ary prime (primary) and weakly (k,n)-absorbing (primary) hyperideals of a commutative Krasner (m,n)-hyperring.
In the figure, it is shown that every weakly (k,n)-absorbing hyperideals is a weakly (k,n)-absorbing primary hyperideal. But the converse of the expression may not be true. Note that $(R, +, \cdot)$ is a Krasner (m,n)-hyperring with $f(x_1^m) = \sum_{i=1}^m x_i$ and $g(y_1^n) = \prod_{i=1}^n y_i$ for each $x_i^m, y_i^n \in R$ in [11]. Consider $H = \mathbb{Z}_{2^3}$. Clearly, $(\mathbb{Z}_{2^3}, +, \cdot)$ is a commutative Krasner (m,n)-hyperring with scalar identity element from the above explanation. Let $k = 2$ and $n = 2$. Note that $< 0 >$ is a weakly $(2,2)$-absorbing primary hyperideal of a Krasner $(m,2)$-hyperring but not a $(2,2)$-absorbing primary hyperideal.

In [9], authors explained that if an hyperideal I is a (k,n)-absorbing primary hyperideal of a Krasner (m,n)-hyperring H, then $rad(I)$ is a (k,n)-absorbing hyperideal of H. But the proposition is not valid for the notion of weakly (k,n)-absorbing primary hyperideal. For instance, see the Example 2.3.

Theorem 3.2. *Let P be a weakly n-ary primary hyperideal of H. Then P is a weakly $(2,n)$-absorbing primary hyperideal.*

Proof. It is seen to be true in a similar manner to [9, Theorem 4.3]. □

Theorem 3.3. *If P is a weakly (k,n)-absorbing primary hyperideal of H, then P is a weakly (u,n)-absorbing primary hyperideal for all $u \geq k$.*

Proof. It can be easily seen that the claim is true in a similar manner to the proof of [9, Theorem 4.4]. □

As a result of the previous two theorems, we give the following corollary.

Corollary 3.4. *If P is a weakly n-ary primary hyperideal of H, then P is a weakly (k,n)-absorbing primary hyperideal for all $k \geq 2$.*

Proposition 3.5. *If P is a weakly n-ary primary hyperideal of H and $g(rad(P)^{(k-1)n-k+2}) \subseteq P$, then P is weakly (k,n)-absorbing hyperideal.*

Proof. Let $0 \neq g(x_1^{kn-k+1}) \in P$ for some $x_1^{kn-k+1} \in H$. Obviously, $0 \neq g(g(x_1^{(k-1)n-k+2}), x_{(k-1)n-k+3}^{kn-k+1}) \in P$. Since P is a weakly n-ary primary hyperideal, then $g(x_1^{(k-1)n-k+2}) \in P$ or $g(x_{(k-1)n-k+3}^{k-k+1}, 1_H^2) \in rad(P)$. If $g(x_1^{(k-1)n-k+2}) \in P$, then the proof is completed. Assume that $g(x_1^{(k-1)n-k+2}) \notin P$ and $g(x_{(k-1)n-k+3}^{kn-k+1}, 1_H^2) \in rad(P)$. Then, there exists $t \in \mathbb{Z}^+$ such that $g((g(x_{(k-1)n-k+3}^{kn-k+1}, 1_H^2))^t, 1_H^{(n-t)}) \in P \Rightarrow x_i \in rad(P)$. It can be seen that $g(x_i^{(k-1)n-k+2+i}) \in g(rad(P)^{(k-1)n-k+2}) \subseteq P$. Hence P is weakly (k,n)-absorbing hyperideal. □

$x_1, ..., \widehat{x_i}, ..., \widehat{x_j}, ..., x_{kn-k+1}$ indicates that x_i and x_j are omitted from the sequence $x_1, ..., x_{kn-k+1}$.

Theorem 3.6. *Let P be a hyperideal of H. If $rad(P)$ is weakly $(k+1,n)$-absorbing hyperideal, then P is weakly $(k+1,n)$-absorbing primary hyperideal for all $k \geq 2$.*

Proof. Let $0 \neq g(x_1^{(k+1)n-(k+1)+1}) \in P$ and $g(x_1^{kn-k+1}) \notin P$ for some $x_1^{(k+1)n-(k+1)+1} \in H$. Then $0 \neq g(x_1^{kn-k}, g(x_{kn-k+1}, ..., x_{(k+1)n-(k+1)+1})) \in P \subseteq rad(P)$, where $x_j = g(x_{kn-k+1}, ..., x_{(k+1)n-(k+1)+1})$ for some $i \in \{1, 2, ..., kn-k, j\}$. By our assumption, $g(x_1, ..., \widehat{x_i}, ..., x_{kn-k}, x_j) \in rad(P)$ if $i \in \{1, 2, ..., kn-k\}$ or $g(x_{kn-k+1}, ..., x_{(k+1)n-(k+1)+1}) \in rad(P)$ if $i = j$. Since $rad(P)$ is a hyperideal, then $g(x_1, ..., x_{kn-k}, x_{(k+1)n-(k+1)+1}) \in rad(P)$. Hence, P is weakly $(k+1,n)$-absorbing primary. □

Theorem 3.7. *Let (H,f,g) and (H',f',g') be two commutative Krasner (m,n)-hyperrings with scalar identities and $\psi : H \to H'$ be a homomorphism. The following holds:*

(1) *If Q is a weakly (k,n)-absorbing primary hyperideal of H', then $\psi^{-1}(Q)$ is a weakly (k,n)-absorbing primary hyperideal of H.*
(2) *Let $\psi : H \to H'$ be an epimorphism and P a weakly (k,n)-absorbing primary hyperideal of H with $kerf(\psi) \subseteq P$. Then $\psi(P)$ is a weakly (k,n)-absorbing primary hyperideal of H'.*

Proof. It is proved in a similar way to Theorem 2.5 by $\psi^{-1}(rad(Q)) = rad(\psi^{-1}(Q))$ and $\psi(rad(P)) \subseteq rad(\psi(P))$. □

We give the following theorem as a result of Theorem 3.7.

Theorem 3.8. *Let P and Q be two proper hyperideals of H such that $Q \subseteq P$. If P is a weakly (k,n)-absorbing primary hyperideal of H, then P/Q is a weakly (k,n)-absorbing primary hyperideal of H/Q.*

4. On Nakayama's lemma for H

We give a version of Nakayama's lemma for a commutative Krasner (m,n)-hyperring. Before that, we give the definition of (m,n)-hypermodule over a Krasner (m,n) hyperring.

Let M be a nonempty set. Then (M,h,k) is called an (m,n)-hypermodule over a Krasner (m,n) hyperring (H,f,g) if (M,h) is an m-ary hypergroup and the map

$$k:\underbrace{H\times\cdots\times H}_{n-1}\times M\longrightarrow \mathcal{P}^*(M)$$

holds the following:

(i) $k\left(r_1^{n-1},h\left(x_1^m\right)\right)=h\left(k\left(r_1^{n-1},x_1\right),\ldots,k\left(r_1^{n-1},x_m\right)\right)$

(ii) $k\left(r_1^{i-1},f\left(s_1^m\right),r_{i+1}^{n-1},x\right)=h\left(k\left(r_1^{i-1},s_1,r_{i+1}^{n-1},x\right),\ldots,k\left(r_1^{i-1},s_m,r_{i+1}^{n-1},x\right)\right)$

(iii) $k\left(r_1^{i-1},g\left(r_i^{i+n-1}\right),r_{i+m}^{n+m-2},x\right)=k\left(r_1^{n-1},k\left(r_m^{n+m-2},x\right)\right)$

(iv) $0=k\left(r_1^{i-1},0,r_{i+1}^{n-1},x\right)$.

It can be seen that an (m,n)-ary hypermodule M is a hypermodule when $m=n=2$. See [3] and [8] for more information.

Definition 4.1. A proper hyperideal P of H is called strongly weakly (k,n)-absorbing if $0\neq g(I_1^{kn-k+1})\subseteq P$ for each hyperideals I_1^{kn-k+1} of H implies that there exist $(k-1)n-k+2$ of I_i's whose g-product is contained by P for $k\in\mathbb{Z}^+$.

By the definition, it is concluded that every strongly weakly (k,n)-absorbing hyperideal is a weakly (k,n)-absorbing hyperideal.

Definition 4.2. Let P be a weakly (k,n)-absorbing hyperideal of H. Then $(a_1,...,a_{k(n-1)+1})$ is called a (k,n)-zero of P if $g(a_1,...,a_{k(n-1)+1})=0$ and none of g-product of the terms $(k-1)n-k+2$ of a_i's is in P.

Theorem 4.3. *Let P be a strongly weakly (k,n)-absorbing hyperideal of H and $(a_1,...,a_{k(n-1)+1})$ a (k,n)-zero of P. Then $g(a_1,...,\widehat{a_{i_1}},...,\widehat{a_{i_2}},...,\widehat{a_{i_t}},...a_{k(n-1)+1},P^t)=0$ for each $i_1,...,i_t\in\{1,...,k(n-1)+1\}$ and $t\in\{1,...,(k-1)n-k+2\}$.*

Proof. We prove this claim with induction on t. Let $t=1$. Then we will show that $g(a_1,...,\widehat{a_{i_1}},...,a_{k(n-1)+1},P)=0$. Assume that $g(a_1,...,\widehat{a_{i_1}},...,a_{k(n-1)+1},P)\neq 0$. Without loss of generality, we omit a_1, that is, $g(a_2,...,a_{k(n-1)+1},P)\neq 0$. Then there is $x\in P$ such that $0\neq g(a_2,...,a_{k(n-1)+1},x)\in P$. We investigate $g(a_2,...,a_{k(n-1)+1},f(a_1,x,0^{m-2}))$. It can be seen that $0\neq g(a_2,...,a_{k(n-1)+1},f(a_1,x,0^{m-2}))\subseteq P$ since $g(a_2,...,a_{k(n-1)+1},f(a_1,x,0^{m-2}))=f(g(a_2,...,a_{k(n-1)+1},a_1),g(a_2,...,a_{k(n-1)+1},x),0^{m-2})\subseteq P$ and thus a g-product containing x, of the terms $(k-1)n-k+2$ of a_i is a subset of P by the assumption. Without loss of generality, assume that $g(a_3,...,a_{k(n-1)+1},f(a_1,x,0^{m-2}))\subseteq P$, that is, $f(g(a_3,...,a_{k(n-1)+1},a_1),...,g(a_3,...,a_{k(n-1)+1},x),0^{m-2})\subseteq P$. We conclude that $g(a_3,...,a_{k(n-1)+1},a_1)\in f(-g(a_3,...,a_{k(n-1)+1},x),...,0^{m-2})\subseteq P$, which is a contradiction. Thus, $g(a_1,...,\widehat{a_{i_1}},...,a_{k(n-1)+1},P)=0$. Now, we assume that the claim holds for all positive integers which are less than $t>1$. Let $g(a_1,...,\widehat{a_{i_1}},...,\widehat{a_{i_2}},...,\widehat{a_{i_t}},...,a_{k(n-1)+1},P^t)\neq 0$. Without loss of generality, we eliminate $a_1,a_2,...,a_t$, that is, $g(a_{t+1},...,a_{k(n-1)+1},P^t)\neq 0$. Then, there are $x_1^t\in P$ such that $0\neq g(a_{t+1},...,a_{k(n-1)+1},x_1^t)\in P$. By induction hypothesis, it can be seen that $0\neq g(a_{t+1},...,a_{k(n-1)+1},f(a_1,x_1,0^{m-2}),...,f(a_t,x_t,0^{m-2}))\subseteq P$. Again, by the hypothesis, $g(a_{t+1},...,a_{k(n-1)+1},f(a_1,x_1,0^{m-2}),...,f(\widehat{a_1,x_1,0^{m-2}})_{i_1}$, $f(\widehat{a_2,x_2,0^{m-2}})_{i_2},...,f(\widehat{a_{n-1},x_{n-1},0^{m-2}})_{i_{n-1}},...,f(a_t,x_t,0^{m-2}))\subseteq P$ or $g(a_{t+1},...,\widehat{a_{i_{t+1}}}$, $\widehat{a_{i_{t+2}}},...,\widehat{a_{i_{t+(n-1)}}},...,a_{k(n-1)+1},f(a_1,x_1,0^{m-2}),...,f(a_t,x_t,0^{m-2}))\subseteq P$ or $g(a_{t+1},...,\widehat{a_{i_{t+1}}}$,

$..., \widehat{a_{i_{t+2}}}, ...\widehat{a_{i_{t+k}}}, ..., a_{k(n-1)+1}, f(a_1, x_1, 0^{m-2}), ..., f(a_1, x_1, \widehat{0^{m-2}})_{i_{t+(k+1)}}, ...,$ $f(a_{n-1-k}, \widehat{x_{n-1-k}, 0^{m-2}})_{i_{t+(n-1-k)}}, ..., f(a_t, x_t, 0^{m-2})) \subseteq P$ for some $1 \leq i \leq t$. Note that in all probability, the g-product containing x_i is a subset of P by the assumption. Thus, $g(a_{t+1}, ..., a_{k(n-1)+1}, ..., a_n, ..., a_t) \in P$ or $g(a_{t+n}, ..., a_{k(n-1)+1}, ..., a_1, ..., a_t) \in P$, a contradiction. Hence, $g(a_1, ..., \widehat{a_{i_1}}, ..., \widehat{a_{i_2}}, ..., \widehat{a_{i_t}}, ...a_{k(n-1)+1}, P^t) = 0$. □

In the following theorem, Nakayamas lemma is considered for a strongly weakly (k,n)-absorbing hyperideal.

Theorem 4.4. *Let P be a strongly weakly (k,n)-absorbing hyperideal of H but is not (k,n)-absorbing hyperideal. Then, the following holds:*

(1) $g(P^{k(n-1)+1}) = 0$.
(2) *If (M,h,k) is an (m,n)-hypermodule over a Krasner (m,n) hyperring (H,f,g) and $M = k(P,M)$, then $M = \{0\}$.*

Proof. By the assumption, P has a (k,n)-zero. Let $(a_1, ..., a_{k(n-1)+1})$ be a (k,n)-zero of P.

(1) Assume that $g(P^{k(n-1)+1}) \neq 0$. Then, there are $x_1^{k(n-1)+1} \in P$ such that $0 \neq g(x_1^{k(n-1)+1}) \in P$. By Theorem 4.3, we can conclude that $0 \neq g(f(a_1, x_1, 0^{m-2}), f(a_2, x_2, 0^{m-2}), ..., f(a_{k(n-1)+1}, x_{k(n-1)+1}, 0^{m-2})) \subseteq P$. As P is a strongly weakly (k,n)-absorbing hyperideal, then there exist $(k-1)n-k+2$ of $f(a_i, x_i, 0^{m-2})$'s whose g-product is contained by P. Without loss of generality, assume that $g(f(a_1, x_1, 0^{m-2}), f(a_2, x_2, 0^{m-2}), ..., f(a_{(k-1)n-k+2}, x_{(k-1)n-k+2}, 0^{m-2})) \subseteq P$. Then, $f(\underbrace{g(a_1^{(k-1)n-k+2}), g(a_1^d, x_1^l), g(x_1^{(k-1)n-k+2})}_{t}, 0^{m-t}) \subseteq P$ where $d+l = (k-1)n-k+2$. We obtain that $g(a_1^{(k-1)n-k+2}) \in f(\underbrace{-g(a_1^d, x_1^l), -g(x_1^{(k-1)n-k+2})}_{t-1}, 0^{m-t+1}) \subseteq P$, that is, $g(a_1^{(k-1)n-k+2}) \in P$, a contradiction. Thus, $g(P^{k(n-1)+1}) = 0$.
(2) Note that $k(g(P^{k(n-1)+1}), M) = \{0\}$. We can write $k(g(P^{k(n-1)+1}), M)$ $= k(g(P^{k(n-1)}), k(P, M)) = k(g(P^{k(n-1)}), M) = \cdots = k(P, k(P, M)) = k(P, M) =$ M by our assumption. Therefore, $M = \{0\}$.

□

A proper hyperideal P of a commutative Krasner (m,n)-hyperring may not be weakly (k,n)-absorbing when it holds $g(P^{k(n-1)+1}) = 0$. For this situation, see the following example.

Example 4.5. Notice that $\mathbb{Z}_{3^{3n}}$ is a commutative Krasner $(2,2)$-hyperring with usual addition and multiplication where $n > 2$ is a positive integer. Consider the hyperideal $P =< 3^n >$. Then, clearly $P \cdot P \cdot P = 0$ but $3 \cdot 3 \cdot 3^{n-2} \in P$ and $3 \cdot 3$, $3 \cdot 3^{n-2} \notin P$, that is, P is not a weakly $(2,2)$-absorbing hyperideal but $P^3 = 0$.

Corollary 4.6. *Let P be a strongly weakly (k,n)-absorbing hyperideal of H but is not (k,n)-absorbing hyperideal. Then $rad(P) = rad(0)$.*

Definition 4.7. A proper hyperideal P of H is called a strongly weakly (k,n)-absorbing primary if $0 \neq g(I_1^{kn-k+1}) \subseteq P$ for each hyperideal I_1^{kn-k+1} of H implies that $g(I_1^{(k-1)n-k+2)})$ $\subseteq P$ or a g-product of $(k-1)n-k+2$ of I_i's, other than $g(I_1^{(k-1)n-k+2)})$, is a subset of $rad(P)$ for $k \in \mathbb{Z}^+$.

Note that every strongly weakly (k,n)-absorbing primary hyperideal of H is a weakly (k,n)-absorbing primary hyperideal.

Definition 4.8. Let P be a weakly (k,n)-absorbing primary hyperideal of H. Then $(a_1,...,a_{k(n-1)+1})$ is called a (k,n)-zero primary element of P if $g(a_1,...,a_{k(n-1)+1}) = 0$, $g(a_1^{(k-1)n-k+2}) \notin P$ and none of g-product of $(k-1)n-k+2$ of a_i's, other than $g(a_1^{(k-1)n-k+2})$, is in $rad(P)$

Theorem 4.9. *Let P be a strongly weakly (k,n)-absorbing primary hyperideal of H and $(a_1,...,a_{k(n-1)+1})$ a (k,n)-zero primary of P. Then $g(a_1,...,\widehat{a_{i_1}},...,\widehat{a_{i_2}},...,\widehat{a_{i_t}},...a_{k(n-1)+1},P^t)$ $= 0$ for each $i_1,...,i_t \in \{1,...,k(n-1)+1\}$ and $t \in \{1,...,(k-1)n-k+2\}$.*

Proof. We prove this claim with induction on t. Let $t=1$. Then we will show that $g(a_1,...,\widehat{a_{i_1}},...,a_{k(n-1)+1},P)=0$. Assume that $g(a_1,...,\widehat{a_{i_1}},...,a_{k(n-1)+1},P) \neq 0$. Without loss of generality, we omit a_1, that is, $g(a_2,...,a_{k(n-1)+1},P) \neq 0$. Then there is $x \in P$ such that $0 \neq g(a_2,...,a_{k(n-1)+1},x) \in P$. Note that $f(g(a_2,...,a_{k(n-1)+1},a_1),g(a_2,...,a_{k(n-1)+1},x),$ $0^{m-2}) \subseteq P$, namely, $0 \neq g(a_2^{(k-1)n-k+1},f(a_1,x,0^{m-2})) \subseteq P$. By the hypothesis, a g-product containing x of the terms $(k-1)n-k+2$ of a_i's is a subset of $rad(P)$. Without loss of generality, assume that $g(a_3,...,a_{k(n-1)+1},f(a_1,x,0^{m-2})) \subseteq rad(P)$, that is, $f(g(a_3,...,a_{k(n-1)+1},a_1),...,g(a_3,...,a_{k(n-1)+1},x),0^{m-2}) \subseteq rad(P)$.
Thus, $g(a_3,...,a_{k(n-1)+1},a_1) \in f(-g(a_3,...,a_{k(n-1)+1},x),...,0^{m-2}) \subseteq rad(P)$, which is a contradiction. Hence, $g(a_1,...,\widehat{a_{i_1}},...,a_{k(n-1)+1},P)=0$. Now, we assume that the claim holds for all positive integers which are less than $t>1$.
Let $g(a_1,...,\widehat{a_{i_1}},...,\widehat{a_{i_2}},...,\widehat{a_{i_t}},...,a_{k(n-1)+1},P^t) \neq 0$. Without loss of generality, we eliminate $a_1,a_1,...,a_t$, that is, $g(a_{t+1},...,a_{k(n-1)+1},P^t) \neq 0$. Then, there are $x_1^t \in P$ such that $0 \neq g(a_{t+1},...,a_{k(n-1)+1},x_1^t) \in P$. By induction hypothesis, it can be seen that $0 \neq g(a_{t+1},...,a_{k(n-1)+1},f(a_1,x_1,0^{m-2}),...,f(a_t,x_t,0^{m-2})) \subseteq P$. Again, by the hypothesis, we get that $g(a_{t+1},...,a_{k(n-1)+1},f(a_1,x_1,0^{m-2}),...,f(a_1,\widehat{x_1,0^{m-2}})_{i_1},...,f(a_2,\widehat{x_2,0^{m-2}})_{i_2},$ $...,f(a_{n-1},\widehat{x_{n-1},0^{m-2}})_{i_{n-1}},...,f(a_t,x_t,0^{m-2})) \subseteq rad(P)$ or $g(a_{t+1},...,\widehat{a_{i_{t+1}}},...,\widehat{a_{i_{t+2}}},...,$ $\widehat{a_{i_{t+(n-1)}}},...,a_{k(n-1)+1},f(a_1,x_1,0^{m-2}),...,f(a_t,x_t,0^{m-2})) \subseteq rad(P)$ for some $1 \leq i \leq t$. Thus, $g(a_{t+1},...,a_{k(n-1)+1},...,a_n,...,a_t) \in rad(P)$ or $g(a_{t+n},...,a_{k(n-1)+1},...,a_1,...,a_t) \in rad(P)$ or $g(a_{t+1},...,\widehat{a_{i_{t+1}}},...,\widehat{a_{i_{t+2}}},...\widehat{a_{i_{t+k}}},...,a_{k(n-1)+1},f(a_1,x_1,0^{m-2}),...,$ $f(a_1,x_1,\widehat{0^{m-2}})_{i_{t+(k+1)}},...,f(a_{n-1-k},\widehat{x_{n-1-k},0^{m-2}})_{i_{t+(n-1-k)}},...,f(a_t,x_t,0^{m-2})) \subseteq rad(P)$ for some $1 \leq i \leq t$. But it contradicts the fact that $(a_1,...,a_{k(n-1)+1})$ is a (k,n)-zero primary of P. Hence, $g(a_1,...,\widehat{a_{i_1}},...,\widehat{a_{i_2}},...,\widehat{a_{i_t}},...a_{k(n-1)+1},P^t)=0$. □

Theorem 4.10. *Let P be a strongly weakly (k,n)-absorbing primary hyperideal of H but is not (k,n)-absorbing primary hyperideal. Then;*

(1) $g(P^{k(n-1)+1})=0$.
(2) *If (M,h,k) is an (m,n)-hypermodule over a Krasner (m,n) hyperring (H,f,g) and $M=k(P,M)$, then $M=\{0\}$.*

Proof. By the assumption, P has a (k,n)-zero. Let $(a_1,...,a_{k(n-1)+1})$ be a (k,n)-zero of P.

(1) Assume that $g(P^{k(n-1)+1}) \neq 0$. Then, there are $x_1^{k(n-1)+1} \in P$ such that $0 \neq g(x_1^{k(n-1)+1}) \in P$. By Theorem 4.9, we can conclude that $0 \neq g(f(a_1,x_1,0^{m-2}),$ $f(a_2,x_2,0^{m-2}),...,f(a_{k(n-1)+1},x_{k(n-1)+1},0^{m-2})) \subseteq P$. As P is a strongly weakly (k,n)-absorbing primary hyperideal, then $g(f(a_1,x_1,0^{m-2})^{(k-1)n-k+2}) \subseteq P$ or a g-product of $(k-1)n-k+2$ of $f(a_i,x_i,0^{m-2})$'s, other than $g(f(a_1,x_1,0^{m-2})^{(k-1)n-k+2})$, is a subset of $rad(P)$. If $g(f(a_1,x_1,0^{m-2})^{(k-1)n-k+2}) = g(f(a_1,x_1,0^{m-2}),$ $f(a_2,x_2,0^{m-2}),...,f(a_{(k-1)n-k+2},x_{(k-1)n-k+2},0^{m-2})) \subseteq P$. Then,
$f(\underbrace{g(a_1^{(k-1)n-k+2}),g(a_1^d,x_1^l),g(x_1^{(k-1)n-k+2})}_{t},0^{m-t}) \subseteq P$ where $d+l=(k-1)n-k+2$.

We obtain that $g(a_1^{(k-1)n-k+2}) \in f(\underbrace{-g(a_1^d, x_1^l), -g(x_1^{(k-1)n-k+2})}_{t-1}, 0^{m-t+1}) \subseteq P$, that is, $g(a_1^{(k-1)n-k+2}) \in P$, a contradiction. Now, assume that a g-product of $(k-1)n-k+2$ of $f(a_i, x_i, 0^{m-2})$'s, other than $g(f(a_1, x_1, 0^{m-2})^{(k-1)n-k+2}))$, is a subset of $rad(P)$. Without loss of generality, consider $g(f(a_2, x_2, 0^{m-2}), f(a_3, x_3, 0^{m-2})$, ..., $f(a_{(k-1)n-k+3}, x_{(k-1)n-k+3}, 0^{m-2})) \subseteq rad(P)$. Then, $f(\underbrace{g(a_2^{(k-1)n-k+3}), g(a_2^d, x_1^l), g(x_1^{(k-1)n-k+2}}_{t}), 0^{m-t}) \subseteq rad(P)$ where $d + l = (k-1)n-k+2$. We obtain that $g(a_2^{(k-1)n-k+3}) \in f(\underbrace{-g(a_2^d, x_1^l), -g(x_1^{(k-1)n-k+2})}_{t-1}, 0^{m-t+1})$ $\subseteq rad(P)$, that is, $g(a_2^{(k-1)n-k+3}) \in rad(P)$, a contradiction. Thus, $g(P^{k(n-1)+1}) = 0$.

(2) The proof can be easily obtained in a similar manner to Theorem 4.4.

□

A proper hyperideal P of H has the property $g(P^{k(n-1)+1}) = 0$ but it does not need to be a weakly (k, n)-absorbing primary hyperideal. For an instance, see Example 4.5.

Corollary 4.11. *Let P be a strongly weakly (k, n)-absorbing primary hyperideal of H but is not (k, n)-absorbing primary hyperideal. Then $rad(P) = rad(0)$.*

5. On cartesian product of commutative Krasner (m, n)-hyperrings

Let (H_1, f_1, g_1) and (H_2, f_2, g_2) be two commutative Krasner (m, n)-hyperrings with scalar identities. In [7], we have that the cartesian product of $H = H_1 \times H_2$ is a Krasner (m, n)-hyperring and for $x_i \in H_1$ and $y_i \in H_2$:
$f = (f_1 \times f_2)((x_1, y_1), ..., (x_m, y_m) = \{(a, b) | a \in f_1(x_1^m), b \in f_2(y_1^m)\}$,
$g = (g_1 \times g_2)((x_1, y_1), ..., (x_n, y_n) = (g_1(x_1^n), g_2(y_1^n))$.

Theorem 5.1. *Let $H = H_1 \times ... \times H_{kn-k+1}$ where each H_i is a commutative Krasner (m, n)-hyperring for $i \in \{1, ..., kn-k+1\}$ with m, n-hyperoperations f_i, g_i, respectively and $P = P_1 \times ... \times P_{kn-k+1}$ be a hyperideal of H where P_i is a proper hyperideal of H_i. If P is a weakly $(k+1, n)$-absorbing hyperideal, then P_i is a weakly (k, n)-absorbing hyperideal of H_i for each $i \in \{1, ..., kn-k+1\}$.*

Proof. Let P be a weakly $(k+1, n)$-absorbing hyperideal of H. Assume that P_i is not a weakly (k, n)-absorbing hyperideal of H_i and $0 \neq g_i(x_i^{kn-k+1}) \in P_i$ for some $x_1^{kn-k+1} \in H_i$. Let $y_1 = (1_{g_1}, ..., 1_{g_{i-1}}, x_1, 1_{g_{i+1}}, ..., 1_{g_{kn-k+1}})$, $y_2 = (1_{g_1}, ..., 1_{g_{i-1}}, x_2, 1_{g_{i+1}}, ..., 1_{g_{kn-k+1}})$, ..., $y_{kn-k+1} = (1_{g_1}, ..., 1_{g_{i-1}}, x_{kn-k+1}, 1_{g_{i+1}}, ..., 1_{g_{kn-k+1}})$, $y_{kn-k} = (1_{g_1}, ...1_{g_{kn-k+1}})$,..., $y_{(k+1)n-(k+1)+1} = (0, ..., 0, 1_{g_i}, 0, ..., 0)$.
It is clear that $0 \neq g(y_1, ..., y_{(k+1)n-(k+1)+1}) \in P$. Notice that every g-product that does not contain $y_{(k+1)n-(k+1)+1}$ is not in P because $1_{g_i} \notin P_i$ for every $i \in \{1, 2, ..., kn-k+1\}$. Then, one of other g-products, containing the element $y_{(k+1)n-(k+1)+1}$, must be in P due to the hypothesis that P is weakly $(k+1, n)$-absorbing hyperideal. Thus, $g_i(x_i^{(k-1)n-k+2}) \in P_i$, a contradiction because of the assumption. Therefore, P_i must be a weakly (k, n)-absorbing hyperideal of H_i for every $i \in \{1, ..., kn-k+1\}$. □

Theorem 5.2. *Let $H = H_1 \times ... \times H_{kn-k+1}$ where each H_i is a commutative Krasner (m, n)-hyperring for $i \in \{1, ..., kn-k+1\}$ with m, n-hyperoperations f_i, g_i, respectively and $P = P_1 \times ... \times P_{kn-k+1}$ be a hyperideal of H where P_i is a proper hyperideal of H_i. If P is a weakly $(k+1, n)$-absorbing primary hyperideal, then P_i is a weakly (k, n)-absorbing primary hyperideal of H_i for each $i \in \{1, ..., kn-k+1\}$.*

Proof. It can be seen similarly to Theorem 5.1. □

Future Work In this paper, the concept of strongly weakly (k,n)-absorbing (primary) hyperideal is defined and the Nakayama's Lemma is applied to Krasner (m,n)-hyperring with this concept. If that each weakly (k,n)-absorbing (primary) hyperideal is strongly weakly (k,n)-absorbing (primary) hyperideal is proved, then these ideas are proved for weakly (k,n)-absorbing (primary) hyperideal.

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