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A Note on Rough Abel Convergence

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Abstract

In this paper, we define a new type of Abel convergence by using the rough convergence of a sequence. We also obtained some results for this convergence.

Keywords: *Abel convergence; Abel summability; rough convergence; rough limits.*

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1. Introduction and Background

The idea of rough convergence of a sequence was first given by Phu [1] in normed linear spaces as follows:

Let (a_n) be a sequence in the normed linear space X, and r be a nonnegative real number. The sequence (a_n) is said to be *rough convergent* to a with the roughness degree r, denoted by $a_n \xrightarrow{r} a$, if for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that $\|a_n - a\| < r + \varepsilon$ for all $n \geq N(\varepsilon)$ [1].

The r-limit set of the sequence (a_n) is denoted by

$$LIM^r a_n = \left\{ a \in X : a_n \xrightarrow{r} a \right\} \text{ [1].}$$

The sequence (a_n) is said to be rough convergent if $LIM^r a_n \neq \varnothing$.

If a sequence is convergent, then it is rough convergent to the same value for each r. The converse of this claim is false, as shown in Example 1.1.

Example 1.1. Let $X = \mathbb{R}^2$ and define a sequence (a_n) as follows:

$$a_n := \left(\frac{(-1)^n}{2}, 0 \right)$$

This sequence is rough convergent to $a = \{(0,0)\}$ for $r \geq \frac{1}{2}$. But it is not convergent to $a = \{(0,0)\}$.

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A sequence (a_n) is said to be rough Cauchy sequence (or ρ-Cauchy sequence) with roughness degree ρ if for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$\|a_m - a_n\| < \rho + \varepsilon \text{ for } m, n \geq N(\varepsilon) \ [1].$$

ρ is also called a Cauchy degree of (a_n).

Proposition 1.1. *([2]) Let (a_n) be rough convergent, i.e., $LIM^r a_n \neq \varnothing$. Then (a_n) is a ρ-Cauchy sequence for every $\rho \geq 2r$. This bound for the Cauchy degree cannot be generally decreased.*

We note that a convergent (or non-convergent) sequence can have different rough limits with a certain degree of roughness. This theory has been generalized by many authors with different theories. Aytar [3] gave the definition of rough statistical convergence of a sequence. The rough ideal convergence of a sequence is given in [4] and [5]. Malik and Maity [6] introduced the rough statistical convergence for double sequences. Laterly, Kişi and Dündar [7] defined the rough I_2-lacunary statistical convergence of double sequences. The concept of rough convergence is expressed in general metric spaces by Debnath and Rakshit [8]. Moreover, Arslan and Dündar [9] extended this concept to 2-normed spaces. On the other hand, Dündar and Ulusu [10] studied on the rough convergence of a sequence of functions defined on amenable semigroups. Kişi and Dündar [11] investigated the rough ΔI-statistical convergence for difference sequences. Recently, the rough convergence of a sequence of sets has also been studied (see [12], [13]).

Our aim is to show that the rough convergence theory can be applied on many types of convergence in the summability theory, such as the Abel convergence. In this way, we think that new research topics can be obtained.

Throughout this paper, we suppose that (a_n) be a sequence of complex numbers. Now let's remind the definition of Abel convergence.

We say that a sequence (a_n) is *Abel convergent* to ℓ if the limit

$$\lim_{t \to 1^-} (1-t) \sum_{n=0}^{\infty} a_n t^n = \ell \text{ for each } t \in (0,1) \ [14].$$

Note that any convergent sequence is Abel convergent to the same value but not conversely ([14]).

Finally, let's give the series formulas that we will use throughout this paper:

$$(1-t) \sum_{n=0}^{\infty} t^n = 1 \text{ and } (1-t) \sum_{n=0}^{\infty} \ell t^n = \ell \text{ for each } t \in (0,1).$$

Hence, we have

$$(1-t) \sum_{n=0}^{\infty} a_n t^n - \ell = (1-t) \sum_{n=0}^{\infty} (a_n - \ell) t^n.$$

In this paper, we first give the definition of the rough Abel convergence. We have proved that they are equivalent by giving an alternative representation of this convergence (see Proposition 2.1). We also show that every rough convergent sequence is rough Abel convergent (see Theorem 2.1). Lastly, we expressed the relationship between rough Abel convergence and Abel convergence (see Theorem 2.2).

2. Main Results

Definition 2.1. A sequence (a_n) is said to be rough *Abel convergent* to ℓ if for every $\varepsilon > 0$ and each $t \in (0,1)$ there is an $N(\varepsilon) \in \mathbb{N}$ such that

$$\left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell \right\| < r + \varepsilon \text{ for all } n \geq N(\varepsilon).$$

In this case, we write $a_n \overset{r-A}{\to} \ell$ as $n \to \infty$.

The r-Abel limit set of the sequence (a_n) is denoted by

$$A - LIM^r a_n = \left\{ \ell \in X : a_n \overset{r-A}{\to} \ell \right\}.$$

Let us now give an alternative representation of the rough Abel convergence of a sequence.

Proposition 2.1. *For every $\varepsilon > 0$ and each $t \in (0,1)$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that*

$$\left\|(1-t)\sum_{n=0}^{\infty} a_n t^n - \ell\right\| < r + \varepsilon \text{ for all } n \geq N(\varepsilon)$$

if and only if the following condition holds:

$$\limsup_{n\to\infty} \left\|(1-t)\sum_{n=0}^{\infty} a_n t^n - \ell\right\| \leq r.$$

Its proof can be given in a similar way by taking

$$f_n(t) = \left\|(1-t)\sum_{n=0}^{\infty} a_n t^n - \ell\right\|$$

instead of the f function in the proof of [12, Proposition 2.2].

Theorem 2.1. *If $a_n \overset{r}{\to} \ell$, then $a_n \overset{r-A}{\to} \ell$.*

Proof. Given $0 < \varepsilon < 1$. Since $a_n \overset{r}{\to} \ell$, for every $\varepsilon > 0$ and each $t \in (0,1)$ there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\|a_n - \ell\| < r + \frac{\varepsilon}{2}$$

for all $n \geq N$. Let $M = \max\{\|a_0 - \ell\|, \|a_1 - \ell\|, \dots \|a_N - \ell\|\}$. Take $\delta = \delta(\varepsilon) = \frac{\varepsilon}{2(N+1)(M+1)}$. If $t \in (1-\delta, 1)$ then

$$\begin{aligned}
\left\|(1-t)\sum_{n=0}^{\infty} a_n t^n - \ell\right\| &= \left\|(1-t)\sum_{n=0}^{\infty}(a_n - \ell)t^n\right\| \\
&\leq \left\|(1-t)\sum_{n=0}^{N}(a_n - \ell)t^n\right\| + \left\|(1-t)\sum_{n=N+1}^{\infty}(a_n - \ell)t^n\right\| \\
&< (1-t)(N+1)M + r + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2(N+1)(M+1)}(N+1)M + r + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + r + \frac{\varepsilon}{2} = r + \varepsilon.
\end{aligned}$$

Consequently, we have $a_n \overset{r-A}{\to} \ell$. □

The next theorem shows the relationship between rough Abel convergence and Abel convergence.

Theorem 2.2. *The sequence (a_n) is rough Abel convergent to ℓ if and only if there exists a sequence (b_n) in $\mathbb{C}$ such that $b_n \overset{A}{\to} \ell$ and $\|a_n - b_n\| \leq r$ for every $n \in \mathbb{N}$.*

Proof. ($\Leftarrow$) Since $b_n \overset{A}{\to} \ell$, for every $\varepsilon > 0$ and each $t \in (0,1)$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$\left\|(1-t)\sum_{n=0}^{\infty} b_n t^n - \ell\right\| < \varepsilon \text{ for all } n \geq N(\varepsilon). \tag{2.1}$$

By assumption $\|a_n - b_n\| \leq r$, we can write for each $t \in (0,1)$

$$\begin{aligned}
\left\|(1-t)\sum_{n=0}^{\infty}(a_n - b_n)t^n\right\| &= |1-t|\left\|\sum_{n=0}^{\infty}(a_n - b_n)t^n\right\| \\
&\leq |1-t|\sum_{n=0}^{\infty}\|a_n - b_n\||t^n| \\
&\leq (1-t)r\sum_{n=0}^{\infty} t^n = r.
\end{aligned}$$

Thus we have

$$\left\|(1-t)\sum_{n=0}^{\infty}(a_n-b_n)\,t^n\right\|\leq r. \tag{2.2}$$

On the other hand,

$$\begin{aligned}\left\|(1-t)\sum_{n=0}^{\infty}a_nt^n-\ell\right\| &= \left\|(1-t)\sum_{n=0}^{\infty}(a_n-b_n+b_n)\,t^n-\ell\right\| \\ &= \left\|(1-t)\sum_{n=0}^{\infty}(a_n-b_n)\,t^n+(1-t)\sum_{n=0}^{\infty}b_nt^n-\ell\right\| \\ &\leq \left\|(1-t)\sum_{n=0}^{\infty}(a_n-b_n)\,t^n\right\|+\left\|(1-t)\sum_{n=0}^{\infty}b_nt^n-\ell\right\|.\end{aligned}$$

From (2.1) and (2.2), we see immediately that

$$\left\|(1-t)\sum_{n=0}^{\infty}a_nt^n-\ell\right\|<r+\varepsilon.$$

This shows that the sequence (a_n) rough Abel converges to ℓ.

($\Rightarrow$) Let $a_n\overset{r-A}{\to}\ell$ and define the sequence (b_n) by

$$b_n:=\begin{cases}\ell & ,\ \|a_n-\ell\|\leq r\\ a_n+r\frac{\ell-a_n}{\|\ell-a_n\|} & ,\ \|a_n-\ell\|>r\end{cases}.$$

It follows that the inequality $\|a_n-b_n\|\leq r$ holds for every $n\in\mathbb{N}$. We also obtain

$$\|b_n-\ell\|\leq\begin{cases}0 & ,\ \|a_n-\ell\|\leq r\\ \|a_n-\ell\|-r & ,\ \|a_n-\ell\|>r\end{cases}.$$

Let us show that $b_n\overset{A}{\to}\ell$.

$$\begin{aligned}\left\|(1-t)\sum_{n=0}^{\infty}b_nt^n-\ell\right\| &= \left\|(1-t)\sum_{n=0}^{\infty}(b_n-\ell)\,t^n\right\| \\ &\leq (1-t)\sum_{n=0}^{\infty}\|b_n-\ell\|\,t^n \\ &\leq (1-t)\sum_{n=0}^{\infty}(\|a_n-\ell\|-r)\,t^n \\ &= (1-t)\sum_{n=0}^{\infty}\|a_n-\ell\|\,t^n-(1-t)r\sum_{n=0}^{\infty}t^n\end{aligned}$$

and thus we have

$$\left\|(1-t)\sum_{n=0}^{\infty}b_nt^n-\ell\right\|\leq(1-t)\sum_{n=0}^{\infty}\|a_n-\ell\|\,t^n-r. \tag{2.3}$$

Since $a_n\overset{r-A}{\to}\ell$, we can write

$$\limsup_{n\to\infty}\left\|(1-t)\sum_{n=0}^{\infty}a_nt^n-\ell\right\|=\limsup_{n\to\infty}\left\|(1-t)\sum_{n=0}^{\infty}(a_n-\ell)\,t^n\right\|\leq r.$$

Taking the limit superior both sides in (2.3), we obtain

$$\limsup_{n\to\infty}\left\|(1-t)\sum_{n=0}^{\infty}b_nt^n-\ell\right\|\leq\limsup_{n\to\infty}\left[(1-t)\sum_{n=0}^{\infty}\|a_n-\ell\|\,t^n\right]-r.$$

This implies that $b_n\overset{A}{\to}\ell$. □

Proposition 2.2. *(i) If $a_n\overset{r-A}{\to}\ell_1$ and $b_n\overset{r-A}{\to}\ell_2$, then $a_n+b_n\overset{2r-A}{\to}\ell_1+\ell_2$.*
(ii) If $a_n\overset{r-A}{\to}\ell$, then $\lambda a_n\overset{|\lambda|r-A}{\to}\lambda\ell$ for each $\lambda\in\mathbb{R}$.

Proof. (i) Suppose that $a_n \overset{r-A}{\to} \ell_1$ and $b_n \overset{r-A}{\to} \ell_2$. Let $\varepsilon > 0$ and $t \in (0,1)$. Then there exist $N_1(\varepsilon), N_2(\varepsilon) \in \mathbb{N}$ such that

$$\left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell_1 \right\| < r + \frac{\varepsilon}{2} \text{ for all } n \geq N_1(\varepsilon)$$

and

$$\left\| (1-t) \sum_{n=0}^{\infty} b_n t^n - \ell_2 \right\| < r + \frac{\varepsilon}{2} \text{ for all } n \geq N_2(\varepsilon).$$

Let $N(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$. Hence, we have

$$\begin{aligned} \left\| (1-t) \sum_{n=0}^{\infty} (a_n + b_n) t^n - (\ell_1 + \ell_2) \right\| &\leq \left\| (1-t) \sum_{n=0}^{\infty} a_n t^n + (1-t) \sum_{n=0}^{\infty} b_n t^n - \ell_1 - \ell_2 \right\| \\ &\leq \left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell_1 \right\| + \left\| (1-t) \sum_{n=0}^{\infty} b_n t^n - \ell_2 \right\| \\ &< r + \varepsilon/2 + r + \varepsilon/2 = 2r + \varepsilon \end{aligned}$$

for all $n \geq N(\varepsilon)$, which completes the proof.

(ii) For $\lambda = 0$ the statement is trivial. Let $\lambda \neq 0$. Since $a_n \overset{r-A}{\to} \ell$, for every $\varepsilon > 0$ and each $t \in (0,1)$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$\left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell \right\| < r + \frac{\varepsilon}{|\lambda|} \text{ for all } n \geq N(\varepsilon).$$

Then we have

$$\begin{aligned} \left\| (1-t) \sum_{n=0}^{\infty} \lambda a_n t^n - \lambda \ell \right\| &= |\lambda| \left\| (1-t) \sum_{n=0}^{\infty} a_n t^n - \ell \right\| \\ &\leq |\lambda| \left(r + \frac{\varepsilon}{|\lambda|} \right) \\ &= |\lambda| r + \varepsilon \end{aligned}$$

for all $n \geq N(\varepsilon)$. Thus, we obtain $\lambda a_n \overset{|\lambda| r-A}{\to} \lambda \ell$ for each $\lambda \in \mathbb{R}$. □

3. Conclusion

The converse of Theorem 2.1 is not true. That is, if a sequence is rough Abel convergent, it may not rough convergent to the same point. The Proposition 2.2 shows that the sum of rough Abel convergent sequences with the same degree of roughness converges with a different degree of roughness. In other words, the roughness degree $2r$ cannot be decreased. It also states that the scalar product of a rough Abel convergent sequence converges with a different degree of roughness. After that, we can examine some properties of the set of rough Abel limit points of a sequence.

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