Konuralp Journal of Mathematics
Volume 1 No. 1 pp. 1–7 (2013) ©KJM

CYCLIC-PARALLEL RICCI TENSOR OF ALMOST S-MANIFOLDS

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Abstract. In this paper, we consider cyclic-parallel almost S-manifolds and we obtain some results.

1. Introduction

An extensive research about contact geometry is done in recent years. We recall the precise definitions. Let M be a $(2n+s)$-dimensional manifold. We say that M is equipped with an f-structure with a parallelizable kernel, more briefly $f.pk$-structure, if there are given on M an f-structure φ, s global vector fields $\xi_1, ..., \xi_s$ and 1-forms $\eta_1, ..., \eta_s$ on M satisfying the following conditions

$$\varphi(\xi_i) = 0, \quad \eta_i \circ \varphi = 0, \quad \varphi^2 = -Id + \sum_{j=1}^{s} \eta_j \otimes \xi_j, \quad \eta_i(\xi_j) = \delta_j^i, \tag{1.1}$$

for all $i, j \in \{1, ..., s\}$; we denote by D the bundle $Im(\varphi)$, and we set $\overline{\xi} := \xi_1 + ... + \xi_s$, $\overline{\eta} := \eta_1 + ... + \eta_s$. The structure (φ, ξ_i, η_j) on M is said to be normal if and only if $N_\varphi = 0$, where N_φ is the $(2,1)$-tensor on M given by $N_\varphi := [\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta_i \otimes \xi_i$. On a manifold equipped with an $f.pk$-structure there always exists a compatible Riemannian metric g in the sense that for each $X, Y \in \Gamma(TM)$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{j=1}^{s} \eta_j(X)\eta_j(Y). \tag{1.2}$$

However such that a metric on M is not unique: we fix one of them; then the structure obtained is called a metric $f.pk$-structure. Let Φ be the Sasaki form of φ defined by $\Phi(X, Y) := g(X, \varphi Y)$ for $X, Y \in \Gamma(TM)$. It may be observed that D is the orthogonal complement of the bundle $\ker(\varphi) = \langle \xi_1, ..., \xi_s \rangle$.

2000 *Mathematics Subject Classification.* 53D10, 53C15, 53C25, 53C35.
Key words and phrases. S-structures, f-structures almost S-manifolds, cyclic-parallel Ricci tensor.

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The metric $f.pk$-manifold $(M, \varphi, \xi_i, \eta_j, g)$ is said to be an almost S-manifold if and only if $d\eta_1 = ... = d\eta_s = \Phi$. Almost S-manifold which are normal are called S-manifolds.

The study of f-manifolds was started by Blair, Goldberg, Yano, Vanzura, cf. [3], [6] [11]. Almost S-structures were studied; without being precisely named, by Cabrerizo, Fernandez, Fernandez, cf., [7]. Then Duggal, Pastore and Ianus, cf. [8], also studied such manifolds and gave them the name "almost S-manifolds". S-manifolds were introduced by Blair cf. [3], who proved that the space of a principal toroidal bundle over a Kaehler manifolds is an S-manifold. S-structures are a natural generalization of Sasakian structures, but unlike Sasakian manifolds, no S-structure can be realized on a simply connected, compact manifold cf. [5]. In [9] there is an example of an even dimensional principal toroidal bundle over a Kaehler manifold which does not carry any Sasakian structure. On the other hand, there is constructed an S-structure on the even dimensional manifold $U(2)$. It is well known that $U(2)$ does not admit a Kaehler structure. We conclude that there exist manifolds such that the best structure we can hope to obtain on them is an S-structure.

On an almost S-manifold $(M, \varphi, \xi_i, \eta_j, g)$ there are defined the $(1,1)$-tensor fields $h_i := \frac{1}{2} L_{\xi_i} \varphi$ for $i = 1, ..., s$ cf. ([7] (2.5)). We use extensively the properties of these tensor fields in the present paper. In particular these operators are self adjoint, traceless, anti-commute with φ and for each $i, j \in \{1, ..., s\}$

$$h_i \xi_j = 0, \quad \eta_i \circ h_j = 0, \tag{1.3}$$

cf. [7]. Moreover the following identities hold, cf. [8],

$$\nabla_X \xi_i = -\varphi X - \varphi h X, \quad \nabla_{\xi_i} \varphi = 0, \quad \nabla_{\xi_i} \xi_j = 0, \tag{1.4}$$

where ∇ is the Levi Civita connection of g, $X \in \Gamma(TM)$ and $i, j \in \{1, ..., s\}$. We shall sometimes use the following curvature identity related to ∇

$$R_{\xi_i X} \xi_j - \varphi\left(R_{\xi_i \varphi X} \xi_j\right) = 2\left(\left(h_i \circ h_j\right) X + \varphi^2 X\right), \tag{1.5}$$

which can be immediately obtained combining the first equation on ([7] pag. 158) and (1.4).

In 1995 Blair, Koufogiorgos and Papantonio, cf. [4], studied contact metric manifolds such that the characteristic vector field belongs to the (κ, μ)-nullity distribution. This concept is generalized for almost S-manifolds by Cappelletti Montano and Di Terlizzi in [1].

In the present paper we are concerned cyclic-parallel Ricci tensor of almost S-manifolds.

2. Preliminaries

Definition 2.1. [1] Let M be an almost S-manifold, κ, μ real constant. We say that M verifies the (κ, μ)-nullity condition if and only if for each $i \in \{1, ..., s\}$, $X, Y \in \Gamma(TM)$ the following identity holds

$$R_{XY} \xi_i = \kappa\left(\overline{\eta}(X) \varphi^2 Y - \overline{\eta}(Y) \varphi^2 X\right) + \mu\left(\overline{\eta}(Y) h_i X - \overline{\eta}(X) h_i Y\right). \tag{2.1}$$

Lemma 2.1. [1] *Let M be an almost S-manifold verifying the (κ, μ)-nullity condition. Then we have*

i) $h_i \circ h_j = h_j \circ h_i$ for each $i, j \in \{1, ..., s\}$,

ii) $k \leq 1$,

iii) if $\kappa < 1$ then, for each $i \in \{1, ..., s\}$, h_i has eigenvalues 0, $\pm\sqrt{1-\kappa}$,

iv) $h_i^2 = (\kappa - 1)\varphi^2$.

Proposition 2.1. [1] *Let M be an almost S-manifold verifying the (κ, μ)-nullity condition. Then*

$$h_1 = ... = h_s. \tag{2.2}$$

Remark 2.1. [1] Throughout all this paper whenever (2.1) holds we put $h := h_1 = ... = h_s$. Then (2.1) becomes

$$R_{XY}\xi_i = \kappa\left(\overline{\eta}(X)\varphi^2 Y - \overline{\eta}(Y)\varphi^2 X\right) + \mu\left(\overline{\eta}(Y)hX - \overline{\eta}(X)hY\right). \tag{2.3}$$

Furthermore, using (2.3), the symmetries properties of the curvature tensor and the symmetry of φ^2 and h, we get

$$R_{\xi_i X}Y = \kappa\left(\overline{\eta}(Y)\varphi^2 X - g\left(X, \varphi^2 Y\right)\overline{\xi}\right) + \mu\left(g(X, hY)\overline{\xi} - \overline{\eta}(Y)hX\right). \tag{2.4}$$

Proposition 2.2. [1] *Let M be an almost S-manifold verifying the (κ, μ)-nullity condition. Then M is an S-manifold if and only if $\kappa = 1$.*

3. Properties of the Curvature

Let $\left(M^{2n+s}, \varphi, \xi_i, \eta_j, g\right)$, $i, j \in \{1, ..., s\}$, be an almost S-manifold. We consider the $(1,1)$-tensor fields defined by

$$l_{ij}(X) = R_{X\xi_i}\xi_j,$$

for each $i, j \in \{1, ..., s\}$, $X \in \Gamma(TM)$ and put $l_i = l_{ii}$.

Lemma 3.1. [1] *Let M be an almost S-manifold. For each $i, j \in \{1, ..., s\}$ the following identities hold*

$$\varphi \circ l_{ij} \circ \varphi - l_{ij} = 2\left(h_i \circ h_j + \varphi^2\right), \tag{3.1}$$

$$\eta_k \circ l_{ij} = 0, \tag{3.2}$$

$$l_{ij}(\xi_k) = 0, \tag{3.3}$$

$$\nabla_{\xi_i} h_j = \varphi - \varphi \circ l_{ij} - \varphi \circ h_i \circ h_j + \varphi \circ (h_j - h_i), \tag{3.4}$$

$$\nabla_{\xi_i} h_i = \varphi - \varphi \circ l_{ij} - \varphi \circ h_i^2. \tag{3.5}$$

Lemma 3.2. [1] *Let M be an almost S-manifold verifying the (κ,μ)-nullity condition. Then for $i,j\in\{1,...,s\}$ we have*

$$\nabla_{\xi_i}h=\mu h\circ\varphi, \tag{3.6}$$

$$l\circ\varphi-\varphi\circ l=2\mu h\circ\varphi, \tag{3.7}$$

$$l\circ\varphi+\varphi\circ l=2\kappa\varphi, \tag{3.8}$$

$$Q\xi_i=2n\kappa\overline{\xi}. \tag{3.9}$$

Lemma 3.3. [1] *Let M be an almost S-manifold verifying the (κ,μ)-nullity condition. Then the following identities hold*

$$(\nabla_X\varphi)Y=g\left(Y,hX-\varphi^2X\right)\overline{\xi}-\overline{\eta}(Y)\left(hX-\varphi^2X\right), \tag{3.10}$$

$$\begin{aligned}(\nabla_Xh)Y-(\nabla_Yh)X=\ &(1-\kappa)\left(2g(X,\varphi Y)\overline{\xi}+\overline{\eta}(X)\varphi Y-\overline{\eta}(Y)\varphi X\right.\\ &(1-\mu)\left(\overline{\eta}(X)\varphi hY-\overline{\eta}(Y)\varphi hX\right),\end{aligned} \tag{3.11}$$

$$\begin{aligned}(\nabla_Xh)Y=\ &((1-\kappa)g(X,\varphi Y)+g(X,h\varphi Y))\overline{\xi}\\ &\overline{\eta}(Y)h(\varphi X+\varphi hX)-\mu\overline{\eta}(X)\varphi hY.\end{aligned} \tag{3.12}$$

Lemma 3.4. [1] *Let M be an almost S-manifold verifying the (κ,μ)-nullity condition with $\kappa<1$ and κ,μ are smooth function. Then the Ricci operator verifies the following identities*

$$Q=s\left[(2(1-n)+\mu n)\varphi^2+(2(n-1)+\mu)h\right]+2n\kappa\overline{\eta}\otimes\overline{\xi}, \tag{3.13}$$

$$Q\circ\varphi-\varphi\circ Q=2s(2(n-1)+\mu)h\circ\varphi. \tag{3.14}$$

Lemma 3.5. *Let M be a $(2n+s)$ dimensional almost S-manifold verifying the (κ,μ)-nullity condition, $\kappa<1$. Then*

(3.15)

$$\begin{aligned}(\nabla_XS)(Y,Z)=\ &s(2(1-n)+\mu n)\{\overline{\eta}(Z)(g(\varphi Y,hX)-g(Y,\varphi X))\\ &-\overline{\eta}(Y)(g(\varphi hX,Z)+g(\varphi X,Z))\}\\ &+s(2(n-1)+\mu)\{\overline{\eta}(Z)((1-\kappa)g(\varphi Y,X)+g(h\varphi Y,X))\\ &+\overline{\eta}(Y)(g(h\varphi X,Z)+(\kappa-1)g(\varphi X,Z))\\ &-\mu\overline{\eta}(X)g(\varphi hY,Z)\}-2nk\overline{\eta}(Z)(g(Y,\varphi hX)+g(Y,\varphi X))\\ &-2\overline{\eta}(Y)(g(\varphi hX,Z)+g(\varphi X,Z)).\end{aligned}$$

Proof. We know that the Ricci operator satisfies

$$(\nabla_X S)(Y,Z) = \nabla_X S(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{3.16}$$

Using (3.13) in (3.16) we have

$$\begin{aligned}(\nabla_X S)(Y,Z) = \quad & s\,(2\,(1-n) + \mu n)\,(g\,(\nabla_X \varphi)\,(\varphi Y)\,, Z) + g\,(\varphi\,(\nabla_X \varphi)\,Y, Z) \\ & s\,(2\,(n-1) + \mu)\,g\,((\nabla_X h)\,Y, Z) \\ & +2n\kappa\overline{\eta}(Z)g\,(Y, \nabla_X \xi_i) + 2n\kappa\overline{\eta}\,(Y)\,g\,(Z, \nabla_X \xi_i)\,.\end{aligned} \tag{3.17}$$

By the use of (1.4), (3.10) and (3.12) in (3.17) we get (3.15). □

4. Almost S-manifolds with Cyclic-Parallel Ricci Tensor

The Ricci tensor S of Riemannian manifold M is said to be cyclic-parallel if

$$C\nabla S = 0,$$

namely

$$(\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) = 0, \tag{4.1}$$

for all vector fields X, Y, Z.

Let M be an η-Einstein manifold whose Ricci tensor S of the form

$$S(X,Y) = Ag\,(X,Y) + B\overline{\eta}(X)\overline{\eta}(Y), \tag{4.2}$$

where A, B are non-zero real numbers and X, Y are vector fields on M. So we have;

Theorem 4.1. *Let M be a $(2n+s)$ dimensional an η-Einstein almost S-manifold of the form (4.2). If the Ricci tensor S of M is cyclic parallel then M is an S-manifold.*

Proof. Let us consider M is an η-Einstein almost S-manifold of the form (4.2). If the Ricci tensor S of M is cyclic parallel then replacing Z with ξ_i in (4.1) we can write

$$(\nabla_{\xi_i} S)(X,Y) + (\nabla_X S)(Y,\xi_i) + (\nabla_Y S)(\xi_i,X) = 0. \tag{4.3}$$

Using (4.2) after some computations we get

$$(\nabla_X S)(Y,Z) = B\left[\overline{\eta}(Z)g(Y,\nabla_X \xi_i) + \overline{\eta}(Y)g(Z,\nabla_X \xi_i)\right],$$

which implies

$$(\nabla_{\xi_i} S)(X,Y) = 0, \tag{4.4}$$

$$(\nabla_X S)(Y,\xi_i) = B\left[g(Y,\nabla_X \xi_i) + \overline{\eta}(Y)g(\xi_i,\nabla_X \xi_i)\right], \tag{4.5}$$

$$(\nabla_Y S)(\xi_i,X) = B\left[g(X,\nabla_Y \xi_i) + \overline{\eta}(X)g(\xi_i,\nabla_Y \xi_i)\right]. \tag{4.6}$$

So substituting (4.4)-(4.6) in (4.3) and using (1.2) and (1.4), the equation (4.3)

$$g(\varphi X + \varphi h X, Y) + g\,(\varphi Y + \varphi h Y, X) = 0, \tag{4.7}$$

which implies

$$g(\varphi Y, hX) = 0. \tag{4.8}$$

Replacing Y by φY in (4.8), we get

$$g\left(\varphi^2 Y, hX\right) = 0.$$

So by the use (1.1) and (1.3), we have

$$g(Y, hX) = 0, \tag{4.9}$$

for all vector fields X and Y and hence we have $h = 0$ which implies M is an S-manifold. □

Theorem 4.2. *Let M be a $(2n+s)$ dimensional a non-Sasakian almost S-manifold. If the Ricci tensor S of M is cyclic parallel then M is either S-manifold or $\kappa = \frac{-1}{4}\frac{s\mu^2 + s4n\mu}{n}$.*

Proof. Let M be an almost S-manifold. Then by [7], $\kappa \leq 1$. But if $\kappa = 1$ then M is Sasakian. Since we suppose M is non-Sasakian we have $\kappa < 1$. So by the use of (3.15) we have

$$(\nabla_{\xi_i} S)(X, Y) = s(2(n-1) + \mu)\mu g(hX, \varphi Y). \tag{4.10}$$

Similarly, using (3.15) we have

$$\begin{aligned}(\nabla_X S)(Y, \xi_i) = \; & (s(2(1-n) + \mu n) + s(2(n-1) + \mu)(1-\kappa) + 2n\kappa) g(X, \varphi Y) \\ & + (s(2(1-n) + \mu n) + s(2(n-1) + \mu) + 2n\kappa) g(hX, \varphi Y),\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}(\nabla_Y S)(\xi_i, X) = \; & (s(2(1-n) + \mu n) + s(2(n-1) + \mu) + 2n\kappa) g(\varphi X, hY) \\ & + (s(2(1-n) + \mu n) + s(2(n-1) + \mu)(1-\kappa) + 2n\kappa) g(\varphi X, Y).\end{aligned} \tag{4.12}$$

So substituting (4.10)- (4.12) into (4.3) we obtain

$$\left[4ns\mu + 4n\kappa + \mu^2\right] g(hX, \varphi Y) = 0. \tag{4.13}$$

Suppose $g(hX, \varphi Y) = 0$. Then replacing Y with φY, the last equation becomes $g\left(hX, \varphi^2 Y\right) = 0$. So using (1.1) we get $g(hX, Y) = 0$ for all vector fields X and Y snd hence; we have $h = 0$ which gives us M is an S-manifold (note that M is non-Sasakian since $n \neq 1$). If $4ns\mu + 4n\kappa + \mu^2 = 0$ then we get $\kappa = \frac{-1}{4}\frac{s\mu^2 + s4n\mu}{n}$. □

Corollary 4.1. *Let M be a $(2n+s)$ dimensional a non-Sasakian manifold with ξ_i, $i \in \{1, ..., s\}$, belonging to κ-nullity distribution. If M is S-manifold and the Ricci tensor S of M is cyclic parallel then M is locally isometric to the product $\mathbf{E}^{n+s} \times \mathbf{S}^n(4)$.*

Proof. Since ξ_i, $i \in \{1, ..., s\}$, belongs to κ-nullity distribution then $\mu = 0$. Hence from Theorem 2., we get $\kappa = 0$. So by [2], M is locally isometric to the product $\mathbf{E}^{n+s} \times \mathbf{S}^n (4)$. □

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