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COMPLEX TORSIONS AND HOLOMORPHIC HELICES

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Abstract. Recently, properties of holomorphic helix of Kahler Frenet curves on n- dimensional M Kahler manifold studied by S. Maeda, H. Tanabe and T. Adachi. In this paper we give some characterizasions for complex torsions by $\tau_{i,j}$ in the Kahler manifold to be general helix, and by considering κ_1, κ_2 curvatures of order 3. Curvatures of Frenet curve on M Kahler manifold are not constant but their ratios are constant. We investigate relationship between $\tau_{1,2}$ and $\tau_{2,3}$ complex torsions which are not seperately constant but their ratios are constant.

1. Introduction

Let M be a n-dimentional Kahler manifold, with complex structure J and Riemannian metric g. For a helix γ on M of order $d(\leq 2n)$ with the associated Frenet frame $\{V_1, \ldots, V_d\}$ and we define $\tau_{i,j}$ called complex torsions by $\tau_{i,j} = g(V_i(s), JV_j(s))$ for $1 \leq i < j \leq d$, γ is a holomorphic helix if all the complex torsions are constant [4]. They are used curvatures κ_i and complex torsions $\tau_{i,j}$ which are constant. A classical result stated by M. A. Lancert in 1802 and first proved by B. De Saint Venant in 1845 is a necessary and sufficient condition that a curve be a general helix is the ratio of curvature of torsion to be constant [7, 8]. In a Kahler manifold, a Frenet curve is called a general helix if $\frac{\tau_{1,2}}{\tau_{2,3}}$ is constant and its first and second curvatures are not constant.

If its first and second curvatures are constant and its third curvature is zero then the Frenet curve is called a *helix*. We obtained the relations between the complex torsions and their own derivations.

2. Preliminaries

2.1. **Complex Torsions.** A smooth curve $\gamma = \gamma(s)$ parametrized by its arclenght s is called a *helix of proper order* d if there exist an orthonormal system $\{V_1 = \dot{\gamma}, V_2, \ldots, V_d\}$ of vector fields along γ and positive constants $\kappa_1(s), \kappa_2(s), \ldots, \kappa_{d-1}(s)$

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Which satisfy the system of ordinary differential equations

$$D_{\dot{\gamma}}V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \qquad j = 1, 2, \dots, d$$

where $V_0 \equiv V_{d+1} \equiv 0$ and $\kappa_0 = \kappa_d = 0$ [1].

Let M be a complex n-dimensional Kahler manifold (K- manifold) with complex structure J. $\{V_1, \dots, V_d, JV_1, \dots, JV_d\}$ system is a basis of tangent space of M. A smooth curve $\gamma = \gamma(s)$ on M parametrized by its arclength s is called a *Kahler Frenet curve*, if it satisfies the following diferential equation

$$D_{\dot{\gamma}}\dot{\gamma} = \kappa(s)J\dot{\gamma} \quad \text{or} \quad D_{\dot{\gamma}}\dot{\gamma} = -\kappa(s)J\dot{\gamma}$$

for some positive C^{∞} function $\kappa = \kappa(s)$, where $D_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the Riemannian connection D of M [5].

For a Frenet curve γ in a K-manifold M of order d with associated Frenet frame $\{V_1, \dots, V_d, JV_1, \dots, JV_d\}$, we define functions $\tau_{i,j}$called *complex torsions* by

$$\tau_{i,j}(s) = \begin{cases} 0 & , i = j, i = 0, j > d \\ \langle V_i(s), JV_j(s) \rangle & , 1 \le i < j \le d \end{cases} \qquad , \|\tau_{i,j}(s)\| \le 1$$

[5].

Definition 2.1. For a curve γ on a K-manifold M of order d we call a *holomophic helix* (H - helis) if all its complex torsions are constant functions.

Let a curve γ on a K-manifold M of order d. In this stuation for

$$D_{\dot{\gamma}}V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \qquad j = 1, 2, \dots, d \quad \text{and} \quad \tau_{i,j}(s) = \langle V_i(s), JV_j(s) \rangle$$

$$D_{\dot{\gamma}}\tau_{i,j}(s) = -\kappa_{i-1}\tau_{i-1,j}(s) + \kappa_i\tau_{i+1,j}(s) - \kappa_{j-1}\tau_{i,j-1}(s) + \kappa_j\tau_{i,j+1}(s) \tag{2.1}$$

[2].

For complex torsions of helix on K-manifold of order 3 from $d = 3$, $1 \le i < j \le 3$, $i = j = 0, \quad i = 1, 2 \quad j = 1, 2, 3$ and from (2.1) we obtain

$$D_{\dot{\gamma}}\tau_{1,2} = \kappa_2\tau_{1,3} \quad , \quad D_{\dot{\gamma}}\tau_{1,3} = -\kappa_2\tau_{1,2} + \kappa_1\tau_{2,3} \quad , \quad D_{\dot{\gamma}}\tau_{2,3} = -\kappa_1\tau_{1,3}$$

or

$$\begin{bmatrix} D_{\dot{\gamma}}\tau_{1,2} \\ D_{\dot{\gamma}}\tau_{1,3} \\ D_{\dot{\gamma}}\tau_{2,3} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & 0 \\ -\kappa_2 & 0 & \kappa_1 \\ 0 & -\kappa_1 & 0 \end{bmatrix} \begin{bmatrix} \tau_{1,2} \\ \tau_{1,3} \\ \tau_{2,3} \end{bmatrix}$$

When γ a Frenet curve on K-manifold M of order 2 and $\tau_{1,2}$ is constant. Really

for $\tau_{1,2} = \langle V_1, JV_2 \rangle$

$$D_{\dot{\gamma}}\langle V_1, JV_2 \rangle = \langle D_{\dot{\gamma}}V_1, JV_2 \rangle + \langle V_1, JD_{\dot{\gamma}}V_2 \rangle = \kappa\langle V_2, JV_2 \rangle - \kappa\langle V_1, JV_1 \rangle = 0$$

Then a Frenet curve of order 2 is a H-helix.

3. Holomorphic Helices

If we give theorems and results which they known related to holomorphic helices of order 3 and 4.

Theorem 3.1. *The complex torsions of a H-helix of proper order on a K-manifold satisfy*

$$\sum_{j=1}^{i-1} \tau_{j,i}^2 + \sum_{j=i+1}^{d} \tau_{i,j}^2 \leq 1$$

For every i [4].

We take H-helices of order3 we need to choose orthonormal vectors $\{V_1, V_2, V_3\}$ which satisfy

$$\begin{array}{ccccccc} \langle V_1, V_1\rangle & = & \langle V_2, V_2\rangle & = & \langle V_3, V_3\rangle & = & 1 \\ \langle V_1, V_2\rangle & = & \langle V_1, V_3\rangle & = & \langle V_2, V_3\rangle & = & 0 \\ \langle JV_1, JV_1\rangle & = & \langle JV_2, JV_2\rangle & = & \langle JV_3, JV_3\rangle & = & 1 \\ \langle JV_1, JV_2\rangle & = & \langle JV_1, JV_3\rangle & = & \langle JV_2, JV_3\rangle & = & 0 \end{array}$$

And then we set $V_1, V_2$and V_3 as

$$\begin{aligned} V_1 &= (1, 0, \ldots, 0) \\ V_2 &= (-i\tau, \sqrt{1-\tau^2}, 0, \ldots 0) \\ V_3 &= (0, \tfrac{-i\rho}{\sqrt{1-\tau^2}}, \tfrac{\sqrt{1-\tau^2-\rho^2}}{\sqrt{1-\tau^2}}, 0, \ldots, 0) \end{aligned}$$

For positive constants $\tau = \tau_{1,2}$ and $\rho = \tau_{2,3}$ with $|\tau| \leq 1$, $\tau^2 + \rho^2 \leq 1$ then we

obtain orthonormal vectors and satisfy $\langle V_1, JV_2\rangle = \tau, \langle V_2, JV_3\rangle = \rho$, $\langle V_1, JV_3\rangle = 0$.

Corollary 3.1. *The complex torsions $\tau_{i,j}$ of a H-helix γ, $\tau_{i,j} = 0$ when $i + j$ is even* [6].

Theorem 3.2. *The complex torsions of a holomorphic helix of odd and even proper order d on a Kahler manifold satisfy the following relations.*

$\tau_{i,j+2k} = 0$	$i = 1, 2, \ldots, d-2k,$	$k = 1, 2, \ldots, (d-1)/2$ (d *odd*) $k = 1, 2, \ldots, (d-2)/2$ (d *even*)
$\kappa_1\tau_{2,d} = \kappa_{d-1}\tau_{1,d-1}$		
$\kappa_1\tau_{2,j} + \kappa_j\tau_{1,j+1} = \kappa_{j-1}\tau_{1,j-1}$	$j = 3, 5, \ldots, d-2$ (d *odd*),	$j = j = 3, 5, \ldots, d-1$ (d *even*)
$\kappa_{i-1}\tau_{i-1,d} + \kappa_{d-1}\tau_{i,d-1} = \kappa_i\tau_{i+1,d}$	$i = 3, 5, \ldots, d-2$ (d *odd*),	$i = 2, 4, \ldots, d-2$ (d *even*)
$\kappa_{i-1}\tau_{i-1,j} + \kappa_{j-1}\tau_{i,j-1} = \kappa_j\tau_{i,j+1} + \kappa_i\tau_{i+1,j}$	$i = 2, 3, \ldots, d-3$	$j = i+2, i+4, \ldots, d-1$

[4].

3.1. **Holomorphic helices of order 3.**

Theorem 3.3. *For $\{V_1, V_2, V_3\}$ orthonormal frame and κ_1, κ_2 pozitive constant on a K-manifold M . There is a H-helix γ with curvatures κ_1, κ_2 if and only if*

$$\begin{cases} \kappa_1\tau_{3,2} + \kappa_2\tau_{1,2} = 0 \\ \tau_{1,3} = 0 \end{cases}, \quad \tau_{1,2} \leq \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}} \text{ for } n \geq 3 \text{ and } \tau_{1,2} = \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}} \text{ for } n = 2$$

[4].

Theorem 3.4. *K-manifold M of order 2 and all complex torsions of H-helix of order 3 with curvatures κ_1 and κ_2 satisfy*

$$\tau_{1,2} = \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}, \quad \tau_{1,3} = 0, \quad \tau_{2,3} = \frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}}$$

or

$$\tau_{1,2} = -\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}, \quad \tau_{1,3} = 0, \quad \tau_{2,3} = -\frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}}$$

[4].

A classical result stated by M. A. Lancert in 1802 and first proved by B. De Saint Venant in 1845 is a necessary and sufficient condition that a curve be a general helix is the ratio of curvature of torsion to be constant [7, 8]. Adhering to this definition we will give the following definition.

Definition 3.1. For Frenet curve γ on a K-manifold of order 3, if the ratio of $\frac{\tau_{1,2}}{\tau_{2,3}}$ is constant, then γ is called a holomorphic helix.

Theorem 3.5. *If γ is a general helices of order 3 on K-manifold. $\frac{\kappa_1}{\kappa_2}$ is constant.*

Proof. $\tau_{i,j} = -\tau_{j,i}$, $\tau_{i,j} = 0$ $(i + j$ even$)$, $-\kappa_2\tau_{1,2} + \kappa_1\tau_{2,3} = 0$. then $\frac{\tau_{1,2}}{\tau_{2,3}} = \frac{\kappa_1}{\kappa_2}$ from hypethesis $\frac{\tau_{1,2}}{\tau_{2,3}}$ =constant then $\frac{\kappa_1}{\kappa_2}$ =constant. □

Theorem 3.6. *γ be a general helix on K-manifold of order 3. Then γ is a general helix if and only if*

$$D^{(3)}_{\dot\gamma}\tau_{1,2} + \lambda D^{(2)}_{\dot\gamma}\tau_{1,2} + \mu D_{\dot\gamma}\tau_{1,2} = 0$$

here $\lambda = -\frac{3\kappa_2'}{\kappa_2}$ *and* $\mu = (\kappa_1^2+\kappa_2^2) - \frac{\kappa_2''}{\kappa_2} - \frac{3(\kappa_2')^2}{\kappa_2^2}$.

Proof. if γ is a general helix

$$\begin{aligned} D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3} \\ D^{(2)}_{\dot\gamma}\tau_{1,2} &= \kappa_2'\tau_{1,3} - \kappa_2^2\tau_{1,2} + \kappa_1\kappa_2\tau_{2,3} \\ D^{(3)}_{\dot\gamma}\tau_{1,2} &= \kappa_2''\tau_{1,3} + \kappa_2'(-\kappa_2\tau_{1,2} + \kappa_1\tau_{2,3}) - 2\kappa_2'\kappa_2\tau_{1,2} \\ &= 3\kappa_2'\left(\frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}\tau_{1,2} + \frac{1}{\kappa_2}D^{(2)}_{\dot\gamma}\tau_{1,2}\right) + \left\{\frac{\kappa_2''}{\kappa_2} - (\kappa_1^2+\kappa_2^2)\right\} \end{aligned}$$

And we obtain

$$D_{\dot\gamma}^{(3)}\tau_{1,2}-\frac{3\kappa_2'}{\kappa_2}D_{\dot\gamma}^{(2)}\tau_{1,2}+\left\{(\kappa_1^2+\kappa_2^2)-\frac{\kappa_2''}{\kappa_2}-\frac{3(\kappa_2')^2}{\kappa_2^2}\right\}D_{\dot\gamma}\tau_{1,2}=0$$

conversely

$$\begin{aligned}D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3}\Longrightarrow\tau_{1,3}=\frac{1}{\kappa_2}D_{\dot\gamma}\tau_{1,2}\\ D_{\dot\gamma}\tau_{1,3} &= -\frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}\tau_{1,2}+\frac{1}{\kappa_2}D_{\dot\gamma}^{(2)}\tau_{1,2}\end{aligned}$$

and

$$D_{\dot\gamma}^{(2)}\tau_{1,2} = \left(-\frac{\kappa_2'}{\kappa_2^2}\right)'D_{\dot\gamma}\tau_{1,2}-\frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}^{(2)}\tau_{1,2}-\frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}^{(2)}\tau_{1,2}+\frac{1}{\kappa_2}D_{\dot\gamma}^{(3)}\tau_{1,2} \tag{3.1}$$

we know that

$$\begin{aligned}D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3}\\ D_{\dot\gamma}^{(2)}\tau_{1,2} &= \kappa_2'\tau_{1,3}-\kappa_2^2\tau_{1,2}+\kappa_1\kappa_2\tau_{2,3}\\ D_{\dot\gamma}^{(3)}\tau_{1,2} &= 3\kappa_2'D_{\dot\gamma}\tau_{1,3}+\Delta D_{\dot\gamma}\tau_{1,2}\end{aligned}$$

Where $\Delta=\frac{\kappa_2''}{\kappa_2}-(\kappa_2^2+\kappa_1^2)$, from (3.1)

$$D_{\dot\gamma}^{(2)}\tau_{1,3}=\left\{\left(-\frac{\kappa_2'}{\kappa_2^2}\right)'+\frac{\Delta}{\kappa_2}\right\}D_{\dot\gamma}\tau_{1,2}-\kappa_2'\tau_{1,2}-\frac{2(\kappa_2')^2}{\kappa_2}\tau_{1,3}+\frac{\kappa_2'\kappa_1}{\kappa_2}\tau_{2,3} \tag{3.2}$$

$D_{\dot\gamma}\tau_{1,3} = -\kappa_2\tau_{1,2}+\kappa_1\tau_{2,3}$ if we find the derivative of the given equation

$D_{\dot\gamma}^{(2)}\tau_{1,3} = -\kappa_2'\tau_{1,2}-\kappa_2D_{\dot\gamma}\tau_{1,2}+\kappa_1'\tau_{2,3}+\kappa_1D_{\dot\gamma}\tau_{2,3}$ and using $D_{\dot\gamma}\tau_{2,3}=-\kappa_1\tau_{1,3}$ we have

$$D_{\dot\gamma}^{(2)}\tau_{1,3}=-\kappa_2'\tau_{1,2}-\kappa_2D_{\dot\gamma}\tau_{1,2}+\kappa_1'\tau_{2,3}-\kappa_1^2\tau_{1,3}$$

By using the equality of (3.2) and (3.3)

$$\begin{aligned}-\kappa_2'\tau_{1,2}-\kappa_2D_{\dot\gamma}\tau_{1,2}+\kappa_1'\tau_{2,3}-\kappa_1^2\tau_{1,3} &= \left\{\left(-\frac{\kappa_2'}{\kappa_2^2}\right)'+\frac{\Delta}{\kappa_2}\right\}D_{\dot\gamma}\tau_{1,2}-\kappa_2'\tau_{1,2}\\ &\quad-\frac{2(\kappa_2')^2}{\kappa_2}\tau_{1,3}+\frac{\kappa_2'\kappa_1}{\kappa_2}\tau_{2,3}\end{aligned}$$

If we product the both sides of the equation with $\tau_{2,3}$ we have the $\kappa_1'=\frac{\kappa_2'\kappa_1}{\kappa_2}$ and then $\kappa_1'\kappa_2-\kappa_2'\kappa_1=0$ and since $\frac{\kappa_1}{\kappa_2}$ is constant then we obtain $\frac{\kappa_1}{\kappa_2}=\frac{\tau_{1,2}}{\tau_{2,3}}$ =constant.

□

Theorem 3.7. *If γ is a helix of order 3 on K-manifold then*

$$D_{\dot\gamma}^{(3)}\tau_{1,2}+(\kappa_2^2+\kappa_1^2)D_{\dot\gamma}\tau_{1,2}=0$$

Proof. Since κ_1, κ_2 are constants and for $d = 3$

$$D_{\dot\gamma}\tau_{1,2} = \kappa_2\tau_{1,3}, \qquad D_{\dot\gamma}\tau_{1,3} = -\kappa_2\tau_{1,2} + \kappa_1\tau_{2,3}, \qquad D_{\dot\gamma}\tau_{2,3} = -\kappa_1\tau_{1,3}$$

then we obtain

$$\begin{array}{rcl} D_{\dot\gamma}\tau_{1,2} &=& \kappa_2\tau_{1,3} \\ D^{(2)}_{\dot\gamma}\tau_{1,2} &=& \kappa_2 D_{\dot\gamma}\tau_{1,3} \\ &=& = -\kappa_2^2\tau_{1,2} + \kappa_1\kappa_2\tau_{2,3} \\ D^{(3)}_{\dot\gamma}\tau_{1,2} &=& -\kappa_2^2(\kappa_2\tau_{1,3}) + \kappa_1\kappa_2(-\kappa_1\tau_{1,3}) \\ &=& -(\kappa_2^2 + \kappa_1^2)D_{\dot\gamma}\tau_{1,2} \end{array}$$

where

$$D^{(3)}_{\dot\gamma}\tau_{1,2} + (\kappa_2^2 + \kappa_1^2)D_{\dot\gamma}\tau_{1,2} = 0$$

□

Corollary 3.2. *If γ is a holomorphic helix κ_1, κ_2 separately constants then $\kappa_1' = 0, \kappa_2' = 0$. From there we find*

$$D^{(3)}_{\dot\gamma}\tau_{1,2} + (\kappa_2^2 + \kappa_1^2)D_{\dot\gamma}\tau_{1,2} = 0$$

3.2. **Holomorphic helices of order 4.** From $D_{\dot\gamma}V_j(s) = -\kappa_{j-1}V_{j-1}(s) + \kappa_j V_{j+1}(s)$, $j = 1, 2, \ldots, d$ and $\tau_{i,j} = \langle V_i, JV_j\rangle$ also for curve of order 4 ($i = 1, 2, 3 \quad j = 1, 2, 3, 4$) then we have

$$\begin{array}{rcl} D_{\dot\gamma}\tau_{1,2} &=& \kappa_2\tau_{1,3} \\ D_{\dot\gamma}\tau_{1,3} &=& -\kappa_2\tau_{1,2} + \kappa_3\tau_{1,4} + \kappa_1\tau_{2,3} \\ D_{\dot\gamma}\tau_{1,4} &=& -\kappa_3\tau_{1,3} + \kappa_1\tau_{2,4} \\ D_{\dot\gamma}\tau_{2,3} &=& -\kappa_1\tau_{1,3} + \kappa_3\tau_{2,4} \\ D_{\dot\gamma}\tau_{2,4} &=& -\kappa_1\tau_{1,4} - \kappa_3\tau_{2,3} + \kappa_2\tau_{3,4} \\ D_{\dot\gamma}\tau_{3,4} &=& -\kappa_2\tau_{2,4} \end{array}$$

so, the matrix form is

$$\begin{bmatrix} D_{\dot\gamma}\tau_{1,2} \\ D_{\dot\gamma}\tau_{1,3} \\ D_{\dot\gamma}\tau_{1,4} \\ D_{\dot\gamma}\tau_{2,3} \\ D_{\dot\gamma}\tau_{2,4} \\ D_{\dot\gamma}\tau_{3,4} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & 0 & 0 & 0 & 0 \\ -\kappa_2 & 0 & \kappa_3 & \kappa_1 & 0 & 0 \\ 0 & -\kappa_3 & 0 & 0 & \kappa_1 & 0 \\ 0 & -\kappa_1 & 0 & 0 & \kappa_3 & 0 \\ 0 & 0 & -\kappa_1 & -\kappa_3 & 0 & \kappa_2 \\ 0 & 0 & 0 & 0 & -\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} \tau_{1,2} \\ \tau_{1,3} \\ \tau_{1,4} \\ \tau_{2,3} \\ \tau_{2,4} \\ \tau_{3,4} \end{bmatrix}$$

and

$$\begin{array}{ccccc} \tau_{3,1} &=& \tau_{4,2} &=& 0 \\ \kappa_2\tau_{2,1} &=& \kappa_3\tau_{4,1} &+& \kappa_1\tau_{3,2} \\ \kappa_2\tau_{4,3} &=& \kappa_1\tau_{4,1} &+& \kappa_3\tau_{3,2} \end{array}$$

Theorem 3.8. *Let M is a 2-dimentional K- manifold. For all H- helix of order 4 of complex torsions with curvatures κ_1, κ_2 and κ_3, satisfy the following equations*

$$\tau_{1,2} = \tau_{3,4} = \tau, \quad \tau_{2,3} = \tau_{1,4} = \frac{\kappa_2\tau}{\kappa_1 + \kappa_3}, \quad \tau_{1,3} = \tau_{2,4} = 0$$

where $\tau = \pm\dfrac{\kappa_1 + \kappa_3}{\sqrt{\kappa_2^2 + (\kappa_1 + \kappa_3)^2}}$

$$\tau_{1,2} = -\tau_{3,4} = \tau, \quad \tau_{2,3} = -\tau_{1,4} = \frac{\kappa_2\tau}{\kappa_1 - \kappa_3}, \quad \tau_{1,3} = \tau_{2,4} = 0$$

when $\kappa_1 \neq \kappa_3$

$\tau = \pm\dfrac{\kappa_1-\kappa_3}{\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}}$ *or* $\tau_{1,2}=\tau_{3,4}=\tau_{1,3}=\tau_{2,4}=0, \quad \tau_{2,3}=-\tau_{1,4}=\pm 1$ *where*

$\kappa_1 = \kappa_3$[4].

Theorem 3.9. *Let* γ *be a general helix on K- manifold M of order 4, so*

$$D_{\dot{\gamma}}^{(3)}\tau_{1,2}+\lambda D_{\dot{\gamma}}^{(2)}\tau_{1,2}+\mu D_{\dot{\gamma}}\tau_{1,2}=0$$

here $\lambda = -\dfrac{3\kappa_2'}{\kappa_2}$ *and* $\mu = \dfrac{3(\kappa_2')^2}{\kappa_2^3}-\dfrac{\kappa_2''}{\kappa_2}+\kappa_1^2+\kappa_2^2-\kappa_3^2.$

Proof.

$$\begin{aligned}
D_{\dot{\gamma}}\tau_{1,2} &= \kappa_2\tau_{1,3}\\
D_{\dot{\gamma}}^{(2)}\tau_{1,2} &= \kappa_2'\tau_{1,3}-\kappa_2^2\tau_{1,2}+\kappa_2\kappa_3\tau_{1,4}+\kappa_2\kappa_1\tau_{2,3}\\
&\text{and}\\
D_{\dot{\gamma}}^{(3)}\tau_{1,2} &= 3\kappa_2' D_{\dot{\gamma}}\tau_{1,3}+(\kappa_2''-\kappa_2^3-\kappa_2\kappa_3^2-\kappa_1^2\kappa_2)\tau_{1,3}+2\kappa_1\kappa_2\kappa_3\tau_{2,4}
\end{aligned}$$

$\kappa_1\tau_{2,d}=\kappa_{d-1}\tau_{1,d-1}$ using this relation, $\kappa_1\tau_{2,4}=\kappa_3\tau_{1,3}$ is obtained and in the above expression

$$2\kappa_1\kappa_2\kappa_3\tau_{2,4} = 2\kappa_2\kappa_3^2\tau_{1,3}$$

is written,

$$\begin{aligned}
D_{\dot{\gamma}}^{(3)}\tau_{1,2} &= 3\kappa_2' D_{\dot{\gamma}}\tau_{1,3}+(\kappa_2''-\kappa_2^3-\kappa_2\kappa_3^2-\kappa_1^2\kappa_2)\tau_{1,3}+2\kappa_2\kappa_3^2\tau_{1,3}\\
&= 3\kappa_2' D_{\dot{\gamma}}\tau_{1,3}(\kappa_2''-\kappa_2^3+\kappa_2\kappa_3^2-\kappa_1^2\kappa_2)\tau_{1,3}
\end{aligned}$$

is obtained and for

$$\begin{aligned}
D_{\dot{\gamma}}\tau_{1,2} &= \kappa_2\tau_{1,3} \Longrightarrow \tau_{1,3}=\frac{1}{\kappa_2}D_{\dot{\gamma}}\tau_{1,2}\\
&\Longrightarrow D_{\dot{\gamma}}\tau_{1,3}=\left(\frac{1}{\kappa_2}\right)' D_{\dot{\gamma}}\tau_{1,2}+\frac{1}{\kappa_2}D_{\dot{\gamma}}^{(2)}\tau_{1,2}
\end{aligned}$$

we find

$$D_{\dot{\gamma}}^{(3)}\tau_{1,2} = \frac{3\kappa_2'}{\kappa_2^2}D_{\dot{\gamma}}^{(2)}\tau_{1,2}+\left\{-\frac{3(\kappa_2')^2}{\kappa_2^2}+\frac{\kappa_2''}{\kappa_2}-\kappa_2^2+\kappa_3^2-\kappa_1^2\right\}D_{\dot{\gamma}}\tau_{1,2}$$

or

$$D_{\dot{\gamma}}^{(3)}\tau_{1,2}+\lambda D_{\dot{\gamma}}^{(2)}\tau_{1,2}+\mu D_{\dot{\gamma}}\tau_{1,2}=0$$

Where $\lambda = -\dfrac{3\kappa_2'}{\kappa_2}$ and $\mu = \dfrac{3(\kappa_2')^2}{\kappa_2^3}-\dfrac{\kappa_2''}{\kappa_2}+\kappa_1^2+\kappa_2^2-\kappa_3^2$

□

Theorem 3.10. *If* γ *is a helix on K- manifold of order 4*

$$D_{\dot{\gamma}}^{(3)}\tau_{1,2}+\left\{\kappa_1^2+\kappa_2^2-\kappa_3^2\right\}D_{\dot{\gamma}}\tau_{1,2}=0$$

Proof.

$$\begin{aligned} D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3} \\ D^{(2)}_{\dot\gamma}\tau_{1,2} &= -\kappa_2^2\tau_{1,2}+\kappa_2\kappa_3\tau_{1,4}+\kappa_2\kappa_1\tau_{2,3} \\ D^{(3)}_{\dot\gamma}\tau_{1,2} &= -\kappa_2^2 D_{\dot\gamma}\tau_{1,2}-\kappa_2\kappa_3^2\tau_{1,3}-\kappa_1^2\kappa_2\tau_{1,3}+2\kappa_1\kappa_2\kappa_3\tau_{2,4} \end{aligned}$$

using the equation $\kappa_1\tau_{2,d}=\kappa_{d-1}\tau_{1,d-1}$, $\kappa_1\tau_{2,4}=\kappa_3\tau_{1,3}$ is obtained and from the above equation

$$2\kappa_1\kappa_2\kappa_3\tau_{2,4} = 2\kappa_2\kappa_3^2\tau_{1,3}$$

and

$$D_{\dot\gamma}\tau_{1,2} = \kappa_2\tau_{1,3} \Longrightarrow \tau_{1,3}=\frac{1}{\kappa_2}D_{\dot\gamma}\tau_{1,2}$$

using the equations,

$$\begin{aligned} D^{(3)}_{\dot\gamma}\tau_{1,2} &= -\kappa_2^2 D_{\dot\gamma}\tau_{1,2}-\frac{\kappa_2\kappa_3^2}{\kappa_2}D_{\dot\gamma}\tau_{1,2}-\frac{\kappa_2\kappa_1^2}{\kappa_2}D_{\dot\gamma}\tau_{1,2}+2\kappa_2\kappa_3^2\frac{1}{\kappa_2}D_{\dot\gamma}\tau_{1,2} \\ &= (-\kappa_2^2+\kappa_3^2-\kappa_1^2)D_{\dot\gamma}\tau_{1,2} \end{aligned}$$

is obtained

$$D^{(3)}_{\dot\gamma}\tau_{1,2}+\left\{\kappa_1^2+\kappa_2^2-\kappa_3^2\right\}D_{\dot\gamma}\tau_{1,2}=0.$$

□

Corollary 3.3. *If γ is a helix, because of κ_1,κ_2 will be constants sperately, $\kappa_1'=0,\kappa_2'=0$. Then we obtain*

$$D^{(3)}_{\dot\gamma}\tau_{1,2}+\left\{\kappa_1^2+\kappa_2^2-\kappa_3^2\right\}D_{\dot\gamma}\tau_{1,2}=0.$$

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