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SOME PROPERTIES OF FINITE {0,1}-GRAPHS

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Abstract. Let G=(V,E) be a connected graph , X be a subset of V, A be a finite subset of non-negative integers and $n(x, y)$ be the total number of neighbours of any two vertices x, y of X. The set X is called A-semiset if $n(x, y) \in A$ for any two vertices x ande y of X. If X is a A-semiset, but not B-semiset for any subset B of A ,the set X is called A-set. The graph G=(V,E) is a A-semigraph and A-graph if V is the A-semiset and A-set, respectively. Mulder [2] observed that $\{0, \lambda\}$-semigraphs(these graphs are called $(0, \lambda)$-graphs by Mulder [2]), $(\lambda \geq 2)$, are regular. Furthermore a lower bound for the degree of $\{0, \lambda\}$-semigraphs with diameter at least four was derived by Mulder [2].

In this paper, we determined basic properties of finite bigraphs with at least one {0,1}-part.

1. Introduction

Let us first recall some definitions and results. For more details, (see [1]). To facilitate the general definition of a graph, we first introduce the concept of the unordered product of a set V with itself. Recall that the ordered product or cartesian product of a set V with itself,denoted by $V \times V$, is defined to be the set of all ordered pairs (s, t), where $s \in V$ and $t \in V$. The symbol $\{s, t\}$ will denote an unordered pair.

A graph G=(V,E) consists of a finite nonempty set V of v vertices together with a prescribed set E of e unordered pairs of distinct vertices of V. If a pair u=$\{x, y\}$ is an edge of G, u is said to joins x and y. We write $u = xy$ and say that vertices x and y are adjacent vertices; the vertex x and the edge u are incident with each other, If two distinct edges u and v are incident with a common vertex, then they are adjacent edges. A vertex z which adjacents to two distinct vertices x and y is called neighbour of x and y. The neighborhood of a vertex x is the set $N(x)$ consists of all vertices which are adjacent to x. The set $N_i(x)$ is the set of vertices at distance i from $x.G\,[W]$ is the subgraph of G induced by the vertex or edge set W. The degree of a vertex p is the number $d\,(p)$ of edges which are incident with

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it. Let X be a subset of V. The integer n , where $n+1=\max\{d(p):p\in X\}$, is called the order of the set X. The minimum degree among the vertices of G=(V,E) is denoted by $\delta(G)$. If G=(V,E) contains a cycle, the girth of G=(V,E) denoted $g(G)$ is the lenght of its shortest cycle.

Definition 1.1. Let G=(V,E) be a connected graph , X be a subset of V, A be a finite subset of non-negative integers and $n\,(x,y)$ be the total number of neighbours of any two different x,y of X. The set X is called A-semiset if $n\,(x,y)\in A$ for any two vertices x ande y of X. If X is a A-semiset, but not B-semiset for any subset B of A ,the set X is called A-set. The graph G=(V,E) is a A-semigraph and A-graph if V is the A-semiset and A-set, respectively. If the set X does not contain edge will be called edge-free.

Mulder [2] observed that $\{0,\lambda\}$-semigraphs(these graphs are called $(0,\lambda)$-graphs by Mulder [2]), $(\lambda\geq 2)$, are regular. Furthermore a lower bound for the degree of $\{0,\lambda\}$-semigraphs with diameter at least four was derived by Mulder [2].

Definition 1.2. A bigraph (or bipartite graph) G=($P\cup L$,E) is a graph whose vertex set $P\cup L$ can be partitioned into subsets P and L in which a way that each edge of E joins a vertex in P to a vertex in L. Its clear that the parts P and L are edge-free.

In this paper, we determined basic properties of finite (0,1)-graphs.

2. Main Results

Corollary 2.1. *Every subgraph of a $\{0,1\}-$ semigraph is a $\{0,1\}-$ semigraph.*
Proof.*Let $G=(V,E)$ be a $\{0,1\}-$ semigraph and let $G'=(V',E')$ be subgraph of G. Since G is a $(0,1)-$ graph and $V'\subset V$ for all $x,y\in V',|N(x)\cap N(y)|\leq 1$. Therefore G' is a $(0,1)-$ graph.*

Corollary 2.2. *G is a $\{0,1\}-$ semigraph if and only if G is C_4-free.*
Proof.*Let G be a $(0,1)-$ graph. Suppose that, G isn't C_4-free. Then, G contains at least one C_4 in which $u_1-u_2-u_3-u_4-u_1$ for $u_1,u_2,u_3,u_4\in V$. Therefore, $N(u_1)\cap N(u_3)=\{u_2\}$ and $N(u_1)\cap N(u_3)=\{u_4\}$ .In this case, $|N(u_1)\cap N(u_3|=2$. This case contradicts with being a $(0,1)-$ graph of G. So, G is C_4-free.*
Conversely, let G be C_4-free. If G isn't a $(0,1)-$ graph, then there are at least two different $u_1,u_2\in V$ such that $|N(u_1)\cap N(u_2|\geq 2$. Let v_1 and v_2 be two different vertices in the set $N(u_1)\cap N(u_2)$. Then $u_1-v_2-u_1-v_1-u_1$ is cycle of four lenght in G. This contradicts with being C_4-free of G. So, G is a $(0,1)-$ graph.

Theorem 2.1. *Let $G=(V,E)$ be a $\{0,1\}-$ semigraph, gird$\,(G)\geq 8$ and $v\in V$. If $P=N_0(v)\cup N_2(v)$ and $L=N_1(v)\cup N_3(v)$, then $G'=[(P\cup L)]$ is bipartite subgraph of G.*
Proof. *Let G be a $\{0,1\}-$ semigraph. Since G' is subgraph of G, G' is a is a $(0,1)-$ graph by proposition 2.1. By the definition of P and $L, P\cap L=nothing$. In this case, we show that $G'=[(P\cup L)]$ is a bipartite $\{0,1\}-$ semigraph with parts P and L.*
(i)Suppose, there are at least two vertices x and y in P such that $\{x,y\}\in E$. If $x=v$, then $y\in N_2(v)$. Thus, there is a u vertex in L which $N(x)\cap N(y)=N(v)\cap N(y)=\{u\}$. Then , $x-u-y-x$ is cycle of three-lenght which contradicts $g(G)\geq 8$. The same contradict occurs for $y=v$, thus $x\neq v\neq y$ and $x,\,y\in N_2(v)$.

Let $P_1 : x - u - v$ and $P_2 : v - t - y$ be two paths. In this case, if $u \neq t$, $P_1 \cup P_2$ is a cycle of lenght 5. This contradicts $g(G) \geq 8$. Therefore $u = t$. In this case, P contains a cycle of lenght 3, contradict $g(G) \geq 8$. So, P is edge-free.
(ii)Suppose, there are at least two different x and y in L for which $\{x, y\} \in E$. There are three cases for x and y.
Case 1: *If $x, y \in N_1(v)$ then, $x - y - v - x$ is a cycle of lenght 3. This contradicts with $g(G) \geq 8$.*
Case 2: *If $x, y \in N_3(v)$, let us consider minumum paths*

$$P_1 : v - x_1 - x_2 - x \text{ and } P_2 : v - y_1 - y_2 - y$$

For $\forall$ $i, j : 1, 2$, if $x_i \neq y_j$, then seven lenght $v - x_1 - x_2 - x - y - y_2 - y_1 - v$ cycle is obtained. This contradicts $g(G) \geq 8$. Then, there is at least one pair (i, j) for which $x_i = y_j$. Thus G contains cycle of lenght 5 or 3.
If $x_i = y_j$, then G consist at least one $x - x_2 - x_1 - y_2 - y - x$ five-lenght cycle, contradicts $g(G) \geq 8$. Similar contradiction is obtained an all other cases.
Case 3: *Let $x \in N_1(v)$ and $y \in N_3(v)$. Suppose $P_1 : v - y_1 - y_2 - y$ is three lenght path. If $x \in P$, then $x - y_2 - y - x$ is a cycle of three-lenght in G, contradiction. Therefore, $x \notin P_1$, $x \neq y_1$ and $x \neq y_2$. Thus $v - x - y - y_2 - y_1 - v$ is a cycle of lenght 5 in G. This contradicts with $g(G) \geq 8$. So, L is edge-free.*

Theorem 2.2. *Let G=(P $\cup$ L,E) be a connected bigraph with parts P and L. If the part P is a {0,1}-semiset, the part L is {0,1}-semiset and G=(P $\cup$ L,E) is {0,1}-semibigraph.*
Proof. *Let G=(P $\cup$ L,E) be a bigraph with parts P and L and let the part P be a {0,1}-semiset. Assume that the part L does not {0,1}-semiset. Then the part L has at least two distinct vertices u and w having at least two distinct common neighbours x and y in the part P. This contradict to choosen of the part P. Thus the part L is {0,1}-semiset and G=(P $\cup$ L,E) is {0,1}-semibigraph.*

Let G=($P \cup L$,E) be a {0,1}-semibigraph with parts P and L. and $|P| = v$, $|L| = b$, the vertices of P will be labelled $p_1, p_2, ..., p_v$ Similary, the vertices of L will be labelled $l_1, l_2, ..., l_b$. To make our notation even more concise we define,

$$r_{ij} = \begin{cases} 0, & If \;\; p_i l_j \notin E \\ 1, & If \;\; p_i l_j \in E \end{cases}$$

we see that the $(i, j)th$ entry of the matrix $A = [r_{ij}]_{n \times m}$ is just the number r_{ij}. The matrix A is called incidence matrix of G. Where, $n = |V(G)|$, $m = |E|$. The matrix.$A' = [r_{ij}]_{vxb}$is called blok matrix of G

Theorem 2.3. *Let $A = [r_{ij}]_{vxb}$ be the blok matrix of a {0,1}-semibigraph G, then the following equations valid.*

(i) ${}_{i=1}^{v} r_{ij} = v_j$, ${}_{j=1}^{v} r_{ij} = b_i$ and
(ii) ${}_{j=1}^{b} v_j = {}_{j=1}^{b} \left({}_{i=1}^{v} r_{ij}\right) = {}_{i=1}^{v} \left({}_{j=1}^{b} r_{ij}\right) = {}_{i=1}^{v} b_i$.

Proof. If we add the 1's in each column, column by column, we get ${}_{i=1}^{b} v_i$. If we add the 1's in each row, row by row, we get ${}_{i=1}^{v} b_i$. But obviously we are just counting the same number of 1's in two different ways so we have the equations
$(i)_{i=1}^{v} r_{ij} = v_j,_{j=1}^{v} r_{ij} = b_i$ and
$(ii)_{j=1}^{b} v_j =_{j=1}^{b} \left({}_{i=1}^{v} r_{ij}\right) =_{i=1}^{v} \left({}_{j=1}^{b} r_{ij}\right) =_{i=1}^{v} b_i$

Theorem 2.4. *If $r_{ij} = 0$ then the number adjacent vertices to p_i and dont have common neihbour to l_j is $d(p_i) - p_{ij}$.*

Proof. *Since $d(p_i)$ is the total number of vertices which are adjacent with p_i and by definition p_{ij} the result is immediate.*

Theorem 2.5. *If G=($P \cup L$,E) be a $\{0,1\}$-semibigraph with parts P and L and the part P be a $\{0,1\}$-semiset and $p_{ij} = d(l_j)$ for every vertex p_i of P and vertex l_j of L such that $r_{ij} = 0$ then P is a$\{1\}$-set.*

Proof. *Since $|L| = b \geq 1$, there is a vertex l_k of L, say. We must show that the set P is $\{1\}$-set, that is, for any distinct two vertices p_i and p_j of P, $n(p_i, p_j) = 1$. Let p_i, p_j be two distinct vertices of P. If $r_{ik} = r_{jk} = 1, n(p_i, p_j) = 1$. If $r_{ik} = 0$ and $r_{jk} = 1$ then by assumption $p_{ik} = d(l_k)$ so that p_i has a common neighbour with vertex which is adjacent to l_k. In particular, p_i and p_j have common neighbour. Thus $n(p_i, p_j) = 1$. Finally, if $r_{ik} = r_{jk} = 0$, using the hypothesis once again, for a vertex q which is adjacent with l_k $n(p_i, q) = 1$. If the vertex p_j is adjacent with common neighbour of vertices p_i and q, $n(p_i, p_j) = 1$ and otherwise, by the hypothesis one last time to get a common neighbour of vertices p_i and p_j. Therefore $n(p_i, p_j) = 1$, that is, P is $\{1\}$-set.*

Theorem 2.6. *Let G=($P \cup L$,E) be a $\{0,1\}$-semibigraph and let $|P| = v$, $|L| = b$, $\delta(L) \geq 2$. P is a$\{1\}-set$ if and only if*

$$ {}_{j=1}^{b} d(l_j)(d(l_j) - 1) \geq v(v-1) $$

Proof. *Suppose that P is $\{1\} - set$.Counting the number of pairs of vertices of P in two different ways. First of all, there are $\begin{pmatrix} v \\ 2 \end{pmatrix}$ pairs of vertices of P (counting $\{p_i, p_j\}$ to be same pair as $\{p_j, p_i\}$). Second way, since P is $\{1\} - set$, there is a uniqe l vertex of L which $l \in N(p_i) \cap N(p_j)$. Thus, the total number of pairs of vertices of P is the total number of pairs of vertices of $N(l)$, for each $l \in L$. Summed over all vertices of L, that is ${}_sumj = 1^b d(l_j)(d(l_j) - 1)/2$.*
So,

$$ {}_{j=1}^{b} d(l_j)(d(l_j) - 1) = v(v-1) $$

Suppose, convercely, that

$$ {}_{j=1}^{b} d(l_j)(d(l_j) - 1) \geq v(v-1).....* $$

We prove that P is $\{1\}-set$ by induction on v.Since $\delta(L) \geq 2$, v is at least two and $b = 1$. In the case, P is $\{1\} - set$. If $v = 3$ there are exactly three possibilities , for $b = 1, 2$ or 3. Of these, only the case $b = 1$, $p_1 = v = 3$ and $b = 3$, $p_1 = p_2 = p_3 = 2$ satisfy inequality. In both of these case, P is $\{1\} - set$.
Suppose then that if the inequality holds for a partial adjacent bigraph G' with part P' and L' which P' is a part with fewer than v vertices then $P'$is $\{1\} - set$.We may assume ${}_{j=1}^{b} d(l_j)(d(l_j) - 1) \geq v(v-1)$ in G, where $v \geq 4$. Let $p \in P$ be and consider the partial adjacent bigraph G' with part P' and L' which is the restiriction of G to $P \setminus \{p\}$. So $P' = P - \{p\}$ and $L' = \{l \in L \mid \{p, l\} \notin E \text{ and } d(l) \geq 3\}$. As $|P'| = v - 1$, we attempt to prove that P is $\{1\} - set$ by showing that approprite inequality above holds. Its right hand side becomes $(v-1)(v-2)$.

In G' ,

$$\begin{aligned}\sum_{l'_j} d(l'_j)(d(l'_j)-1) &= \sum_{l_j\notin N(p)} d(l_j)(d(l_j)-1) + \sum_{l_j\in N(p),d(l_j)\geq 3} d(l'_j)(d(l'_j)-1)\\ &= \sum_{l_j\notin N(p)} d(l_j)(d(l_j)-1) + \sum_{l_j\in N(p),d(l_j)\geq 3} (d(l_j)-1)(d(l_j)-2)\\ &= \sum_{l_j\notin N(p)} d(l_j)(d(l_j)-1) + \sum_{l_j\in N(p),d(l_j)\geq 3} d(l_j)(d(l_j)-1)\\ &\quad -2\left(\sum_{l_j\in N(p),d(l_j)\geq 3} (d(l_j)-1)\right)\end{aligned}$$

In G,

$$\begin{aligned}\sum_{l_j} d(l_j)(d(l_j)-1) &= \sum_{l_j\notin N(p)} d(l_j)(d(l_j)-1) + \sum_{l_j\in N(p),d(l_j)\geq 3} d(l_j)(d(l_j)-1)\\ &\quad + \sum_{l_j\in N(p),d(l_j)=2} d(l_j)(d(l_j)-1)\\ \sum_{l_j\notin N(p)} d(l_j)(d(l_j)-1) &= \sum_{l_j} d(l_j)(d(l_j)-1) - \sum_{l_j\in N(p),d(l_j)\geq 3} d(l_j)(d(l_j)-1)\\ &\quad - \sum_{l_j\in N(p),d(l_j)=2} d(l_j)(d(l_j)-1)\end{aligned}$$

Substituting in the above, we get

$$\begin{aligned}\sum_{l'_j} d(l'_j)(d(l'_j)-1) &= \sum_{l_j} d(l_j)(d(l_j)-1) - 2\sum_{l_j\in N(p),d(l_j)\geq 3} (d(l_j)-1)\\ &\quad - \sum_{l_j\in N(p),d(l_j)=2} d(l_j)(d(l_j)-1)\\ &= \sum_{l_j} d(l_j)(d(l_j)-1) - 2\left(\sum_{l_j\in N(p)} (d(l_j)-1)\right)\end{aligned}$$

By hypohesis, $\sum_{l_j} d(l_j)(d(l_j)-1) \geq v(v-1)$. *By counting total degree of the vertices of* L *which adjacent to* p*, it becomes evident that* $\sum_{l_j\in N(p)}(d(l_j)-1) \leq v-1$ *and so* $-2\left(\sum_{l_j\in N(p)}(d(l_j)-1)\right) \geq -2(v-1)$*. Therefore,*

$$\sum_{l'_j} d(l'_j)(d(l'_j)-1) \geq v(v-1) - 2(v-1) = (v-1)(v-2)$$

as desired. By our induction hypothesis, $P' = P - \{p\}$ *is* $\{1\}-$ *set in* G'*. In the case, show that , for each* p' *of* P'*,there is a exactly one common vertex of* p *and* p'*. Let* p'' *be arbitrarily vertex of* P *which* $p'' \neq p$ *and* $p'' \neq p'$*.* $P'' = P - \{p''\}$ *is a restriction of* P *with* $v-1$ *vertices and the and the argument used above shows that* $P'' = P - \{p''\}$ *is* $\{1\}-$ *set. Hence there is exactly one common vertex of* p *and* p'*.*

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