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THE HADAMARD TYPE INEQUALITIES FOR $m-$CONVEX FUNCTIONS

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Abstract. In this paper, we obtained some new Hadamard-Type inequalities for functions whose derivatives absolute values are $m-$convex. Some applications to special means of real numbers are given.

1. INTRODUCTION

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality is well known as the Hermite-Hadamard inequality for convex functions.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In recent years many authors have established several inequalities connected to Hermite-Hadamard inequality. For recent results, refinements, counterparts, generalizations and new Hadamard-type inequalities see [3], [4] and [5].

A function $f : I \to \mathbb{R}$ is said to be convex function on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In [2], G. Toader defined $m-$convexity as the following:

Definition 1.1. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m-$convex where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m-concave if -f is m-convex.

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For recent results related to above definitions we refer interest of readers to [6],[7],[8].

The following theorems which were obtained by Kavurmacı *et al.* contains the Hadamard-type integral inequalities in [1].

Theorem 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex function on $[a,b]$, then the following inequality holds:*

(1.1)
$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{b-a}{12}\left[\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(a)| + |f'(b)|\right].$$

Theorem 1.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex function on $[a,b]$, for some fixed $q \geq 1$, then the following inequality holds:*

$$(1.2) \qquad \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{b-a}{8}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + 2|f'(a)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + 2|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right].$$

The main purpose of this paper is to establish refinements inequalities of right-hand side of Hadamard's type for $m-$convex functions.

2. MAIN RESULTS

In [1], in order to prove some inequalities related to Hermite-Hadamard inequality Kavurmacı *et al.* used the following lemma.

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du$$
$$= \frac{(x-a)^2}{b-a}\int_a^b (1-t)f'(tx+(1-t)a)dt + \frac{(b-x)^2}{b-a}\int_a^b (t-1)f'(tx+(1-t)b)dt.$$

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o where $a, b \in I$ with $a < b$. If $|f'|$ is $m-$convex function on $[a,b]$ for some fixed $m \in (0,1]$, then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\left[\frac{|f'(x)| + 2m\left|f'\left(\frac{a}{m}\right)\right|}{6}\right] + \frac{(b-x)^2}{b-a}\left[\frac{|f'(x)| + 2m\left|f'\left(\frac{b}{m}\right)\right|}{6}\right].$$

Proof. From Lemma 1 and using the property of modulus we get;

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|\\ \leq\ & \frac{(x-a)^2}{b-a}\int_0^1(1-t)\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1(1-t)\left|f'(tx+(1-t)b)\right|dt.\end{aligned}$$

Since $|f'|$ is $m-$convex, we can write;

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|\\ \leq\ & \frac{(x-a)^2}{b-a}\int_0^1(1-t)\left[t\left|f'(x)\right|+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|\right]dt\\ &+\frac{(b-x)^2}{b-a}\int_0^1(1-t)\left[t\left|f'(x)\right|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt\\ =\ & \frac{(x-a)^2}{b-a}\left[\left|f'(x)\right|\int_0^1(t-t^2)dt+m\left|f'\left(\frac{a}{m}\right)\right|\int_0^1(1-t)^2dt\right]\\ &+\frac{(b-x)^2}{b-a}\left[\left|f'(x)\right|\int_0^1(t-t^2)dt+m\left|f'\left(\frac{b}{m}\right)\right|\int_0^1(1-t)^2dt\right]\\ =\ & \frac{(x-a)^2}{b-a}\left[\frac{\left|f'(x)\right|+2m\left|f'\left(\frac{a}{m}\right)\right|}{6}\right]+\frac{(b-x)^2}{b-a}\left[\frac{\left|f'(x)\right|+2m\left|f'\left(\frac{b}{m}\right)\right|}{6}\right].\end{aligned}$$

This completes the proof. □

Corollary 2.1. *In Theorem 3, if we choose $x=\frac{a+b}{2}$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{b-a}{12}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+m\left|f'\left(\frac{a}{m}\right)\right|+m\left|f'\left(\frac{b}{m}\right)\right|\right].$$

Remark 2.1. In Corollary 1, if we choose $m=1$, the inequality in (1.1) is obtained.

Theorem 2.2. *Let $f:I\subset\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable mapping on I^o where $a,b\in I$ with $a<b$. If $|f'|^q$ is $m-$convex function on $[a,b]$ and $p>1$, then the following* *inequality holds:*

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
\leq\ &\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(x-a)^2}{b-a}\left(\frac{|f'(x)|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right. \\
&\left.+\frac{(b-x)^2}{b-a}\left(\frac{|f'(x)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $m\in(0,1]$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the property of modulus we can write;

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
\leq\ &\frac{(x-a)^2}{b-a}\int_0^1(1-t)\left|f'(tx+(1-t)a)\right|dt \\
&+\frac{(b-x)^2}{b-a}\int_0^1(1-t)\left|f'(tx+(1-t)b)\right|dt.
\end{aligned}$$

By using the Hölder inequality we have:

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
\leq\ &\frac{(x-a)^2}{b-a}\left(\int_0^1(1-t)^pdt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2}{b-a}\left(\int_0^1(1-t)^pdt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is $m-$convex function:

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f'(x)\right|^q+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f'(x)\right|^q+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(x-a)^2}{b-a}\left(\frac{\left|f'(x)\right|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(b-x)^2}{b-a}\left(\frac{\left|f'(x)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Corollary 2.2. *In Theorem 4, if we choose* $x=\frac{a+b}{2}$, *we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\cdot\frac{b-a}{4}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

In Corollary 2, if we choose $m=1$ and $\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\leq 1$, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{b-a}{4}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

Theorem 2.3. *Let* $f:I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I^o *where* $a,b\in I$ *with* $a<b$. *If* $|f'|^q$ *is* $m-$*convex function on* $[a,b]$ *for some fixed* $m\in(0,1]$, *then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{(x-a)^2}{2(b-a)}\left(\frac{\left|f'(x)\right|^q+2m\left|f'\left(\frac{a}{m}\right)\right|^q}{3}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{2(b-a)}\left(\frac{\left|f'(x)\right|^q+2m\left|f'\left(\frac{b}{m}\right)\right|^q}{3}\right)^{\frac{1}{q}}$$

where $q\geq 1$.

Proof. From Lemma 1 and using the property of modulus we get;

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\int_0^1 (1-t)\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)\left|f'(tx+(1-t)b)\right|dt.$$

By using the Power-mean inequality, we have

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'(x)|^q$ is $m-$convex, we have

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left[t\left|f'(x)\right|^q+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left[t\left|f'(x)\right|^q+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$= \frac{(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'(x)\right|^q\int_0^1 (t-t^2)dt+m\left|f'\left(\frac{a}{m}\right)\right|^q\int_0^1 (1-t)^2dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'(x)\right|^q\int_0^1 (t-t^2)dt+m\left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1 (1-t)^2dt\right)^{\frac{1}{q}}$$
$$= \frac{(x-a)^2}{2(b-a)}\left(\frac{\left|f'(x)\right|^q+2m\left|f'\left(\frac{a}{m}\right)\right|^q}{3}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{2(b-a)}\left(\frac{\left|f'(x)\right|^q+2m\left|f'\left(\frac{b}{m}\right)\right|^q}{3}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 2.3. *In Theorem 5, if we choose* $x = \frac{a+b}{2}$, *we get*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{b-a}{8}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + 2m\left|f'\left(\frac{a}{m}\right)\right|^q}{3} \right)^{\frac{1}{q}} + \left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + 2m\left|f'\left(\frac{b}{m}\right)\right|^q}{3} \right)^{\frac{1}{q}} \right].$$

Remark 2.2. In Corollary 3, if we choose $m = 1$, the inequality in (1.2) is obtained.

3. APPLICATIONS TO SPECIAL MEANS

We now consider the means for arbitrary real numbers α, β $(\alpha \neq \beta)$. We take

(1) *Arithmetic mean* :

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}^+.$$

(2) *Logarithmic mean*:

$$L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|,\ \alpha,\beta \neq 0,\ \alpha,\beta \in \mathbb{R}^+.$$

(3) *Generalized log − mean*:

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{Z}\backslash\{-1,0\},\ \alpha,\beta \in \mathbb{R}^+.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

Proposition 3.1. *Let* $a, b \in \mathbb{R}^+$, $a < b$, $m \in (0,1]$ *and* $n \in \mathbb{Z}, n > 1$. *Then, we have*

$$|A(a^n,b^n) - L_n^n(a,b)| \leq n.\frac{b-a}{12}\left[\left|\frac{a+b}{2}\right|^{n-1} + m\left|\frac{a}{m}\right|^{n-1} + m\left|\frac{b}{m}\right|^{n-1} \right].$$

If we choose $m = 1$, *we obtain*

$$|A(a^n,b^n) - L_n^n(a,b)| \leq n.\frac{b-a}{12}\left[\left|\frac{a+b}{2}\right|^{n-1} + |a|^{n-1} + |b|^{n-1} \right].$$

Proof. The assertion follows from Corollary 1 applied to the m-convex mapping $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. □

Proposition 3.2. *Let* $a, b \in \mathbb{R}^+$, $a < b$, $m \in (0,1]$ *and* $n \in \mathbb{Z}\backslash\{-1,0\}$. *Then, for all* $q \geq 1$, *we have*

$$|A(a^n,b^n) - L_n^n(a,b)| \leq n.\frac{b-a}{8}\left(\left[\frac{\left|\frac{a+b}{2}\right|^{q(n-1)} + 2m\left|\frac{a}{m}\right|^{q(n-1)}}{3}\right]^{\frac{1}{q}} + \left[\frac{\left|\frac{a+b}{2}\right|^{q(n-1)} + 2m\left|\frac{b}{m}\right|^{q(n-1)}}{3}\right]^{\frac{1}{q}} \right).$$

If we choose $m = 1$, *we obtain*

$$|A(a^n,b^n) - L_n^n(a,b)| \leq n.\frac{b-a}{8}\left(\left[\frac{\left|\frac{a+b}{2}\right|^{q(n-1)} + 2|a|^{q(n-1)}}{3}\right]^{\frac{1}{q}} + \left[\frac{\left|\frac{a+b}{2}\right|^{q(n-1)} + 2|b|^{q(n-1)}}{3}\right]^{\frac{1}{q}} \right).$$

Proof. The assertion follows from Corollary 3 applied to the m-convex mapping $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. □

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