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ON WEIGHTED MONTOGOMERY IDENTITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this paper, we extend the weighted Montogomery identities for the Riemann-Liouville fractional integrals. We also use this Montogomery identities to establish some new Ostrowski type integral inequalities.

1. Introduction

The inequality of Ostrowski [18] gives us an estimate for the deviation of the values of a smooth function from its mean value. More precisely, if $f : [a, b] \to \mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for every $x \in [a, b]$. Moreover the constant $1/4$ is the best possible.

For some generalizations of this classic fact see the book [8, p.468-484] by Mitrinovic, Pecaric and Fink. A simple proof of this fact can be done by using the following identity [8]:

If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ with the first derivative f' integrable on $[a, b]$, then Montgomery identity holds:

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt + \int_a^b P_1(x,t) f'(t)dt, \tag{1.1}$$

where $P_1(x, t)$ is the Peano kernel defined by

$$P_1(x,t) := \begin{cases} \dfrac{t-a}{b-a}, & a \le t < x \\[2ex] \dfrac{t-b}{b-a}, & x \le t \le b. \end{cases}$$

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Recently, several generalizations of the Ostrowski integral inequality are considered by many authors; for instance covering the following concepts: functions of bounded variation, Lipschitzian, monotonic, absolutely continuous and n-times differentiable mappings with error estimates with some special means together with some numerical quadrature rules. For recent results and generalizations concerning Ostrowski's inequality, we refer the reader to the recent papers [3], [6], [9]-[11], [13]-[15].

In [1] and [16], the authors established some inequalities for differentiable mappings which are connected with Ostrowski type inequality by used the Riemann-Liouville fractional integrals, and they used the following lemma to prove their results:

Lemma 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I° with $a, b \in I$ $(a < b)$ and $f' \in L_1[a, b]$, then*

(1.2)

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^\alpha f(b) - J_a^{\alpha-1}(P_2(x,b)f(b)) + J_a^\alpha(P_2(x,b)f^{'}(b)), \quad \alpha \geq 1,$$

where $P_2(x,t)$ is the fractional Peano kernel defined by

$$P_2(x,t) = \begin{cases} \dfrac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & a \leq t < x \\[2ex] \dfrac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & x \leq t \leq b. \end{cases}$$

In this article, we use the Riemann-Liouville fractional integrals to establish some new weighted integral inequalities of Ostrowski's type. From our results, the weighted and the classical Ostrowski's inequalities can be deduced as some special cases.

2. Fractional Calculus

Firstly, we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [7], [12].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ with $a \geq 0$ is defined as

$$\begin{aligned} J_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \\ J_a^0 f(x) &= f(x). \end{aligned}$$

Recently, many authors have studied a number of inequalities by used the Riemann-Liouville fractional integrals, see ([1], [2], [4], [5], [16], [17]) and the references cited therein.

3. Main Results

Throughout this work, we assume that the weight function $w:[a,b]\to[0,\infty)$, is integrable, nonnegative and

$$m(a,b)=\int_a^b w(t)dt<\infty.$$

In order to prove our main results, we need the following identities:

Lemma 3.1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° with $a,b\in I$ $(a<b)$, $\alpha\geq 1$ and $f'\in L_1[a,b]$, then the generalization of the weighted Montgomery identity for fractional integrals holds:*

$$\begin{aligned} m\left(a,b\right)f\left(x\right) &= \left(b-x\right)^{1-\alpha}\Gamma\left(\alpha\right)J_a^{\alpha}\left(w\left(b\right)f\left(b\right)\right) \\ & -J_a^{\alpha-1}\left(\Omega_w\left(x,b\right)f\left(b\right)\right)+J_a^{\alpha}\left(\Omega_w\left(x,b\right)f^{'}\left(b\right)\right) \end{aligned} \tag{3.1}$$

where $\Omega_w\left(x,t\right)$ is the weighted fractional Peano kernel defined by

$$\Omega_w\left(x,t\right):=\begin{cases}\left(b-x\right)^{1-\alpha}\Gamma\left(\alpha\right)\int_a^t w\left(u\right)du, & t\in[a,x) \\ \left(b-x\right)^{1-\alpha}\Gamma\left(\alpha\right)\int_b^t w\left(u\right)du, & t\in[x,b].\end{cases} \tag{3.2}$$

Proof. By definition of $\Omega_w\left(x,t\right)$, we have

$$\begin{aligned} & J_a^{\alpha}\left(\Omega_w\left(x,b\right)f^{'}\left(b\right)\right) \\ &= \frac{1}{\Gamma\left(\alpha\right)}\int_a^b\left(b-t\right)^{\alpha-1}\Omega_w\left(x,t\right)f'\left(t\right)dt \\ &= \left(b-x\right)^{1-\alpha}\left[\int_a^x\left(b-t\right)^{\alpha-1}\left(\int_a^t w\left(u\right)du\right)f^{'}\left(t\right)dt\right. \\ &= \left.+\int_x^b\left(b-t\right)^{\alpha-1}\left(\int_b^t w\left(u\right)du\right)f^{'}\left(t\right)dt\right] \\ &= \left(b-x\right)^{1-\alpha}\left(J_1+J_2\right). \end{aligned} \tag{3.3}$$

Integrating by parts, we can state:

$$
\begin{aligned}
J_1 &= (b-x)^{\alpha-1}\left(\int_a^x w(u)\,du\right) f(x) \\
&\quad + (\alpha-1)\int_a^x (b-t)^{\alpha-2}\left(\int_a^t w(u)\,du\right) f(t)\,dt - \int_a^x (b-t)^{\alpha-1} w(t) f(t)\,dt
\end{aligned}
\tag{3.4}
$$

and similary,

$$
\begin{aligned}
J_2 &= (b-x)^{\alpha-1}\left(\int_x^b w(u)\,du\right) f(x) \\
&\quad + (\alpha-1)\int_x^b (b-t)^{\alpha-2}\left(\int_b^t w(u)\,du\right) f(t)\,dt - \int_x^b (b-t)^{\alpha-1} w(t) f(t)\,dt.
\end{aligned}
\tag{3.5}
$$

Adding (3.4) and (3.5), we obtain (3.1) which this completes the proof. □

Remark 3.1. If we choose $\alpha = 1$ and $w(u) = 1$, the formula (3.1) reduces to the classical Montgomery Identity given by (1.1).

Remark 3.2. If we choose $w(u) = 1$, the formula (3.1) reduces to the fractional Montgomery Identity given by (1.2).

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable on (a,b) such that $f' \in L_1[a,b]$, where $a < b$. If $\left|f'(x)\right| \leq M$ for every $x \in [a,b]$ and $\alpha \geq 1$, then the following Ostrowski fractional inequality holds:*

$$
\begin{aligned}
&\left| m(a,b) f(x) - (b-x)^{1-\alpha}\Gamma(\alpha) J_a^{\alpha}(w(b) f(b)) - J_a^{\alpha-1}(\Omega_w(x,b) f(b)) \right| \\
&\leq \frac{M(b-x)^{1-\alpha}}{\alpha}\left[A(x) - (b-x)^{\alpha} B(x)\right]
\end{aligned}
\tag{3.6}
$$

where

$$
A(x) = \int_a^x (b-u)^{\alpha-1} w(u)\,du - \int_x^b (b-u)^{\alpha} w(u)\,du
$$

and

$$
B(x) = \int_a^x w(u)\,du - \int_x^b w(u)\,du.
$$

Proof. From Lemma 3.1, we get

$$\begin{aligned}
&\left| m\left(a,b\right) f\left(x\right) - \Gamma\left(\alpha\right)\left(b-x\right)^{1-\alpha} J_a^{\alpha}\left(w\left(b\right) f\left(b\right)\right) - J_a^{\alpha-1}\left(\Omega_w\left(x,b\right) f\left(b\right)\right)\right| \\
&\leq \frac{1}{\Gamma\left(\alpha\right)}\left|\int_a^b \left(b-t\right)^{\alpha-1}\Omega_w\left(x,t\right) f'\left(t\right) dt\right| \\
&\leq \frac{M}{\Gamma\left(\alpha\right)}\int_a^b \left(b-t\right)^{\alpha-1}\left|\Omega_w\left(x,t\right)\right| dt \qquad (3.7)\\
&= M\left(b-x\right)^{1-\alpha}\left(\int_a^x \left(b-t\right)^{\alpha-1}\left(\int_a^t w\left(u\right) du\right) dt + \int_x^b \left(b-t\right)^{\alpha-1}\left(\int_t^b w\left(u\right) du\right) dt\right) \\
&= M(b-x)^{1-\alpha}\left\{J_3 + J_4\right\}.
\end{aligned}$$

Now, using the change of order of integration we get

$$\begin{aligned}
J_3 &= \int_a^x \left(b-t\right)^{\alpha-1}\left(\int_a^t w\left(u\right) du\right) dt \\
&= \int_a^x w(u)\int_u^x \left(b-t\right)^{\alpha-1} dtdu \\
&= \frac{1}{\alpha}\left[\int_a^x \left(b-u\right)^{\alpha-1} w\left(u\right) du - \left(b-x\right)^{\alpha}\int_a^x w\left(u\right) du\right]
\end{aligned}$$

and similarly,

$$\begin{aligned}
J_4 &= \int_x^b \left(b-t\right)^{\alpha-1}\left(\int_t^b w\left(u\right) du\right) dt \\
&= \int_x^b w(u)\int_x^u \left(b-t\right)^{\alpha-1} dtdu \\
&= \frac{1}{\alpha}\left[\left(b-x\right)^{\alpha}\int_x^b w\left(u\right) du - \int_x^b \left(b-u\right)^{\alpha} w\left(u\right) du\right].
\end{aligned}$$

Using J_3 and J_4 in (3.7), we obtain (3.6). □

Remark 3.3. We note that in the special cases, if we take $w\left(u\right) = 1$ in Theorem 3.1, then it reduces Theorem 4.1 proved by Anastassiou et. al. [1]. So, our results are generalizations of the corresponding results of Anastassiou et. al. [1].

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