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$\phi-$CONFORMALLY FLAT $C-$MANIFOLDS

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Abstract. In this paper, we have studied $\phi-$conformally flat, $\phi-$conharmonically flat and $\phi-$projectively flat $C-$manifolds.

1. Preliminaries

Let (M^n,g), $n = dimM$, $n \geq 3$, be a connected Riemannian manifold of class C^∞ and ∇ be its Riemannian connection. The Riemannian-Christoffel curvature tensor R, the Weyl conformal curvature tensor C (see [8]), the conharmonic curvature tensor K (see [13]) and the projective curvature tensor P (see [8]) of (M^n,g) are defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1}$$

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)\mathcal{S}X - g(X,Z)\mathcal{S}Y] + \\ &\quad \frac{\tau}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{1.2}$$

$$\begin{aligned} K(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)\mathcal{S}X - g(X,Z)\mathcal{S}Y] \end{aligned} \tag{1.3}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)\mathcal{S}X - g(X,Z)\mathcal{S}Y] \tag{1.4}$$

respectively, where $\mathcal{S}$ is the Ricci operator, defined by $S(X,Y) = g(\mathcal{S}X,Y)$, S is the Ricci tensor, $\tau = tr(S)$ is the scalar curvature and $X,Y,Z \in \chi(M)$, $\chi(M)$ being the vector fields of M.

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In this paper, we have studied $\phi-$conformally flat $C-$ manifolds. We show that there are no exist $\phi-$conformally flat and $\phi-$projectively flat $C-$manifolds unless the dimension of structure vector field s is 1. Similarly, we obtain that there is no exist $\phi-$conharmonically flat $C-$manifolds unless $s=4$.

2. $C-$manifolds

We need the following definition which is given in [4].

Let (M,g) be a Riemannian manifold with $dim(M)=2m+s$. Then M is said to be an $C-$manifold if there exist on M an $\phi-$structure ϕ [16] of rank $2m$ and s global vector fields $\xi_1,...,\ \xi_s$ (structure vector fields) such that [4]

(i) If $\eta_1,...,\eta_s$ are dual $1-$forms of $\xi_1,...,\ \xi_s$, then:

$$\phi\xi_i=0,\quad \eta_i\circ\phi=0,\quad \eta_i(\xi_i)=1,\quad \phi^2=-I+\sum_{i=1}^{s}\xi_i\otimes\eta_i \tag{2.1}$$

$$g(X,Y)=g(\phi X,\phi Y)+\sum_{i=1}^{s}\eta_i\,(X)\,\eta_i\,(Y) \tag{2.2}$$

$$g(\xi_i,X)=\eta_i(X) \tag{2.3}$$

for any $X,Y\in\chi(M)$ and $i=1,...,s$.

(ii) The $\phi-$structure ϕ is normal, that is

$$[\phi,\phi]+2\sum_{i=1}^{s}\xi_i\otimes d\eta_i=0$$

where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ.

(iii) $\eta_1\wedge...\wedge\eta_s\wedge(d\eta_i)^n\neq 0$ and $d\eta_i=0$, for any i. Examples of $C-$manifolds are given in [4].

In a $C-$manifold M, besides the relations (1.1) and (1.2) the following also hold [9] :

$$(\nabla_X\phi)\,Y=0$$

$$\nabla_X\xi_i=0$$

$$\begin{aligned}R(\xi_i,X)Y=0\\ R(\xi_i,X)\xi_\beta=0\end{aligned} \tag{2.4}$$

$$S(\xi_i,X)=2m\sum_{\beta=1}^{s}\eta_\beta(X) \tag{2.5}$$

$$S(\phi X,\phi Y)=S(X,Y) \tag{2.6}$$

An $C-$manifold M is said to be $\eta-$Einstein if its Ricci tensor S is of the form

$$S(X,Y)=ag(X,Y)+b\sum_{i=1}^{s}\eta_i(X)\eta_i(Y) \tag{2.7}$$

for any vector fields X and Y, where, $a,\ b$ are functions on M^n

3. Main Results

In this section we consider $\phi-$conformally flat, $\phi-$conharmonically flat and $\phi-$projectively flat $C-$ manifolds.

Let C be the Weyl conformal curvature tensor of M^n. Since at each point $p \in M^n$ the tangent space $T_P(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \phi(T_p(M^n)) \oplus L(\xi_p)$, where $L(\xi_p)$ is a $1-$dimensional linear subspace of $T_p(M^n)$ generated by ξ_p, we have a map:

$$C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \phi(T_p(M^n)) \oplus L(\xi_p).$$

It may be natural to consider the following particular cases:

(1) $C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to L(\xi_p)$, that is, the projection of the image of C in $\phi(T_p(M^n))$ is zero.

(2) $C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \phi(T_p(M^n))$, that is, the projection of the image of C in $L(\xi_p)$ is zero.

(3) $C : \phi(T_p(M^n)) \times \phi(T_p(M^n)) \times \phi(T_p(M^n)) \to L(\xi_p)$, that is, when C is restricted to $(T_p(M^n)) \times \phi(T_p(M^n)) \times \phi(T_p(M^n))$, the projection of the image of in $\phi(T_p(M^n))$ is zero. This condition is equivalent to

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0 \tag{3.1}$$

(see [8]).

Definition 3.1. A differentiable manifold (M^n, g), $n > 3$, satisfying the condition (3.1) is called $\phi-$conformally flat.

The cases (1) and (2) were considered in ([18]) and ([19]) respectively. The case (3) was considered in ([8]) for the case M^n is a $K-$contact manifold.

Furthermore in [1], the authors studied $(k, \mu)-$contact metric manifolds satisfying (3.1). Now our aim is to find the characterization of $C-$manifolds satisfying the condition (3.1).

Theorem 3.1. *Let M be an $2m + s-$dimensional, $(s > 1)$, $C-$manifold. Then There is no exist $\phi-$conformally flat $C-$manifolds.*

Proof. Suppose that (M, g), $(s > 1)$, is a $\phi-$conformally flat $C-$manifold. It is easy to see that $\phi^2 C(\phi X, \phi Y)\phi Z = 0$ holds if and only if

$$g(C(\phi X, \phi Y)\phi Z, \phi W) = 0,$$

for any $X, Y, Z, W \in \chi(M)$. So by the use of (1.2) $\phi-$conformally flat means

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) \quad = \quad & \frac{1}{2m+s-2}[g(\phi Y, \phi Z)S(\phi X, \phi W) \\ & -g(\phi X, \phi Z)S(\phi Y, \phi W) + g(\phi X, \phi W)S(\phi Y, \phi Z) \\ & -g(\phi Y, \phi W)S(\phi X, \phi Z)] \\ & -\frac{\tau}{(2m+s-1)(2m+s-2)}[g(\phi Y, \phi Z)g(\phi X, \phi W) \\ & -g(\phi X, \phi Z)g(\phi Y, \phi W)] \end{aligned} \tag{3.2}$$

Let $\{w_1, ..., w_{2m}, \xi_1, ..., \xi_s\}$ be a local orthonormal basis of vector fields in M. Using that $\{\phi w_1, ..., \phi w_{2m}, \xi_1, ..., \xi_s\}$ is also a local orthonormal basis, if we put $X = W = w_i$ in (3.2) and sum up with respect to i, then

$$
\begin{aligned}
\sum_{i=1}^{2m} g(R(\phi w_i,\phi Y)\phi Z,\phi w_i) &= \frac{1}{2m+s-2}\sum_{i=1}^{2m}[g(\phi Y,\phi Z)S(\phi w_i,\phi w_i) \\
&\quad -g(\phi w_i,\phi Z)S(\phi Y,\phi w_i)+g(\phi w_i,\phi w_i)S(\phi Y,\phi Z) \\
&\quad -g(\phi Y,\phi w_i)S(\phi w_i,\phi Z)] \\
&\quad -\frac{\tau}{(2m+s-1)(2m+s-2)}\sum_{i=1}^{2m}[g(\phi Y,\phi Z)g(\phi w_i,\phi w_i) \\
&\quad -g(\phi w_i,\phi Z)g(\phi Y,\phi w_i)]
\end{aligned}
\tag{3.3}
$$

It can be easily verify that

$$\sum_{i=1}^{2m} g(R(\phi w_i,\phi Y)\phi Z,\phi w_i)=S(\phi Y,\phi Z), \tag{3.4}$$

$$\sum_{i=1}^{2m} S(\phi w_i,\phi w_i)=\tau \tag{3.5}$$

$$\sum_{i=1}^{2m} g(\phi w_i,\phi Z)S(\phi Y,\phi w_i)=S(\phi Y,\phi Z) \tag{3.6}$$

$$\sum_{i=1}^{2m} g(\phi w_i,\phi w_i)=2m \tag{3.7}$$

and

$$\sum_{i=1}^{2m} g(\phi w_i,\phi Z)g(\phi Y,\phi w_i)=g(\phi Y,\phi Z) \tag{3.8}$$

So by virtue of (3.4)–(3.8) the equation (3.3) can be written as

$$
\begin{aligned}
S(\phi Y,\phi Z) &= \frac{1}{2m+s-2}[\tau g(\phi Y,\phi Z)-2S(\phi Y,\phi Z)+2mS(\phi Y,\phi Z)] \\
&\quad -\frac{\tau}{(2m+s-1)(2m+s-2)}[2mg(\phi Y,\phi Z)-g(\phi Y,\phi Z)]
\end{aligned}
$$

$$S(\phi Y,\phi Z)=\frac{\tau}{2m+s-1}g(\phi Y,\phi Z). \tag{3.9}$$

Then by making use of (2.2) and (2.6), the equation (3.9) takes the form

$$S(Y,Z)=\frac{\tau}{2m+s-1}g(Y,Z)-\frac{\tau}{2m+s-1}\sum_{i=1}^{s}\eta_i(Y)\,\eta_i(Z)\,. \tag{3.10}$$

Therefore from (3.10), by contraction, we obtain $s=1$ which is a contradiction. This completes the proof of the theorem. □

From Theorem 3.1, we have following corollary.

Corollary 3.1. *Let M be an $(2m+1)-$dimensional $\phi-$conformally flat $C-$manifold. Then M is an $\eta-$Einstein manifold.*

Definition 3.2. A differentiable manifold $(M^n, g),\ n > 3$, satisfying the condition

$$\phi^2 K(\phi X, \phi Y)\phi Z = 0 \tag{3.11}$$

is called $\phi-$conharmonically flat.

In [2], the authors considered $(k, \mu)-$contact manifolds satisfying (3.11). Now we will study the condition (3.11) on $C-$ manifolds.

Theorem 3.2. *Let M be an $(2m+s)-$dimensional, $(s > 4)$, $C-$manifold There is no exist $\phi-$conharmonically flat $C-$manifold.*

Proof. Assume that $(M, g),\ (s > 4)$, is a $\phi-$conharmonically flat $C-$manifold. It can be easily seen that $\phi^2 K(\phi X, \phi Y)\phi Z = 0$ holds if and only if

$$g(K(\phi X, \phi Y)\phi Z, \phi W) = 0,$$

for any $X, Y, Z, W \in \chi(M)$. Using (1.3) $\phi-$conformally flat means

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) &= \frac{1}{2m+s-2}[g(\phi Y, \phi Z)S(\phi X, \phi W) - g(\phi X, \phi Z)S(\phi Y, \phi W) \\ &\quad +g(\phi X, \phi W)S(\phi Y, \phi Z) - g(\phi Y, \phi W)S(\phi X, \phi Z)]. \end{aligned} \tag{3.12}$$

Similar to the proof of Theorem 3.1, we can suppose that $\{w_1, ..., w_{2m}, \xi_1, ..., \xi_s\}$ is a local orthonormal basis of vector fields in M. By using the fact that $\{\phi w_1, ..., \phi w_{2m}, \xi_1, ..., \xi_s\}$ is also a local orthonormal basis, if we put $X = W = w_i$ in (3.12) and sum up with respect to i, then

$$\begin{aligned} \sum_{i=1}^{2m} g(R(\phi w_i, \phi Y)\phi Z, \phi w_i) &= \frac{1}{2m+s-2}\sum_{i=1}^{2m}[g(\phi Y, \phi Z)S(\phi w_i, \phi w_i) - g(\phi w_i, \phi Z)S(\phi Y, \phi w_i) \\ &\quad +g(\phi w_i, \phi w_i)S(\phi Y, \phi Z) - g(\phi Y, \phi w_i)S(\phi w_i, \phi Z)]. \end{aligned} \tag{3.13}$$

So by the use of (3.4)-(3.7) the equation (3.13) turns into

$$S(\phi Y, \phi Z) = \frac{\tau}{2m+s-2}g(\phi Y, \phi Z) - \frac{(2m-2)}{2m+s-2}S(\phi Y, \phi Z) \tag{3.14}$$

Thus applying (2.2) and (2.6) into (3.14) we get

$$S(Y, Z) = \frac{\tau}{4m+s-4}g(Y, Z) - \frac{\tau}{4m+s-4}\sum_{i=1}^{s}\eta_i(Y)\,\eta_i(Z) \tag{3.15}$$

from (3.10), by contraction, we obtain $n = 4$ which is a contradiction. □

From Theorem 3.2, we have following corollary.

Corollary 3.2. *Let M be an $(2m+4)-$dimensional $\phi-$conharmonically flat $C-$manifold. Then M is an $\eta-$Einstein manifold.*

Similar to Definition 3.1 and Definition 3.2 we can state the following:

Definition 3.3. A differentiable manifold $(M^n, g),\ n > 3$, satisfying the condition

$$\phi^2 P(\phi X, \phi Y)\phi Z = 0 \tag{3.16}$$

is called $\phi-$projectively flat.

Theorem 3.3. *Let M be an $2m+s$ dimensional, $(s \geq 2)$, $C-$manifold. There not exist $\phi-$projectively flat $C-$manifold.*

Proof. We assume that M be an $2m+s$ $-$dimensional, $(s\geq 2)$, $\phi-$projectively flat $C-$manifold. It can be easily seen that $\phi^2 P(\phi X,\phi Y)\phi Z=0$ holds if and only if

$$g(R(\phi X,\phi Y)\phi Z,\phi W)=0,$$

for any $X,Y,Z,W\in\chi(M)$. Using ((1.1) and (1.4) $\phi-$projectively flat means

$$g(R(\phi X,\phi Y)\phi Z,\phi W)=\frac{1}{2m+s-2}[g(\phi Y,\phi Z)S(\phi X,\phi W)-g(\phi X,\phi Z)S(\phi Y,\phi W). \tag{3.17}$$

In a manner similar to the method in the proof of Theorem 2, choosing $\{w_1,...,w_{2m},\xi_1,...,\xi_s\}$ as a local orthonormal basis of vector fields in M and using the fact that $\{\phi w_1,...,\phi w_{2m},\xi_1,...,\xi_s\}$ is also a local orthonormal basis, putting $X=W=w_i$ in (3.17) and summing up with respect to i, then we have

$$\sum_{i=1}^{2m} g(R(\phi w_i,\phi Y)\phi Z,\phi w_i)=\frac{1}{2m+s-2}\sum_{i=1}^{2m}[g(\phi Y,\phi Z)S(\phi w_i,\phi w_i)-g(\phi w_i,\phi Z)S(\phi Y,\phi w_i). \tag{3.18}$$

So applying (3.4)–(3.6) into (3.18) we get

$$S(\phi Y,\phi Z)=\frac{\tau}{2m+s-1}g(\phi Y,\phi Z)$$

Hence by virtue of (2.2) and (2.6) we obtain

$$S(Y,Z)=\frac{\tau}{2m+s-1}g(Y,Z)-\frac{\tau}{2m+s-1}\sum_{i=1}^{s}\eta_i(Y)\,\eta_i(Z) \tag{3.19}$$

Therefore from (3.19), by contraction, we obtain $s=1$ which is a contraction. Hence, the proof is completed. □

From Theorem 3.3, we have following corollary.

Corollary 3.3. *Let M be an $(2m+1)$ $-$dimensional $\phi-$projectively flat $C-$manifold. Then M is an $\eta-$Einstein manifold.*

References

[1] K. Arslan, C. Murathan and C. Ozgür, *On $\phi-$Conformally flat contact metric manifolds*, Balkan J. Geom. Appl. (BJGA), 5 (2) (2000), 1–7.

[2] K. Arslan, C. Murathan and C. Ozgür, *On contact manifolds satisfying certain curvature conditions*, Proceedings of the Centennial "G. Vranceanu" and the Annual Meeting of the Faculty of Mathematics (Bucharest, 2000). An. Univ. Bucureşti Mat. Inform., 49 (2) (2000), 17–26.

[3] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2002.

[4] D. E. Blair., *Geometry of manifolds with structural group $U(n)xO(s)$*, J. *Diff. Geom.*, 4(1970), 155-167.

[5] D. E. Blair., *On a generalization of the Hopf fibration*, *An. St. Univ. "Al. I. Cuza"Iasi*, 17(1971), 171-177.

[6] D. E. Blair., G. D. Ludden and K. Yano, *Differential geometric structures on principal torodial bundles*, *Trans. Am. Math. Soc.*, 181(1973), 175-184.

[7] D. E. Blair, *Contact manifolds in Riemannian Geometry*, Lecture Notes in Math. Springer–Verlag, Berlin–Heidelberg–New–York, 509 (1976).

[8] J.L. Cabrerizo, L.M. Fernandez, M. Fernandez and G. Zhen, *The structure of a class of K–contact manifolds*, Acta Math. Hungar, 82 (4) (1999), 331–340.

[9] J.L. Cabrerizo, L.M. Fernandez, M. Fernandez, *The curvature tensor fields on $f-$manifolds with complemented frames*, An. şt. Univ. "Al. I. Cuza" Iaşi Matematica, 36(1990), 151-162.

[10] I. Mihai and R. Rosca, *On Lorentzian P–Sasakian manifolds*, Classical Analysis, World Scientific Publ, Singapore (1992), 155–169.
[11] I. Sato, *On a structure similar to almost contact structure*, Tensor N.S, 30 (1976), 219–224.
[12] I. Sato, *On a structure similar to almost contact structure II*, Tensor N.S, 31 (1977), 199–205.
[13] H. Singh, Q. Khan, *On special weakly symmetric Riemannian manifolds*, Publ. Math. Debrecen, Hungary 58(2001), 523–536.
[14] C. Özgür, *$\phi-$conformally flat Lorentzian para-Sasakian manifolds*, Radovi Mathematicki, 12(2003), 99-106.
[15] Y. Ishii, *On conharmonic transformations*, Tensor N.S, 7 (1957), 73–80.
[16] K. Yano, *On a structure defined by a tensor field f of type $(1,1)$ satisfing $f^3+f=0$ Tensor*, 14(1963), 99-109.
[17] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Math, Vol 3, World Sci, 1984.
[18] G. Zhen, *On conformal symmetric K–contact manifolds*, Chinese Quart. J. of Math, 7 (1992), 5–10.
[19] G. Zhen, J.L. Cabrerizo, L.M. Fernandez and M. Fernandez, *On $\xi-$conformally flat contact metric manifolds*, Indian J. Pure Appl. Math, 28 (1997), 725–734.

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