Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 1–8 (2014) ©KJM

MODULES WITH VALUES IN THE SPACE OF ALL DERIVATIONS OF AN ALGEBRA

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Abstract. In this paper, we construct a groupoid associated to a module with values in the space of all derivations of a unital algebra. More precisely, for a pair $(\mathcal{A},\mathcal{G})$ consisting of an algebra $\mathcal{A}$ with a unit, a module $\mathcal{G}$ over the center $Z(\mathcal{A})$ of $\mathcal{A}$ together with a homomorphism of $Z(\mathcal{A})$-modules from $\mathcal{G}$ to the space of all derivations $Der(\mathcal{A})$ of $\mathcal{A}$, we associate a groupoid. We discuss on the equivalence relation induced from this groupoid.

1. Introduction

The concept of a groupoid is a generalization of the concept of a group, the main difference being that not any two elements of a groupoid are composable. Note that groupoids generalize not only the notion of a group but also the notion of a group action. A groupoid can be endowed with the algebraic, geometric or topological structures and in this case we can study the compatibility among these structures and groupoid.

Note that the theory of groupoids has developed in different fields of mathematics. The algebraic, topological and differentiable groupoids play an important role in algebra, measure theory, harmonic analysis, differential geometry and symplectic geometry. This can also be seen from a look at the list of references (see [3, 5, 6, 7, 8, 9]).

A set $\mathcal{H}^{(1)}$ has the structure of a *groupoid* with the set of units $\mathcal{H}^{(0)}$, if there are defined maps $\Delta : \mathcal{H}^{(0)} \to \mathcal{H}^{(1)}$, an involution $\imath : \mathcal{H}^{(1)} \to \mathcal{H}^{(1)}$ and denoted by $\imath(\alpha) = \alpha^{-1}$, a map $r : \mathcal{H}^{(1)} \to \mathcal{H}^{(0)}$, a map $s : \mathcal{H}^{(1)} \to \mathcal{H}^{(0)}$ and an associative multiplication $(\alpha, \beta) \mapsto \alpha\beta$ defined on the set

$$\mathcal{H}^{(2)} = \{(\alpha, \beta) \in \mathcal{H}^{(1)} \times \mathcal{H}^{(1)} \bigm| s(\alpha) = r(\beta)\},$$

satisfying the conditions

(i) $s(\alpha) = r(\alpha^{-1}),\ \ \alpha\alpha^{-1} = \Delta(r(\alpha)),$

(ii) $r(\Delta(t)) = t = s(\Delta(t)),\ \ \alpha\Delta(s(\alpha)) = \alpha,\ \ \Delta(r(\alpha))\alpha = \alpha,$

for all $\alpha \in \mathcal{H}^{(1)}$ and $t \in \mathcal{H}^{(0)}$.

2000 *Mathematics Subject Classification.* 53C12, 58B34.

Key words and phrases. Groupoid, Lie algebra, Lie algebroid.

It is known that for an arbitrary groupoid $(\mathcal{H}^{(1)}, \mathcal{H}^{(0)})$ there is an equivalence relation on the unit set $\mathcal{H}^{(0)}$. Namely, for two elements $x, y \in \mathcal{H}^{(0)}$ the relation $x \sim y$ iff $s^{-1}(x) \cap r^{-1}(y) \neq \emptyset$ is an equivalence relation on the unit set $\mathcal{H}^{(0)}$.

In [1], a method of associating a groupoid to a smooth manifold was introduced. In this paper, we use the same method to construct a groupoid associated to a module with values in the space of all derivations of a unital algebra. The focus in this paper is on the several examples.

Let $\mathcal{A}$ be an algebra with a unit. Let $Der(\mathcal{A})$ denote the space of all derivations of $\mathcal{A}$, i.e., the space of all linear mappings X of $\mathcal{A}$ into itself satisfying the Leibniz rule $X(ab) = X(a)b + aX(b)$. The space $Der(\mathcal{A})$ is in a natural way a module over the center $Z(\mathcal{A})$ of $\mathcal{A}$. Furthermore, the space $Der(\mathcal{A})$ is also a Lie algebra with Lie bracket $[X, Y] = XY - YX$.

Consider a pair $(\mathcal{A}, \mathcal{G})$ consisting of a module $\mathcal{G}$ over the center $Z(\mathcal{A})$ of $\mathcal{A}$ together with a linear map from $\mathcal{G}$ to $Der(\mathcal{A})$, which is also a homomorphism of $Z(\mathcal{A})$-modules. In this paper, such pairs are called $\mathcal{A}$-pairs.

We now give a brief summary of how the paper is organized.

In **Section** 2, we begin with our basic construction. We construct a groupoid associated to an $\mathcal{A}$-pair and we shall discuss on the equivalence relation induced from this groupoid. In the case when $\mathcal{G}$ is a Lie algebra, we will give the conditions that the equivalence classes are abelian Lie subalgebras of the Lie algebra $\mathcal{G}$.

In **Section** 3, we compute and investigate the equivalence classes for several examples. This section is devoted to the central algebras, foliation manifolds and the endomorphism algebra of a vector bundle.

Our basic reference for groupoids is [2], and for an extensive use of them one can refer to [5].

Throughout this paper, all smooth manifolds are assumed to be real, Hausdorff, and finite-dimensional. All vector fields on manifolds are assumed to be smooth. If M is a smooth manifold, let $\Im(M)$ be the Lie algebra of all vector fields on M and let $C^\infty(M)$ be the algebra of all smooth functions on M.

2. Groupoid associated to an $\mathcal{A}$-pair

In this section, we will introduce and construct a groupoid associated to an $\mathcal{A}$-pair. Let $\mathcal{A}$ be an algebra with a unit.

Definition 2.1. By an *$\mathcal{A}$-pair* we mean a pair $(\mathcal{A}, \mathcal{G})$, where $\mathcal{G}$ is a module over the center $Z(\mathcal{A})$ of $\mathcal{A}$ together with a homomorphism of $Z(\mathcal{A})$-modules $T : \mathcal{G} \to Der(\mathcal{A})$.

In this paper, $X(a)$ denotes $T(X)(a)$, for all $X \in \mathcal{G}$ and $a \in \mathcal{A}$. Using Definition 2.1, for all $X \in \mathcal{G}$ and $a \in Z(\mathcal{A})$ we have $X(a) \in Z(\mathcal{A})$.

Example 2.1. The pair $(\mathcal{A}, Der(\mathcal{A}))$ is an $\mathcal{A}$-pair.

Consider an $\mathcal{A}$-pair $(\mathcal{A}, \mathcal{G})$. The set of all invertible elements in the algebra $\mathcal{A}$ is denoted by $Inv(\mathcal{A})$, that is, for all $a \in Inv(\mathcal{A})$ there exists an element $a^{-1} \in \mathcal{A}$ such that $aa^{-1} = 1 = a^{-1}a$.

Let $\Gamma_{\mathcal{A}} = (Z(\mathcal{A}) \cap Inv(\mathcal{A})) \times Z(\mathcal{A})$. Fix an element $X \in \mathcal{G}$. Let

$$\mathcal{G}_X^{(1)}(\mathcal{A}) = \{(Y, a, b) \mid Y \in \mathcal{G},\ (a, b) \in \Gamma_{\mathcal{A}},\ Y(a) = X(b) + b\}.$$

We have to show that the pair $(\mathcal{G}_X^{(1)}(\mathcal{A}), \mathcal{G})$ has the structure of a groupoid. Define the map $\Delta : \mathcal{G} \to \mathcal{G}_X^{(1)}(\mathcal{A})$ by

$$\Delta(Y) := (Y, 1, 0).$$

Since $Y(1) = X(0) + 0$, it follows that Δ is well-defined. Define $r, s : \mathcal{G}_X^{(1)}(\mathcal{A}) \to \mathcal{G}$ by

$$r(Y, a, b) := Y, \qquad s(Y, a, b) := aY - bX.$$

Define the involution $\imath : \mathcal{G}_X^{(1)}(\mathcal{A}) \to \mathcal{G}_X^{(1)}(\mathcal{A})$ by

$$\imath(Y, a, b) = (Y, a, b)^{-1} := (aY - bX, a^{-1}, -a^{-1}b).$$

We have

$$\begin{aligned}(aY - bX)(a^{-1}) - X(-a^{-1}b) &= aY(a^{-1}) + a^{-1}Y(a) - a^{-1}b \\ &= Y(1) - a^{-1}b = -a^{-1}b.\end{aligned}$$

This shows that $\imath$ is well-defined. Define the multiplication by

$$(Y, a, b)(aY - bX, c, d) := (Y, ac, bc + d).$$

We have

$$\begin{aligned}Y(ac) - X(bc + d) &= (Y(a) - X(b))c + aY(c) - bX(c) - X(d) \\ &= bc + (aY - bX)(c) - X(d) = bc + d.\end{aligned}$$

It is straightforward to check the following axioms are true:

$$\begin{aligned}&s(\alpha) = r(\alpha^{-1}), \qquad r(\Delta(Y)) = Y = s(\Delta(Y)),\\ &\alpha\alpha^{-1} = \Delta(r(\alpha)),\\ &\alpha\Delta(s(\alpha)) = \alpha, \qquad \Delta(r(\alpha))\alpha = \alpha,\end{aligned}$$

for all $\alpha \in \mathcal{G}_X^{(1)}(\mathcal{A})$ and $Y \in \mathcal{G}$.

For any fixed $X \in \mathcal{G}$, we obtain an equivalence relation on the module $\mathcal{G}$ (see Section 1 above). We say that two elements Y and W of $\mathcal{G}$ are equivalent iff there exists a pair $(a, b) \in \Gamma_{\mathcal{A}}$ such that

$$aW = bX + Y, \qquad W(a) = X(b) + b.$$

Let $[(X, Y)]_{\mathcal{A}}$ be the equivalence class of any $Y \in \mathcal{G}$.

Let us give an example. In the following, we consider the case for $(C^\infty(\mathbb{R}^2), \Im(\mathbb{R}^2))$ (see Example 2.1 above).

Example 2.2. The vector field X on $\mathbb{R}^2$ defined in terms of the identity chart x by

$$X = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}$$

has integral curves $\gamma(t) = (z_1 \exp t, z_2 \exp t)$ starting at the point (z_1, z_2). Let $\zeta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ be the zero vector field, defined by $\zeta(f) = 0$ for each $f \in C^\infty(\mathbb{R}^2)$. We have

$$[(X, \zeta)]_{C^\infty(\mathbb{R}^2)} = \{\frac{g}{f}X \mid (f, g) \in \Gamma_{C^\infty(\mathbb{R}^2)},\ gX(f) - fX(g) = fg\}.$$

Assume that $W = \frac{g}{f}X \in [(X, \zeta)]_{C^\infty(\mathbb{R}^2)}$ such that g is a non-zero function. Since $\frac{d}{dt}(\frac{g}{f} \circ \gamma) = X(\frac{g}{f}) \circ \gamma$, it follows that

$$f(z_1, z_2)g(z_1 \exp t, z_2 \exp t) \exp t = f(z_1 \exp t, z_2 \exp t)g(z_1, z_2),$$

on $\mathbb{R}$, for all $(z_1, z_2) \in \mathbb{R}^2$. Hence, we have

$$f(z_1, z_2)g(z_1 s, z_2 s)s = f(z_1 s, z_2 s)g(z_1, z_2),$$

for all $(z_1, z_2) \in \mathbb{R}^2$ and all $s > 0$. Since $\frac{g}{f} \neq 0$ we can choose $z \in \mathbb{R}^2$ such that $\frac{g(z)}{f(z)} \neq 0$. Then $\lim_{s\to 0} |\frac{g(sz)}{f(sz)}|$ would be infinite and this would imply that $\frac{g}{f}$ is not continuous at 0 and this is a contradiction. Hence, we have $[(X,\zeta)]_{C^\infty(\mathbb{R}^2)} = \{\zeta\}$.

In addition, in the case when $\mathcal{G}$ is a Lie algebra we will give the conditions that the equivalence classes are Lie subalgebras of the Lie algebra $\mathcal{G}$.

The following theorem is the main result of this paper.

Theorem 2.1. *Consider an $\mathcal{A}$-pair $(\mathcal{A}, \mathcal{G})$, where $\mathcal{G}$ is also a Lie algebra with a Lie bracket $[,]$. Moreover, assume that for all $a \in Z(\mathcal{A})$ and $X, Y \in \mathcal{G}$ we have $[X, aY] = X(a)Y + a[X, Y]$. Let $X, Y \in \mathcal{G}$. Then the following properties are equivalent:*
(i) $[(X, Y)]_{\mathcal{A}}$ is an abelian Lie subalgebra of the Lie algebra $\mathcal{G}$,
(ii) $[(X, Y)]_{\mathcal{A}} = \{aX \mid a \in Z(\mathcal{A}),\ X(a) = -a\}$,
(iii) there is an element $t \in Z(\mathcal{A})$ such that $Y = -tX$ and $X(t) = -t$.

Proof. It is easy to check that $(i) \implies (ii)$ and $(ii) \implies (iii)$. It suffices to prove that $(iii) \implies (i)$. Let $t \in Z(\mathcal{A})$ such that $Y = -tX$ and $X(t) = -t$. Assume that $W, Z \in [(X, Y)]_{\mathcal{A}}$ and $\lambda \in \mathbb{C}$. We have to show that $W + Z \in [(X, Y)]_{\mathcal{A}}, \lambda W \in [(X, Y)]_{\mathcal{A}}$ and $[W, Z] = 0$. Choose (a, b) and (c, d) in $\Gamma_{\mathcal{A}}$ such that

$$cZ = (d - t)X, \qquad aW = (b - t)X,$$

and

$$Z(c) = X(d) + d, \qquad W(a) = X(b) + b.$$

Let $(s, h) = (ac, ad + bc + t(1 - c - a))$. It follows that

$$s(W + Z) - hX = Y.$$

Also, we have

$$\begin{aligned}(W + Z)(s) - X(h) &= X(b)c + cb + (b - t)X(c) + (d - t)X(a) \\ &\quad + aX(d) + ad - X(a)d - aX(d) - X(b)c \\ &\quad - bX(c) - X(t) + X(t)c + tX(c) + X(t)a \\ &\quad + tX(a) \\ &= ad + bc - X(t) + X(t)c + X(t)a = h.\end{aligned}$$

So, we have $W + Z \in [(X, Y)]_{\mathcal{A}}$. On the other hand, it is simple to see that $\lambda W \in [(X, Y)]_{\mathcal{A}}$. Also, we have

$$\begin{aligned}ac[W, Z] = (b - t)X(d - t)X &- (b - t)\underbrace{X(c)c^{-1}(d - t)}_{X(d)+d}X \\ &- (d - t)X(b - t)X + (d - t)\underbrace{X(a)a^{-1}(b - t)}_{X(b)+b}X \\ &= (b - t)X(d - t)X - (b - t)(X(d) + d)X \\ &- (d - t)X(b - t)X + (d - t)(X(b) + b)X = 0.\end{aligned}$$

Since $ac \in Inv(\mathcal{A})$, it follows that $[W, Z] = 0$. □

Let M be a smooth manifold and X a vector field on it. A non-zero function $h \in C^\infty(M)$ such that $X(h) = \lambda h$, for some real number λ, is said to be an *eigenfunction* of the vector field X and λ is called the corresponding *eigenvalue*. Note that a non-zero function $h \in C^\infty(M)$ is an eigenfunction of a vector field X corresponding to a zero eigenvalue if and only if it is constant on the range of every integral curve.

Example 2.3. (i) Any vector field X on a compact manifold M has all its eigenvalues zero. Let $\zeta : C^\infty(M) \to C^\infty(M)$ be the zero vector field. Using Theorem 2.1, one gets $[(X,\zeta)]_{C^\infty(M)} = \{\zeta\}$.

(ii) The vector field X on $\mathbb{R}^2$ defined in terms of the identity chart x by

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$$

has every eigenvalue zero, since its integral curves

$$\gamma(t) = (a\sin(t+b), a\cos(t+b)), \quad \gamma(0) = (a\sin(b), a\cos(b)),$$

are all periodic, that is, there exists $r > 0$ such that $\gamma(t_1) = \gamma(t_2)$ if and only if $t_1 - t_2 = kr$, for some $k \in \mathbb{Z}$. Let ζ be the zero vector field on $\mathbb{R}^2$. Hence, using Theorem 2.1, we have $[(X,\zeta)]_{C^\infty(\mathbb{R}^2)} = \{\zeta\}$.

3. Examples

Let us compute and investigate the equivalence classes for some examples. First, we consider the case where the center of the algebra $\mathcal{A}$ is the set $Z(\mathcal{A}) = \mathbb{C}1$. For example, consider the algebra of $n \times n$ complex matrices $M_n(\mathbb{C})$.

Example 3.1. Consider an $\mathcal{A}$-pair $(\mathcal{A},\mathcal{G})$, assume that $Z(\mathcal{A}) = \mathbb{C}1$. We see that $X(Z(\mathcal{A})) = 0$, for all $X \in \mathcal{G}$. Let $X, Y \in \mathcal{G}$ and $W \in [(X,Y)]_\mathcal{A}$. It follows that

$$W(a) = X(b) + b, \qquad aW = bX + Y,$$

for $a = \alpha 1, b = \beta 1$, where $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq 0$. Therefore, we have $b = 0$ and $W = \frac{1}{\alpha} Y$. Thus $[(X,Y)]_\mathcal{A} \subset (\mathbb{C} - \{0\})Y$. On the other hand, it is simple to see that $(\mathbb{C} - \{0\})Y \subset [(X,Y)]_\mathcal{A}$. Hence, we have

$$[(X,Y)]_\mathcal{A} = (\mathbb{C} - \{0\})Y,$$

for all $X, Y \in \mathcal{G}$.

Recall that a p-dimensional *foliation* $\mathcal{F}$ on a n-dimensional smooth manifold M consists of the partition of M into maximal integral submanifolds (*leaves*) of an integrable, p-dimensional subbundle $F = T\mathcal{F}$ of the tangent bundle TM. The vector fields on M which are tangent to the leaves of $\mathcal{F}$ form a Lie subalgebra of the Lie algebra $\Im(M)$, which we denote by $\Im(\mathcal{F})$. In other words, $\Im(\mathcal{F})$ consists of the sections of the tangent bundle $T\mathcal{F}$ of the foliation $\mathcal{F}$. A smooth function φ on M is called *basic* if it is constant along the leaves. Equivalently, a function φ is basic if $X(\varphi) = 0$ whenever $X \in \Im(\mathcal{F})$, briefly $\Im(\mathcal{F})(\varphi) = 0$. We refer to [4], for details on foliations.

Example 3.2. Let $(M,\mathcal{F})$ be a foliation manifold. The basic functions on $(M,\mathcal{F})$ form a subalgebra $\mathcal{A}$ of $C^\infty(M)$:

$$\mathcal{A} = \{\varphi \in C^\infty(M) \bigm| \Im(\mathcal{F})(\varphi) = 0\}.$$

In general, the Lie subalgebra $\Im(\mathcal{F})$ is not a Lie ideal in $\Im(M)$, but it is clearly a Lie ideal in the Lie subalgebra

$$\mathcal{G} = \{Z \in \Im(M) \bigm| [\Im(\mathcal{F}), Z] \subset \Im(\mathcal{F})\}.$$

Remark that $\mathcal{G}$ is a module over the algebra of basic functions. Also, the definition of the Lie bracket implies that the derivative of a basic function in the direction of a vector field in $\mathcal{G}$ is again basic. Therefore, we can define a linear map $T : \mathcal{G} \to Der(\mathcal{A})$ by $T(Z)(\varphi) = Z(\varphi)$. We obtain that the pair $(\mathcal{A}, \mathcal{G})$ is an $\mathcal{A}$-pair. Let $X \in \Im(\mathcal{F}) \subset \mathcal{G}$ and one gets

$$[(X, X)]_{\mathcal{A}} = \{\varphi X \bigm| \varphi \in Inv(\mathcal{A})\}.$$

In the following, we consider the case for the pair

$$(End(\mathcal{E}), Der(End(\mathcal{E}))),$$

which $End(\mathcal{E})$ is the algebra of the endomorphisms of a vector bundle $\mathcal{E}$ over a smooth manifold M. We compute the equivalence class for $X, Y \in Der(End(\mathcal{E}))$ which are also homomorphisms of $C^\infty(M)$-modules.

Example 3.3. Let $\mathcal{E}$ be a finite-dimensional complex (or real) vector bundle over a smooth manifold M. We denote by $End(\mathcal{E})$ the algebra of the endomorphisms of this bundle. Any element $\varphi \in End(\mathcal{E})$ can be considered as a section of the bundle of endomorphisms. Therefore, for any element $\varphi \in End(\mathcal{E})$ and any point $p \in M$ we have $\varphi_p \in End(\mathcal{E}_p)$. The center of the algebra $End(\mathcal{E})$ is the set

$$C^\infty(M) \cdot 1 = Z(End(\mathcal{E})).$$

Assume that derivations $X, Y : End(\mathcal{E}) \to End(\mathcal{E})$ are homomorphisms of $C^\infty(M)$-modules, i.e., $X(f \cdot \varphi) = f \cdot X(\varphi)$ and $Y(f \cdot \varphi) = f \cdot Y(\varphi)$, for all $f \in C^\infty(M)$ and $\varphi \in End(\mathcal{E})$. Hence, we obtain

$$X(Z(End(\mathcal{E}))) = 0 = Y(Z(End(\mathcal{E}))).$$

Let $W \in [(X, Y)]_{End(\mathcal{E})}$. Hence, there exists a pair (a, b) such that

$$W(a) = X(b) + b, \qquad aW = bX + Y,$$

where $a = f \cdot 1, b = g \cdot 1$, $f \in Inv(C^\infty(M))$ and $g \in C^\infty(M)$. Therefore, we have $W(Z(End(\mathcal{E}))) = 0$ and

$$W = (\frac{1}{f} \cdot 1)Y \in (Inv(C^\infty(M)) \cdot 1)Y.$$

Also, it is simple to see that $(Inv(C^\infty(M)) \cdot 1)Y \subset [(X, Y)]_{End(\mathcal{E})}$. Hence, we obtain

$$[(X, Y)]_{End(\mathcal{E})} = (Inv(C^\infty(M)) \cdot 1)Y,$$

for all derivations $X, Y : End(\mathcal{E}) \to End(\mathcal{E})$ that are homomorphisms of $C^\infty(M)$-modules.

As another example we would like to investigate the equivalence classes for derivations of the endomorphism algebra $End(\mathcal{E})$ of a vector bundle $\mathcal{E}$ over a smooth manifold M. In the following, we investigate the equivalence classes for a type of derivations of the algebra $End(\mathcal{E})$ that are not homomorphisms of $C^\infty(M)$-modules.

Recall that a Lie algebroid may be thought of as a generalization of the tangent bundle of a manifold. Just as Lie algebras are in some sense the infinitesimal versions of Lie groups, Lie algebroids are objects that play a similar role for Lie groupoids. A *Lie algebroid* over the manifold M is the triple $(\mathcal{K}, [,], \mu)$ where $\mathcal{K}$

is a vector bundle over M, whose $C^\infty(M)$-module of sections $\Gamma(\mathcal{K})$ is equipped with a Lie algebra structure $[,]$ and $\mu : \mathcal{K} \to TM$ is a bundle map which induces a Lie algebra homomorphism (also denoted μ) from $\Gamma(\mathcal{K})$ to $\Im(M)$, satisfying the Leibnitz rule

$$[X, fY] = \mu(X)(f)Y + f[X, Y],$$

for all $f \in C^\infty(M)$ and $X, Y \in \Gamma(\mathcal{K})$. Here, the map $\mu : \mathcal{K} \to TM$ is called an *anchor map* (see [10, 11]).

Example 3.4. Let M be a smooth manifold and $(\mathcal{K}, [,], \mu)$ be a Lie algebroid over M. Consider a $\mathcal{K}$-connection on a vector bundle $\mathcal{E}$ over M. So, there exists a $\mathbb{R}$-bilinear map $\nabla : \Gamma(\mathcal{K}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ such that

$$\nabla_{fX}(s) = f\nabla_X(s), \qquad \nabla_X(fs) = \mu(X)(f)s + f\nabla_X(s),$$

for all $f \in C^\infty(M)$, $X \in \Gamma(\mathcal{K})$ and $s \in \Gamma(\mathcal{E})$. For any $X \in \Gamma(\mathcal{K})$ define a derivation $D_X : End(\mathcal{E}) \to End(\mathcal{E})$ as

$$D_X(\varphi)(s) = \nabla_X(\varphi(s)) - \varphi(\nabla_X(s)),$$

for all $\varphi \in End(\mathcal{E})$ and $s \in \Gamma(\mathcal{E})$. It is simple to see that

$$D_X(f \cdot \varphi) = \mu(X)(f) \cdot \varphi + f \cdot D_X(\varphi),$$

for all $f \in C^\infty(M)$ and $\varphi \in End(\mathcal{E})$. Thus, one gets

$$D_X(f \cdot 1) = \mu(X)(f) \cdot 1,$$

for all $f \in C^\infty(M)$. Take $X, Y \in \Gamma(\mathcal{K})$ and $W \in [(X, Y)]_{C^\infty(M)}$. There exists a pair $(f, g) \in \Gamma_{C^\infty(M)}$ such that

$$fW = gX + Y, \qquad \mu(W)(f) = \mu(X)(g) + g.$$

We show that $D_W \in [(D_X, D_Y)]_{End(\mathcal{E})}$. For all $\varphi \in End(\mathcal{E})$ and $s \in \Gamma(\mathcal{E})$ we have

$$\begin{aligned}((f \cdot 1)D_W - (g \cdot 1)D_X)(\varphi)(s) &= \nabla_{fW-gX}(\varphi(s)) - \varphi(\nabla_{fW-gX}(s)) \\ &= \nabla_Y(\varphi(s)) - \varphi(\nabla_Y(s)) \\ &= D_Y(\varphi)(s),\end{aligned}$$

which implies that $(f \cdot 1)D_W - (g \cdot 1)D_X = D_Y$. Also, we have

$$\begin{aligned}D_W(f \cdot 1) - D_X(g \cdot 1) &= \mu(W)(f) \cdot 1 - \mu(X)(g) \cdot 1 \\ &= (\mu(W)(f) - \mu(X)(g)) \cdot 1 \\ &= g \cdot 1,\end{aligned}$$

hence one gets $D_W \in [(D_X, D_Y)]_{End(\mathcal{E})}$. So, we can define the surjective map $F : [(X, Y)]_{C^\infty(M)} \to [(D_X, D_Y)]_{End(\mathcal{E})}$ by $F(W) = D_W$. Now, we can check that if the anchor map μ is injective then

$$[(X, Y)]_{C^\infty(M)} \simeq [(D_X, D_Y)]_{End(\mathcal{E})},$$

for all $X, Y \in \Gamma(\mathcal{K})$.

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