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# FRAMED-COMPLEX SUBMERSIONS

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Abstract. In this paper, we introduce the concept of framed-complex submersion from a framed metric manifold onto an almost Hermitian manifold. We investigate the influence of a given structure defined on the total manifold on the determination of the corresponding structure on the base manifold. Moreover, we provide an example, investigate various properties of the O'Neill's tensors for such submersions, find the integrability of the horizontal distribution. We also obtain curvature relations between the base manifold and the total manifold.

## 1. Introduction

The theory of Riemannian submersion was introduced by O'Neill and Gray in [15] and [11], respectively. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. Riemannian submersions were considered between almost complex manifolds by Watson in [19] under the name of almost Hermitian submersion. He showed that if the total manifold is a Kähler manifold, the base manifold is also a Kähler manifold. Riemannian submersions between almost contact manifolds were studied by Chinea in [2] under the name of almost contact submersions. Since then, Riemannian submersions have been used as an effective tool to describe the structure of a Riemannian manifold equipped with a differentiable structure. For instance, Riemannian submersions have been also considered for quaternionic Kähler manifolds [12]. This kind of submersions have been studied with different names by many authors(see [7], [8],[9], [10], [13],[14], [18], and more).

On the other hand, let $(M, g)$ be a Riemannian manifold equipped with a framed metric structure, i.e. an endomorphism $\varphi$ of the tangent bundle such that $\varphi^3+\varphi = 0$ and which is compatible with $g$; the compatibility means that for each $X, Y \in TM$ we have $g(\varphi X, Y) = -g(X, \varphi Y)$ [21]. Moreover we assume that the kernel of $\varphi$ is of constant rank and parallelizable, i.e. there exist global vector fields $\xi_1, ..., \xi_s$ spanning ker$\varphi$. Such manifolds are necessarily of dimension $2m + s$ where $2m$ is the

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rank of $\varphi$. The study of such manifolds was started by Blair, Goldberg and Yano ([1], [5], [6]). In this paper, we define framed-complex submersions from a framed metric manifold onto an almost Hermitian manifold and study the geometry of such submersions. We observe that framed-complex submersion has also rich geometric properties.

The paper is organized as follows. In section 2, we collect basic definitions, some formulas and results for later use. In section 3, we introduce the notion of framed-complex submersions and give an example of framed-complex submersion. Moreover, we investigate properties of O'Neill's tensors and show that such tensors have nice algebraic properties for framed-complex submersions. We find the integrability of the horizontal distribution. In section 4 is focused on the transference of structures defined on the total manifold. Finally, we obtain relations between bisectional curvatures and sectional curvatures of the base manifold, the total manifold and the fibres of a framed-complex submersion.

## 2. Preliminaries

In this section, we are going to recall main definitions and properties of framed metric manifolds, almost Hermitian manifolds and Riemannian submersions.

**2.1. Framed metric manifolds.** Let $M$ be a $(2m+s)$- dimensional framed metric manifold [20](or almost s-contact metric manifold[17]) with a framed metric structure $(\varphi, \xi_j, \eta_j, g)$, $j \in \{1, ..., s\}$, that is, $\varphi$ is a $(1,1)-$tensor field defining a $\varphi-$structure of rank $2m$; $\xi_1, ..., \xi_s$ are $s$ vector fields; $\eta_1, ..., \eta_s$ are $s$ 1-forms and $g$ is a Riemannian metric on $M$ such that

$$\varphi^2 = -I + \sum_{j=1}^{s} \eta_j \otimes \xi_j, \quad \eta_j(\xi_i) = \delta_i^j, \quad \varphi(\xi_j) = 0, \quad \eta_j \circ \varphi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{j=1}^{s} \eta_j(X)\eta_j(Y), \tag{2.2}$$

$$\Phi(X, Y) = g(X, \varphi Y) = -\Phi(Y, X), \tag{2.3}$$

$$g(X, \xi_j) = \eta_j(X) \tag{2.4}$$

for all $X, Y \in \Gamma(TM)$ and $i, j \in \{1, ..., s\}$[20].

A framed metric structure is called normal[20]if

$$[\varphi, \varphi] + 2d\eta_j \otimes \xi_j = 0, \tag{2.5}$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$ given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{2.6}$$

We note that a framed metric manifold $(M^{2m+s}, g, \varphi, \xi_j, \eta_j)$ is called

(a) *almost* $\mathcal{S}-$*manifold*, if $d\eta_j = \Phi$;
(b) $\mathcal{K}-$*manifold*, if $d\Phi = 0$ and normal;
(c) $\mathcal{S}-$*manifold*, if $d\eta_j = d\Phi$ and normal;
(d) *almost* $\mathcal{C}-$*manifold*, if $d\eta_j = 0$, $d\Phi = 0$;

(e) $\mathcal{C}-$*manifold*, if $d\eta_j = 0$, $d\Phi = 0$ and normal([1], [3]).

We have the following relation between the Levi-Civita connection and fundamental 2-form of $M$.

**Lemma 2.1.** [3]. *Let $(M, \varphi, \xi_j, \eta_j, g)$ be a framed metric manifold. Then we have*

$$\begin{aligned} 2g((\nabla_X \varphi)Y, Z) &= 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) + +g(N^{(1)}(Y, Z), \varphi X) \\ &\quad + \sum_{j=1}^{s} \{N^{(2)}(Y, Z)\eta_j(X) + 2d\eta_j(\varphi Y, X)\eta_j(Z) \\ &\quad -2d\eta_j(\varphi Z, X)\eta_j(Y)\}, \end{aligned} \tag{2.7}$$

*where the tensor field $N^{(2)}$ is defined defined by $N^{(2)}(X, Y) = (L_{\varphi X}\eta_j)(Y) - (L_{\varphi Y}\eta_j)(X)$, where $\mathcal{L}$ denotes the Lie derivative, for any $X, Y, Z$ vector fields on $M$.*

*On $\mathcal{S}-$ manifolds we have*[16]

$$\begin{aligned} (\nabla_X \Phi)(Y, Z) &= \frac{1}{2}\sum_{i=1}^{s}[\eta_i(Y)g(X, Z) - \eta_i(Z)g(X, Y)] \\ &\quad -\frac{1}{2}\sum_{i,j=1}^{s} \eta_j(X)[\eta_i(Y)\eta_j(Z) - \eta_i(Z)\eta_j(Y)]. \end{aligned} \tag{2.8}$$

It is easy to see that if $M$ is a framed metric manifold, then the following identities are well known:

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) + 2\sum_{j=1}^{s} d\eta_j(X, Y)\xi_j, \tag{2.9}$$

$$(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi(\nabla_X Y), \tag{2.10}$$

$$(\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi)Z) = -g(Z, (\nabla_X \varphi)Y), \tag{2.11}$$

$$(\nabla_X \eta_j)Y = g(Y, \nabla_X \xi_j). \tag{2.12}$$

**2.2. Almost Hermitian manifolds.** Let $M$ be an even-dimensional differentiable manifold. An almost Hermitian structure on $M$ is by definition a pair $(J, g)$ of an almost complex structure $J$ and a Riemannian metric $g$ satisfying

$$J^2(X) = -X, \quad g(JX, JY) = g(X, Y) \tag{2.13}$$

for any vector fields $X, Y$ on $M$. A manifold with such a structure $(J, g)$ is called an almost Hermitian manifold. The fundamental 2-from $\Phi$ of an almost Hermitian structure is defined by

$$\Phi(X, Y) = g(X, JY)$$

for any vector fields $X, Y$ and is skew-symmetric[20].

The Nijenhuis(or the torsion) tensor of an almost complex structure $J$ is defined by

$$N(X, Y) = -[X, Y] + [JX, JY] - J[X, JY] - J[JX, Y] \tag{2.14}$$

for any vector fields $X, Y$ on $M$.
An almost Hermitian manifold $(M, g, J)$ is called

(a) *Kähler* if $\nabla J = 0$;
(b) *almost Kähler* if $d\Phi = 0$;
(c) *nearly Kähler* if $(\nabla_X J)X = 0$;
(d) *Hermitian* if $N = 0$[20], where $N$ is the Nijenhuis tensor of $J$.

**2.3. Riemannian Submersions.** Let $(M, g)$ and $(B, g')$ be two Riemannian manifolds. A surjective $C^\infty-$map $\pi : M \to B$ is a $C^\infty-$submersion if it has maximal rank at any point of $M$. Putting $\mathcal{V}_x = Ker\pi_{*x}$, for any $x \in M$, we obtain an integrable distribution $\mathcal{V}$, which is called vertical distribution and corresponds to the foliation of $M$ determined by the fibres of $\pi$. The complementary distribution $\mathcal{H}$ of $\mathcal{V}$, determined by the Riemannian metric $g$, is called horizontal distribution. A $C^\infty-$submersion $\pi : M \to B$ between two Riemannian manifolds $(M, g)$ and $(B, g')$ is called a Riemannian submersion if, at each point $x$ of $M$, $\pi_{*x}$ preserves the length of the horizontal vectors. A horizontal vector field $X$ on $M$ is said to be basic if $X$ is $\pi-$related to a vector field $X'$ on $B$. It is clear that every vector field $X'$ on $B$ has a unique horizontal lift $X$ to $M$ and $X$ is basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion $\pi : M \to B$ determines two $(1, 2)$ tensor fields $T$ and $A$ on $M$, by the formulas:

$$T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{2.15}$$

and

$$A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{2.16}$$

for any $E, F \in \Gamma(TM)$, where $v$ and $h$ are the vertical and horizontal projections (see [4]). From (2.15) and (2.16), one can obtain

$$\nabla_U X = T_U X + h(\nabla_U X); \tag{2.17}$$

$$\nabla_X U = v(\nabla_X U) + A_X U; \tag{2.18}$$

$$\nabla_X Y = A_X Y + h(\nabla_X Y), \tag{2.19}$$

for any $X, Y \in \Gamma(\mathcal{H})$, $U \in \Gamma(\mathcal{V})$. Moreover, if $X$ is basic then

$$h(\nabla_U X) = h(\nabla_X U) = A_X U. \tag{2.20}$$

We note that for $U, V \in \Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma(\mathcal{H})$, $A_X Y = \frac{1}{2}v[X, Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that $A$ is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X, Y \in \Gamma(\mathcal{H})$ and $T$ is symmetric on the vertical distribution: $T_U V = T_V U$, for $U, V \in \Gamma(\mathcal{V})$.

We now recall the following result which will be useful for later.

**Lemma 2.2.** *(see* [4],[15]*). If $\pi : M \to B$ is a Riemannian submersion and $X, Y$ basic vector fields on $M$, $\pi-$related to $X'$ and $Y'$ on $B$, then we have the following properties*

(1) $h[X,Y]$ *is a basic vector field and* $\pi_* h[X,Y] = [X', Y'] \circ \pi$;
(2) $h(\nabla_X Y)$ *is a basic vector field* $\pi-$*related to* $(\nabla'_{X'} Y')$, *where* $\nabla$ *and* $\nabla'$ *are the Levi-Civita connection on* $M$ *and* $B$;
(3) $[E, U] \in \Gamma(\mathcal{V})$, *for any* $U \in \Gamma(\mathcal{V})$ *and for any basic vector field* $E$.

## 3. Framed-Complex Submersions

In this section, we define the notion of framed-complex submersion, give an example and study the geometry of such submersions. We now define a $(\varphi, J)-$holomorphic map between framed metric manifolds and almost Hermitian manifolds.

**Definition 3.1.** Let $(M^{2m+s}, \varphi, (\xi_j, \eta_j)_{j=1}^s, g)$ be a framed metric manifold and $(B^{2n}, J)$ be an almost complex manifold, respectively. The map $\pi : M \to B$ is $(\varphi, J)-$holomorphic if $\pi_* \circ \varphi = J \circ \pi_*$.

By using the above definition, we are ready to give the following notion.

**Definition 3.2.** Let $(M^{2m+s}, \varphi, (\xi_j, \eta_j)_{j=1}^s, g)$ be a framed metric manifold and $(B, J, g')$ be an almost Hermitian manifold. A Riemannian submersion $\pi : M \to B$ is called a framed-complex submersion if it is $(\varphi, J)$-holomorphic, as well.

Consider $R^{2m+s}$ with its standard coordinates $x_1, ..., x_m, y_1, ..., y_m, z_1, ..., z_s$. We introduce on $R^{2m+s}$ a framed metric structure $(\varphi, \xi_j, \eta_j, g)$ by setting

$$\eta_j = dz_j, \;\; \xi_j = \frac{\partial}{\partial z_j}, \;\; g = 2\sum_{i=1}^m ((dx_i)^2 + (dy_i)^2) + \sum_{j=1}^s (\eta_j \otimes \eta_j)$$

and $\varphi$ given, with respect to the frame $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_m}, \frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_s})$ by the $(2m+s) \times (2m+s)-$matrix

$$\begin{pmatrix} 0 & -I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On the other hand, the canonical almost complex structure on $R^{2n}$ is given by

$$J(x_1, ..., x_{2n}) = (-x_{2n}, -x_{2n-1}, ..., x_2, x_1),$$

where Riemannian metric is standard inner product defined on $R^{2n}$.

We now give an example for a framed-complex submersion.

**Example 3.1.** Consider the following submersion defined by

$$\begin{array}{rcl} \pi : R^{4+2} & \to & R^2 \\ (x_1, x_2, y_1, y_2, z_1, z_2) & \to & (x_1 + x_2, y_1 + y_2). \end{array}$$

Then, the kernel of $\pi_*$ is

$$\mathcal{V} = Ker\pi_* = Span\{V_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, V_2 = -\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, \xi_1 = \frac{\partial}{\partial z_1}, \xi_2 = \frac{\partial}{\partial z_2}\}$$

and the horizontal distribution is spanned by

$$\mathcal{H} = (Ker\pi_*)^\perp = Span\{X = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Y = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\}.$$

Hence, we have

$$g(X,X) = g'(\pi_*X, \pi_*X) = 4, \quad g(Y,Y) = g'(\pi_*Y, \pi_*Y) = 4.$$

Thus, $\pi$ is a Riemannnian submersion. Moreover, we can easily obtain that $\pi$ satisfies

$$\pi_*\varphi X = J\pi_*X, \quad \pi_*\varphi Y = J\pi_*Y.$$

Thus, $\pi$ is a framed-complex submersion.

As an obvious consequence of Definition 3.2 we obtain:

**Proposition 3.1.** *Let $\pi : M \to B$ be a framed-complex submersion from a framed metric manifold $M$ onto an almost Hermitian manifold $B$. If $X, Y$ are basic vector fields on $M$, $\pi-$related to $X', Y'$ on $B$, then, we have*

(i) *$h(\nabla_X\varphi)Y$ is the basic vector field $\pi-$related to $(\nabla'_{X'}J)Y'$;*
(ii) *$\varphi X$ is the basic vector field $\pi-$related to $JX'$.*

Next proposition shows that a framed-complex submersion puts some restrictions on the distributions $\mathcal{V}$ and $\mathcal{H}$.

**Proposition 3.2.** *Let $\pi : M \to B$ be a framed-complex submersion from a framed metric manifold $M$ onto an almost Hermitian manifold $B$. Then, the horizontal and vertical distributions are $\varphi-$ invariant.*

*Proof.* Consider a vertical vector field $U$; it is known that $\pi_*(\varphi U) = J(\pi_*U)$. Since $U$ is vertical and $\pi$ is a Riemannian submersion, we have $\pi_*U = 0$ from which $\pi_*(\varphi U) = 0$ follows and implies that $\varphi U$ is vertical, being in the kernel of $\pi_*$. As concerns the horizontal distribution, let $X$ be a horizontal vector field. We have $g(\varphi X, U) = -g(X, \varphi U) = 0$ because $\varphi U$ is vertical and $X$ is horizontal. From $g(\varphi X, U) = 0$ we deduce that $\varphi X$ is orthogonal to $U$ and then $\varphi X$ is horizontal. □

**Proposition 3.3.** *Let $\pi : M \to B$ be a framed-complex submersion from a framed metric manifold $M$ onto an almost Hermitian manifold $B$. Then, we have*

(i) *$\pi^*\Phi' = \Phi$ holds on the horizontal distribution, only;*
(ii) *Each $\xi_j$ is vertical vector field, $j \in \{1, ..., s\}$;*
(iii) *$\eta_j(X) = 0$, for all horizontal vector fields $X$.*

*Proof.* We prove only statement (i), the other assertions can be obtained in a similar way. If $X$ and $Y$ are basic vector fields on $M$, $\pi-$related to $X', Y'$ on $B$, then using the definition of a framed-complex submersion, we have

$$\begin{aligned} \pi^*\Phi'(X,Y) &= \Phi'(\pi_*X, \pi_*Y) = g'(\pi_*X, J\pi_*Y) = g'(\pi_*X, \pi_*\varphi Y) \\ &= \pi^*g'(X, \varphi Y) = g(X, \varphi Y) = \Phi(X,Y) \end{aligned}$$

which gives the proof of assertion(i). □

We now check the properties of the tensor fields $T$ and $A$ for a framed-complex submersion, we will see that such tensors have extra properties for such submersions.

**Lemma 3.1.** *Let $\pi : M \to B$ be a framed-complex submersion. If the total space is a $\mathcal{C}-$manifold, then we have*

(i) $T_U\varphi V = \varphi T_U V$;

(ii) $T_{\varphi U}V = \varphi T_U V$,

*for* $U, V \in \Gamma(\mathcal{V})$.

*Proof.* We only prove (i), the other assertion can be obtained in a similar way. Let $U$ and $V$ be vertical vector fields, and $X$ horizontal. Since $M$ is a $\mathcal{C}-$manifold, from (2.7) we get

$$2g((\nabla_U \varphi)V, X) = 0.$$

Then, since the vertical and the horizontal distributions are $\varphi-$invariant, from (2.15) we obtain

$$g(T_U \varphi V - \varphi T_U V, X) = 0.$$

Hence, we have

$$T_U \varphi V = \varphi T_U V.$$

□

For the tensor field $A$ we have the following.

**Lemma 3.2.** *Let* $\pi : M \to B$ *be a framed-complex submersion. If the total space is a* $\mathcal{C}-$*manifold, then we have*

(i) $A_X \varphi Y = \varphi A_X Y$;

(ii) $A_{\varphi X} Y = \varphi A_X Y$;

(iii) $A_{\varphi X} X = 0$;

*for* $X, Y \in \Gamma(\mathcal{H})$.

Using (2.8) we have the following.

**Lemma 3.3.** *Let* $\pi : M \to B$ *be a framed-complex submersion. If the total space is a* $\mathcal{S}-$*manifold, then we have*

(i) $T_U \varphi V = \varphi T_U V$;

(ii) $T_{\varphi U} V = \varphi T_U V$,

(iii) $A_X \varphi Y = \varphi A_X Y$;

(iv) $A_{\varphi X} Y = \varphi A_X Y$;

(v) $A_{\varphi X} X = 0$;

*for* $X, Y \in \Gamma(\mathcal{H})$ *and* $U, V \in \Gamma(\mathcal{V})$.

We shall be interested with the tensor $T$ which is a usefully tool in the study of the fibres.
If $T_U \varphi V = \varphi T_U V$, then the fibres are minimal and if $T = 0$ they are totally geodesic.

**Theorem 3.1.** *Let* $\pi : M \to B$ *be a framed-complex submersion from a* $\mathcal{S}-$*manifold or a* $\mathcal{C}-$*manifold* $M$ *onto an almost Hermitian manifold* $B$*. If for all* $U, V$, $T_{\varphi U} \varphi V + T_U V = 0$, *then T=0.*

*Proof.* Let $U$ and $V$ be vertical vector fields. From Lemma 3.1, we get $T_{\varphi U}\varphi V = \varphi T_{\varphi U}V$. Using again Lemma 3.1, we have

$$T_{\varphi U}\varphi V = \varphi T_{\varphi U}V = \varphi^2 T_U V = -T_U V + \sum_{j=1}^{s} \eta_j(T_U V)\xi_j, j \in \{1, ..., s\}.$$

On the other hand, it can be shown that $T_U \xi_j = 0$ and then $\eta_j(T_U V) = 0$ which gives $T_{\varphi U}\varphi V = -T_U V$ from which $T_{\varphi U}\varphi V + T_U V = 0$ follows. Thus, we get $T = 0$. □

**Corollary 3.1.** *Let $\pi : M \to B$ be a framed-complex submersion. If the total space is a $\mathcal{K}-$ manifold or an almost $\mathcal{C}-$manifold, then the fibres are totally geodesic.*

We now investigate the integrability of the horizontal distribution $\mathcal{H}$.

**Theorem 3.2.** *Let $\pi : M \to B$ be a framed-complex submersion from a $\mathcal{K}-$manifold $M$ onto an almost Hermitian manifold $B$. Then, the horizontal distribution is integrable.*

*Proof.* Let $X$ and $Y$ be basic vector fields. It suffices to prove that $v([X, Y]) = 0$, for basic vector fields on $M$. Since $M$ is a $\mathcal{K}-$manifold, it implies $d\Phi(X, Y, V) = 0$, for any vertical vector $V$. Then, one obtains

$$X(\Phi(Y, V)) - Y(\Phi(X, V)) + V(\Phi(X, Y))$$
$$-\Phi([X, Y], V) + \Phi([X, V], Y) - \Phi([Y, V], X) = 0.$$

Since $[X, V], [Y, V]$ are vertical and the two distributions are $\varphi-$invariant, the last two and the first two terms vanish. Thus, one gets

$$g([X, Y], \varphi V) = V(g(X, \varphi Y)).$$

On the other hand, if $X$ is basic then $h(\nabla_V X) = h(\nabla_X V) = A_X V$, thus we have

$$\begin{aligned} V(g(X, \varphi Y)) &= g(\nabla_V X, \varphi Y) + g(\nabla_V \varphi Y, X) \\ &= g(A_X V, \varphi Y) + g(A_{\varphi Y} V, X). \end{aligned}$$

Since $A$ are skew-symmetric and alternating operator, we get $V(g(X, \varphi Y)) = 0$. This proves the assertion. □

**Corollary 3.2.** *Let $\pi : M \to B$ be a framed-complex submersion. If the total space is a $\mathcal{S}-$manifold, an almost $\mathcal{C}-$manifold or a $\mathcal{C}-$manifold, then the horizontal distribution is integrable.*

## 4. Transference of Structures

In this section, we investigate what kind of almost Hermitian structures are defined on the base manifold, when the total manifold has some special framed structures.

As the fibres of a framed-complex submersion is an invariant submanifold of $M$ with respect to $\varphi$, we have the following.

**Proposition 4.1.** *Let $\pi : (M^{2m+s}, \varphi, \xi, \eta, g) \to (B^{2n}, J, g')$ be a framed-complex submersion from a framed metric manifold $M$ onto an almost Hermitian manifold $B$. Then, the fibres are framed metric manifolds.*

*Proof.* Denoting by $F$ the fibres, it is clear that $dimF = 2(m-n)+s = 2r+s$, where $r = m-n$. We define a framed metric structure $(\hat{g}, \hat{\varphi}, \hat{\eta_j}, \hat{\xi_j})$, where $j = 1, ..., s$, by setting $\varphi = \hat{\varphi}$, $\eta_j = \hat{\eta_j}$and $\xi_j = \hat{\xi_j}$. Then, we get

$$\hat{\varphi}^2 U = \varphi^2 U = -U + \sum_{j=1}^{s} \eta_j(U)\xi_j,$$

for $U \in \Gamma(\mathcal{V})$.

On the other hand, for $U, V \in \Gamma(\mathcal{V})$ we obtain

$$\begin{aligned} \hat{g}(\hat{\varphi}V, \hat{\varphi}U) &= \hat{g}(\varphi V, \varphi U) = -\hat{g}(V, \varphi^2 U) = -\hat{g}(V, -U + \sum_{j=1}^{s} \eta_j(U)\xi_j) \\ &= \hat{g}(V, U) - \sum_{j=1}^{s} \hat{\eta}_j(U)\hat{\eta}_j(V), \end{aligned}$$

which gives the proof of assertion. □

In the sequel, we show that base space is a Hermitian manifold if the total space is a normal.

**Theorem 4.1.** *Let $\pi : M \to B$ be a framed-complex submersion. If the framed metric structure of $M$ is normal, then the base space $B$ is a Hermitian manifold.*

*Proof.* Let $X$ and $Y$ be basic vector fields on $M$, $\pi-$related to $X'$ and $Y'$ on $B$. From (2.5), we have

$$\pi_*(N^{(1)}(X, Y)) = \pi_*([\varphi, \varphi](X, Y) + \sum_{j=1}^{s} 2d\eta_j(X, Y)\xi_j).$$

On the other hand, $\pi_*\varphi = J\pi_*$ implies that

$$\begin{aligned} \pi_*([\varphi, \varphi](X, Y)) &= \pi_*(\varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]) \\ &= -[\pi_* X, \pi_* Y] + \sum_{j=1}^{s} \eta_j([X, Y])\pi_*\xi_j + [\pi_*\varphi X, \pi_*\varphi Y] - J\pi_*[\varphi X, Y] \\ &- J\pi_*[X, \varphi Y] \\ &= -[X', Y'] + [JX', JY'] - J[JX', Y'] \\ &- J[X', JY']. \end{aligned}$$

Then, we have

$$\pi_*([\varphi, \varphi](X, Y)) = N'(X', Y') = 0,$$

which shows that $B$ is a Hermitian manifold. □

**Proposition 4.2.** *Let $\pi : M \to B$ be a framed-complex submersion. If the total space $M$ is an almost $\mathcal{C}-$ manifold, then the base space $B$ is an almost Kähler manifold.*

*Proof.* Let $X$, $Y$ and $Z$ be basic vector fields on $M$ $\pi-$related to $X'$, $Y'$ and $Z'$ on $B$. Since $M$ is an almost $\mathcal{C}-$manifold, we have $d\Phi(X, Y, Z) = 0$. Then, we obtain

$$\begin{gathered} X(\Phi(Y, Z)) - Y(\Phi(X, Z)) + Z(\Phi(X, Y)) \\ -\Phi([X, Y], Z) + \Phi([X, Z], Y) - \Phi([Y, Z], X) = 0. \end{gathered}$$

On the other hand, by direct calculations, we get

$$\begin{aligned} 0 &= g(\nabla_X Y, \varphi Z) + g(Y, \nabla_X \varphi Z) - g(\nabla_Y X, \varphi Z) - g(X, \nabla_Y \varphi Z) \\ &+ g(\nabla_Z X, \varphi Y) + g(X, \nabla_Z \varphi Y) - g([X,Y], \varphi Z) \\ &+ g([X,Z], \varphi Y) - g([Y,Z], \varphi X). \end{aligned}$$

Then, by using $\pi_*\varphi = J\pi_*$, we get

$$\begin{aligned} &0 = g'(\nabla'_{X'}Y', JZ') + g'(Y', \nabla'_{X'}JZ') - g'(\nabla'_{Y'}X', JZ') - g'(X', \nabla'_{Y'}JZ') \\ &+g'(\nabla'_{Z'}X', JY') + g'(X', \nabla'_{Z'}JY') - g'([X', Y'], JZ') \\ &+g'([X', Z'], JY') - g'([Y', Z'], JX') \\ &0 = X'(\Phi'(Y', Z')) - Y'(\Phi'(X', Z')) + Z'(\Phi'(X', Y')) \\ &-\Phi'([X', Y'], Z') + \Phi'([X', Z'], Y') - \Phi'([Y', Z'], X') \\ &0 = d\Phi'(X', Y', Z'). \end{aligned}$$

Thus, if the total space $M$ is an almost $\mathcal{C}-$manifold, then the base space $B$ is an almost Kähler manifold. □

**Corollary 4.1.** *Let $\pi : M \to B$ be a framed-complex submersion. If the total space $M$ is a $\mathcal{K}-$ manifold or a $\mathcal{S}-$ manifold , then the base space $B$ is an almost Kähler manifold.*

We also have the following result which shows that the other structures can be mapped onto the base manifold.

**Proposition 4.3.** *Let $\pi : M \to B$ be a framed-complex submersion. If the total space $M$ is a $\mathcal{C}-$ manifold, then the base space $B$ is a Kähler manifold.*

*Proof.* Let $X$, $Y$ and $Z$ be basic vector fields on $M$ $\pi-$related to $X'$, $Y'$ and $Z'$ on $B$. Since $M$ is a $\mathcal{C}-$ manifold, from (2.7) we get

$$g((\nabla_X\varphi)Y, Z) = g(\nabla_X\varphi Y - \varphi\nabla_X Y, Z) = 0.$$

Since $\pi$ is a Riemannian submersion, we obtain

$$g'(\pi_*(\nabla_X\varphi Y - \varphi\nabla_X Y), \pi_*(Z)) = 0.$$

Then, by using $\pi_*\varphi = J\pi_*$, we get

$$g'(\pi_*(\nabla_X\varphi Y - J\pi_*(\nabla_X Y)), \pi_*(Z)) = 0.$$

On the other hand, from Proposition 3.1, we know that if $X$ is $\pi-$related to $X'$, then $\varphi X$ is $\pi-$related to $JX'$. Also, from Lemma 2.2, it follows $h(\nabla_X\varphi Y)$ and $h(\nabla_X Y)$ are $\pi-$related to $\nabla'_{X'}JY'$ and $\nabla'_{X'}Y'$. Thus, we have

$$g'(\nabla'_{X'}JY' - J\nabla'_{X'}Y', Z') = 0.$$

Then, from (2.10) we get $g'(\nabla'_{X'}J)Y', Z') = 0$. Hence, we have $(\nabla'_{X'}J)Y' = 0$ which proves the assertion. □

**Corollary 4.2.** *Let $\pi : M \to B$ be a framed-complex submersion. If the total space $M$ is an almost $\mathcal{C}-$manifold, a $\mathcal{K}-$ manifold, an almost $\mathcal{S}-$ manifold, a $\mathcal{C}-$manifold or a $\mathcal{S}-$ manifold , then the fibres inherit from the total space a structure of the same type.*

## 5. Curvature Relations for Framed-Complex Submersions

We begin this section relating the $\varphi-$holomorphic bisectional and sectional curvatures of the total space, the base space and the fibres of a framed-complex submersion.

Let $\pi$ be a framed-complex submersion between a framed metric manifold $M$ and an almost Hermitian manifold $N$. We denote the Riemannian curvatures of $M, N$ and any fibre $\pi^{-1}(x)$ by $R, R'$ and $\hat{R}$, respectively. For $X, Y, Z, W \in \Gamma(\mathcal{H})$, we have

$$R^*(X,Y,Z,W) = R'(\pi_*X, \pi_*Y, \pi_*Z, \pi_*W) \circ \pi.$$

Let $\pi : M \to N$ be a framed-complex submersion from a framed metric manifold $(M, \varphi, \xi_j, \eta_j, g)$ onto an almost Hermitian manifold $(N, J, g')$. We denote by $B$ the $\varphi-$holomorphic bisectional curvature, defined for any pair of vectors $X$ and $Y$ on $M$ orthogonal to $\xi_j$ by the formula:

$$B(X,Y) = \frac{R(X, \varphi X, Y, \varphi Y)}{\|X\|^2\|Y\|^2}.$$

The $\varphi-$holomorphic sectional curvature is $H(X) = B(X,X)$ for any vector $X$ orthogonal to $\xi_j$, $j \in \{1, ..., s\}$. We denote by $B'$ and $H'$ the $\varphi-$holomorphic bisectional and $\varphi-$holomorphic sectional curvatures of $N$. Similarly, $\hat{B}$ and $\hat{H}$ denote the bisectional and the sectional holomorphic curvatures of a fibre.

The following is a translation of the results of Gray[11] and O'Neill[15] to the present situation:

**Proposition 5.1.** *Let $\pi : M \to N$ a framed-complex submersion from a framed metric manifold $M$ onto an almost Hermitian manifold $N$. Let $U$ and $V$ be unit vertical vectors, and $X$ and $Y$ unit horizontal vectors orthogonal to $\xi_j$. Then, we have*

$$\begin{aligned}
(a)B(U,V) &= \hat{B}(U,V) - g(T_UV, T_{\varphi U}\varphi V) + g(T_{\varphi U}V, T_U\varphi V);\\
(b)B(X,U) &= g((\nabla_U A)_X \varphi X, \varphi U) - g((\nabla_{\varphi U}A)_X\varphi X, U)\\
&+ g(A_XU, A_{\varphi X}\varphi U) - g(A_X\varphi U, A_{\varphi X}U)\\
&- g(T_UX, T_{\varphi U}\varphi X) + g(T_{\varphi U}X, T_U\varphi X);\\
(c)B(X,Y) &= B'(X',Y') \circ \pi - 2g(A_X\varphi X, A_Y\varphi Y)\\
&+ g(A_{\varphi X}Y, A_X\varphi Y) - g(A_XY, A_{\varphi X}\varphi Y).
\end{aligned}$$

Using Proposition 5.1, we have the following result.

**Proposition 5.2.** *Let $\pi : M \to N$ a framed-complex submersion from a framed metric manifold $M$ onto an almost Hermitian manifold $N$. Let $U$ be unit vertical vector, and $X$ unit horizontal vector orthogonal to $\xi_j$. Then, one has:*

(a) $H(U) = \hat{H}(U) + \|T_U\varphi U\|^2 - g(T_{\varphi U}\varphi U, T_UU)$;
(b) $H(X) = H'(X') \circ \pi - 3\|A_X\varphi X\|^2$.

**Theorem 5.1.** *Let $\pi : M \to N$ a framed-complex submersion. If the total space is a $\mathcal{K}-$manifold, a$\mathcal{S}-$manifold, an almost $\mathcal{C}-$manifold or a $\mathcal{C}-$manifold, then we have*

(a) $B(U,V) = \hat{B}(U,V)$;

(b) $H(U) = \hat{H}(U)$,

*where* $U$ *and* $V$ *are unit vertical vectors orthogonal to* $\xi_j$.

*Proof.* (a) Since the fibres are totally geodesic, we have $T = 0$. Then using Proposition 5.1(a) we get $B(U, V) = \hat{B}(U, V)$.
In a similar way, we obtain (b). □

Since the horizontal distribution $\mathcal{H}$ is integrable, we get $A = 0$. Then, we have the following result.

**Theorem 5.2.** *Let* $\pi : M \to N$ *a framed-complex submersion. If the total space is a* $\mathcal{K}-$*manifold, a* $\mathcal{S}-$*manifold, an almost* $\mathcal{C}-$*manifold or a* $\mathcal{C}-$*manifold, then we have*

(a) $B(X, Y) = B'(X', Y') \circ \pi$;
(b) $H(X) = H'(X') \circ \pi$,

*where* $X$ *and* $Y$ *are unit horizontal vectors orthogonal to* $\xi_j$.

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