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SEMIRADICAL EQUALITY

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Abstract. Semiprime radical of a module is defined and the relation between the intersection of prime submodules and the intersection of semiprime submodules is investigated. Semiradical formula is defined and it is shown that cartesian product of $M_1 \times M_2$ satisfies the semiradical formula if and only if M_1 and M_2 satisfy the semiradical formula.

1. Introduction

Throughout all rings are commutative and all modules are unitary. Let R be a ring and M be an R-module. A proper submodule N of M is prime if whenever $rm \in N$, for some $r \in R, m \in M$ then $m \in N$ or $rM \subseteq N$. A proper submodule N of an R-module M is semiprime, if whenever $r^k m \in N$ for some $r \in R, m \in M$ and $k \in \mathbb{Z}^+$, then $rm \in N$. Also, for any submodule N of M the envelope of N in M is defined as the set

$$E_M(N) = \{rm : r \in R, m \in M \quad and \quad r^k m \in N \quad for \quad some \quad k \in \mathbb{Z}^+\}.$$

It is easy to show that a proper submodule N is semiprime if and only if $\langle E_M(N) \rangle = N$. Also, it is clear that every prime submodule is semiprime but the converse is not true in general; to show this with an example let's give the following Theorem of Ylmaz and Klarslan Cansu.

Theorem 1.1. *([4], Theorem 2.5) Let $N = Q_1 \cap Q_2 \cap \cdots \cap Q_k$ be minimal primary decomposition of N where $\sqrt{Q_i : M} = p_i$ for all $i = 1, 2, \ldots, k$ and $S = \{1, 2, \ldots, k\}$. Then*

$$\langle E_M(N) \rangle = N + (\bigcap_{i=1}^{k} p_i)M + \sum_{T \subset S} (\bigcap_{i \in T} p_i)(\bigcap_{i \in S \setminus T} Q_i)$$

where the summation runs over each non-empty subset T of S.

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Now, we can give the example. The computer algebra system **SINGULAR** was used during the computations.

If $R = \mathbb{Q}[x, y, z]$, $M = R^3$ and $N = \langle z\mathbf{e}_1, y\mathbf{e}_1, xy\mathbf{e}_2, xy\mathbf{e}_3, xz\mathbf{e}_2 + x^2 z\mathbf{e}_3\rangle$. Then primary decompostion of N is $N = Q_1 \cap Q_2 \cap Q_3$ where

$$\begin{aligned} Q_1 &= \langle \mathbf{e}_1, y\mathbf{e}_2, y\mathbf{e}_3, x\mathbf{e}_3 + \mathbf{e}_2\rangle \text{ is } \langle y\rangle - \text{primary}, \\ Q_2 &= \langle z\mathbf{e}_1, z\mathbf{e}_2, z\mathbf{e}_3, y\mathbf{e}_1, y\mathbf{e}_2, y\mathbf{e}_3\rangle \text{ is } \langle z, y\rangle - \text{primary and} \\ Q_3 &= \langle \mathbf{e}_1, x\mathbf{e}_2, x\mathbf{e}_3\rangle \text{ is } \langle x\rangle - \text{primary}. \end{aligned}$$

By Theorem 1.1,

$$\begin{aligned} \langle E_M(N)\rangle = N &+ (p_1 \cap p_2 \cap p_3)M + p_1(Q_2 \cap Q_3) + p_2(Q_1 \cap Q_3) + p_3(Q_1 \cap Q_2) \\ &+ (p_1 \cap p_2)Q_3 + (p_1 \cap p_3)Q_2 + (p_2 \cap p_3)Q_1 \\ &= \langle z\mathbf{e}_1, y\mathbf{e}_1, xy\mathbf{e}_2, xy\mathbf{e}_3, xz\mathbf{e}_2 + x^2 z\mathbf{e}_3\rangle = N. \end{aligned}$$

Hence, N is a semiprime submodule of M with $N : M = \langle xy\rangle$. On the other hand N is not a prime submodule; since $r = z$ and $m = (0, x, x^2)$ gives $rm = z(0, x, x^2) = (0, xz, x^2 z) \in N$ but $r = z \notin N : M$ and $m = (0, x, x^2) \notin N$.

If N is a proper submodule of an R-module M, then the prime radical of N, $rad_M(N)$, is the intersection of all prime submodules containing N. If it is necessary to indicate the underlying ring, the prime radical of N is denoted by $rad_{_RM}(N)$. The semiprime radical of N, denoted by $srad_M(N)(srad_{_RM}(N))$, is defined as the intersection of all semiprime submodules of M containing N. If there is no semiprime submodule containing N, then $srad_M(N) = M$.

A module M satisfies the radical formula (s.t.r.f.) if for any submodule N of M, $rad_M(N) = \langle E_M(N)\rangle$. In the same manner we define, an R-module M satisfies the semiradical formula (s.t.s.r.f.) if for any submodule N of M, $srad_M(N) = \langle E_M(N)\rangle$. Since intersection of semiprime submodules is semiprime, $srad_M(N)$ is the unique smallest semiprime submodule of M containing N.

We know that for an ideal I of R, $\sqrt{\sqrt{I}} = \sqrt{I}$; but the envelope of a submodule does not satisfy an equation similiar to this one. If $R = \mathbb{Q}[x, y, z]$, M is an R-module $R \oplus R$ and $N = \langle z^2\mathbf{e}_1, z^2\mathbf{e}_2, yz\mathbf{e}_2, y^2\mathbf{e}_1 + z\mathbf{e}_2, y^2\mathbf{e}_2, y\mathbf{e}_1 + x^3\mathbf{e}_2\rangle$ is an R-submodule of M. Since N is $\langle z, y\rangle$-primary,

$$\langle E_M(N)\rangle = N + \langle z, y\rangle M = \langle z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2, x^3\mathbf{e}_2\rangle.$$

Since $\langle E_M(N)\rangle = Q_1 \cap Q_2$, where

$$\begin{aligned} Q_1 &= \langle \mathbf{e}_2, z\mathbf{e}_1, y\mathbf{e}_1\rangle \text{ is } \langle z, y\rangle - \text{primary}, \\ Q_2 &= \langle z\mathbf{e}_1, z\mathbf{e}_2, z\mathbf{e}_3, y\mathbf{e}_1, y\mathbf{e}_2, x^3\mathbf{e}_1, x^3\mathbf{e}_2\rangle \text{ is } \langle x, y, z\rangle - \text{primary}, \end{aligned}$$

Theorem 1.1 implies that $\langle E_M(\langle E_M(N)\rangle)\rangle = \langle z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2, x\mathbf{e}_2\rangle \neq \langle E_M(N)\rangle$.

In [2], Azizi and Nikseresht defined the kth envelope of N recursively by $E_0(N) = N, E_1(N) = E_M(N), E_2(N) = E_M(\langle E_M(N)\rangle)$ and $E_k(N) = E_M(\langle E_{k-1}(N)\rangle)$ for every submodule N of M. It is easy to show that

$$N = \langle E_0(N)\rangle \subseteq \langle E_1(N)\rangle \subseteq \langle E_2(N)\rangle \subseteq \cdots\cdots \subseteq \langle E_\infty(N)\rangle \subseteq srad_M(N) \subseteq rad_M(N)$$

where $\langle E_\infty(N)\rangle = \bigcup_{k=0}^{\infty} \langle E_k(N)\rangle$. It is clear that $\langle E_\infty(N)\rangle$ is semiprime and thus $\langle E_\infty(N)\rangle = srad_M(N)$.

When we consider the chain

$$N = \langle E_0(N)\rangle \subseteq \langle E_1(N)\rangle \subseteq \langle E_2(N)\rangle \subseteq \cdots\cdots \subseteq \langle E_\infty(N)\rangle = srad_M(N) \subseteq rad_M(N),$$

it seems meaningfull to focus on the submodules $srad_M(N)$ and $rad_M(N)$ and investigate the conditions where the equality $srad_M(N) = rad_M(N)$ occurs.

Here, semiradical equality is defined and some equivalent conditions for a ring to satisfy the semiradical equality are stated.

2. Semiradical Equality

It is clear that intersection of prime submodules is semiprime, but the converse is not true in general, see [1].

Lemma 2.1. *Let M be an R-module. Then every semiprime submodule is an intersection of prime submodules if and only if $srad_M(N) = rad_M(N)$ for any submodule N of M.*

Proof. (⇒) Since intersection of semiprime submodules is semiprime , $srad_M(N)$ is a semiprime submodule. Hence it is obvious.

(⇐) Let K be a semiprime submodule of M. Then $K = srad_M(K) = rad_M(K)$. Hence K is an intersection of prime submodules. □

Lemma 2.2. *Let N be a submodule of an R-module M such that M/N is projective. Then $srad_M(N) = rad_M(N)$.*

Proof. Since M/N is projective, $rad_{M/N}(0) = \langle E_{M/N}(0)\rangle$ by [1] Lemma 8. Then we have,$rad_M(N) = \langle E_M(N)\rangle$ which implies that $srad_M(N) = rad_M(N)$. □

Corollary 2.1. *Let N be a submodule of an R-module M such that M/N is projective. Then $srad_M(N) = radRM + N$.*

Proof. Clear by [5] Theorem 2.7 and the above lemma. □

We say that a module M satisfies the semiradical equality if for every submodule N of M, $srad_M(N) = rad_M(N)$. It is said that a ring R satisfies the semiradical equality if every R-module satisfies the semiradical equality. Since arithmetical rings satisfy the radical formula [3], an arithmetical ring satisfies the semiradical equality.

Proposition 2.1. *The followings are equivalent.*

(i) *The ring R satisfies the semiradical equality.*
(ii) *for any ideal I of R, the ring R/I satisfies the semiradical equality.*
(iii) *for any non-maximal semiprime ideal P of R, the ring R/P satisfies the semiradical equality.*

Proof. ($i \Rightarrow ii$) Let M be an R/I-module. By Lemma 2.1, it is enough to show that every semiprime R/I-module is an intersection of prime submodules. Let K be a semiprime submodule of an R/I-module M. Then K is a semiprime submodule of M as an R-module. So, $K = srad_{_RM}(K) = rad_{_RM}(K)$.

It is easy to see that every submodule of M is a prime R-submodule if and only if it is a prime R/I-submodule. Hence $rad_{_RM}(K) = rad_{_{R/I}M}(K)$ and thus $K = rad_{_{R/I}M}(K)$.

($iii \Rightarrow i$) Let N be a semiprime submodule of an R-module M with $N : M = P$ where P is a non-maximal semiprime ideal. Consider M/N as an R/P-module. Then by our assumption, $srad_{M/N}(0) = rad_{M/N}(0)$. Hence $srad_M(N) = rad_M(N)$.

□

Corollary 2.2. *If for any non-maximal semiprime ideal P of R; R/P is a Prüfer domain, then R satisfies the semiradical equality.*

Lemma 2.3. *A ring R satisfies the semiradical equality if and only if every free R-module satisfies the semiradical equality.*

Proof. Let M be an R-module. Then there exists a free R-module F such that $M \cong F/K$. By our assumption, for any submodule N of M

$$\begin{aligned} srad_{F/K}(N/K) &= srad_F(N)/K \\ &= rad_F(N)/K \\ &= rad_{F/K}(N/K). \end{aligned}$$

Hence M satisfies the semiradical equality. □

3. Semiprime Submodules of Cartesian Product of Modules

Let $R = R_1 \times R_2$ where each R_i is a commutative ring with nonzero identity. Let M_i be an R_i-module for $i = 1, 2$ and $M = M_1 \times M_2$ be the R-module with action $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$ where $r_i \in R_i, m_i \in M_i$. These notations are fixed for this section.

Note that since our action is $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$ where $r_i \in R_i, m_i \in M_i$, every submodule of $M_1 \times M_2$ is of the form $N_1 \times N_2$ with N_1 is a submodule of M_1 and N_2 is a submodule of M_2.

Proposition 3.1. *Let R and M be as above. Then*

(i) *If N_1 is semiprime submodule of M_1, then $N_1 \times M_2$ is semiprime submodule of $M_1 \times M_2$.*
(ii) *If N_2 is semiprime submodule of M_2, then $M_1 \times N_2$ is semiprime submodule of $M_1 \times M_2$.*

Proof. (i) Let $r = (r_1, r_2) \in R$, $m = (m_1, m_2) \in M$ and $r^k m \in N_1 \times M_2$ for some $k \in \mathbb{Z}^+$. Since N_1 is semiprime submodule of M_1, $r_1m_1 \in N_1$. Then $(r_1m_1, r_2m_2) = rm \in N_1 \times M_2$ which implies that $N_1 \times M_2$ is semiprime submodule of M.

(ii) Similiar to case (i). □

Lemma 3.1. *Let R and M be as above. Then $Q_1 \times Q_2$ is a semiprime submodule of M if and only if Q_i is semiprime submodule of M_i for all $i = 1, 2$.*

Proof. Let $(r_1, r_2)^k(m_1, m_2) \in Q_1 \times Q_2$ where $m_i \in M_i$, $r_i \in R_i$ and $k \in \mathbb{Z}^+$ and Q_1 and Q_2 be semiprime submodules of M_1 and M_2 respectively. Since Q_1 and Q_2 are semiprime, $r_i m_i \in Q_i$ for $i = 1, 2$ which implies that $Q_1 \times Q_2$ is semiprime.
Now assume that $Q_1 \times Q_2$ is semiprime submodule of $M_1 \times M_2$. Let $r_1 \in R_1$, $m_1 \in M_1$ with $r_1^k m_1 \in Q_1$. Then $(r_1, 1)^k(m_1, 0) \in Q_1 \times Q_2$. Since $Q_1 \times Q_2$ is semiprime, $(r_1, 1)(m_1, 0) = (r_1 m_1, 0) \in Q_1 \times Q_2$ implies that Q_1 is semiprime submodule of M_1. Similarly it can be shown that Q_2 is semiprime submodule of M_2. □

Lemma 3.2. *Let $N = N_1 \times N_2$ be a submodule of M where N_i is a submodule of M_i for $i = 1, 2$. Then $N : M = (N_1 : M_1) \times (N_2 : M_2)$*

Proof. Let $x = (x_1, x_2) \in (N : M)$. Then $xM \subseteq N$ which means that

$$(x_1, x_2)(m_1, m_2) = (x_1 m_1, x_2 m_2) \in N_1 \times N_2$$

for all $m_1 \in M_1$ and $m_2 \in M_2$. So, $x_1 m_1 \in N_1$ and $x_2 m_2 \in N_2$. Hence

$$x_1 \in (N_1 : M_1), \quad x_2 \in (N_2 : M_2)$$

and thus $x = (x_1, x_2) \in (N_1 : M_1) \times (N_2 : M_2)$.

Conversely, let $y = (y_1, y_2) \in (N_1 : M_1) \times (N_2 : M_2)$. Then $y_1 M_1 \subseteq N_1$ and $y_2 M_2 \subseteq N_2$. Hence for all $m_1 \in M_1, m_2 \in M_2$,

$$(y_1, y_2)(m_1, m_2) = (y_1 m_1, y_2 m_2) \in N_1 \times N_2.$$

This implies that $y \in (N_1 \times N_2) : (M_1 \times M_2) = (N : M)$. □

Let N be a semiprime submodule of an R-module M. If $p = \sqrt{N : M}$ is a prime ideal, then N is called p -semiprime submodule.

Lemma 3.3. *Let $N = N_1 \times N_2$ be a submodule of M. Then*

(i) *N is $p \times R_2$ semiprime submodule of M iff N_1 is p-semiprime submodule of M_1 and $N_2 = M_2$.*
(ii) *N is $R_1 \times p$ semiprime submodule of M iff N_2 is p-semiprime submodule of M_2 and $N_1 = M_1$.*

Proof. (i) Suppose $N = N_1 \times N_2$ is semiprime submodule of $M_1 \times M_2$. By Lemma 3.1, N_1 is semiprime submodule of M_1.

Since $N : M = p \times R_2$, N_1 is p-semiprime and $N_2 : M_2 = R_2$ implies that $N_2 = M_2$.

Other side is clear by Proposition 3.1 and Lemma 3.2.

(ii) Similiar to case (i). □

If p_1 and p_2 are prime ideals of R_1 and R_2 respectively, it is not true in general that $p_1 \times p_2$ is prime ideal of $R_1 \times R_2$, for example if we take $R_1 = R_2 = \mathbb{Z}$, $p_1 = 2\mathbb{Z}$ and $p_2 = 3\mathbb{Z}$, then $2\mathbb{Z} \times 3\mathbb{Z}$ is not a prime ideal of $\mathbb{Z} \times \mathbb{Z}$ since $\mathbb{Z}_2 \times \mathbb{Z}_3$ is not an integral domain. So, if we try to generalize Lemma 3.3, we only get the following lemma.

Lemma 3.4. *Let $N = N_1 \times N_2$ be a submodule of M. If $N_1 \times N_2$ is $p_1 \times p_2$-semiprime submodule, then N_i is p_i-semiprime submodule of M_i for $i = 1, 2$.*

Proof. Assume that $N_1 \times N_2$ is $p_1 \times p_2$-semiprime submodule of M. By Lemma 3.1, N_1 and N_2 are semiprime submodules of M_1 and M_2 respectively. Since $p_1 \times p_2$ is prime ideal, by Lemma 3.2

$$\begin{aligned}(N_1 \times N_2) : (M_1 \times M_2) &= \sqrt{N : M} = p_1 \times p_2 \\ (N_1 : M_1) \times (N_2 : M_2) &= p_1 \times p_2.\end{aligned}$$

Hence, $N_1 : M_1 = p_1$ and $N_2 : M_2 = p_2$. Since p_1 and p_2 are prime ideals,

$$N_1 : M_1 = \sqrt{N_1 : M_1} = p_1 \quad \text{and}$$
$$N_2 : M_2 = \sqrt{N_2 : M_2} = p_2.$$

Thus, N_i is p_i-semiprime submodule of M_i for $i = 1, 2$. □

Proposition 3.2. *Let $N = N_1 \times N_2$ be a submodule of M. Then*

$$srad_M(N) = srad_{M_1}(N_1) \times srad_{M_2}(N_2)$$

Proof. Let $Q_1 \times Q_2$ be a semiprime submodule of M containing $N_1 \times N_2$. By Lemma 3.1, Q_i is semiprime submodule of M_i containing N_i for $i = 1, 2$. Then

$$srad_{M_1}(N_1) \times srad_{M_2}(N_2) \subseteq srad_M(N_1 \times N_2)$$

since $srad_{M_1}(N_1) \times srad_{M_2}(N_2) \subseteq Q_1 \times Q_2$.

Since $srad_{M_i}(N_i)$ is the minimal semiprime submodule of M_i containing N_i, Lemma 3.1 implies that $srad_{M_1}(N_1) \times srad_{M_2}(N_2)$ is a semiprime submodule of $M_1 \times M_2$ which contains $N_1 \times N_2$. Hence

$$srad_M(N) \subseteq srad_{M_1}(N_1) \times srad_{M_2}(N_2)$$

□

Corollary 3.1. *Let $N = N_1 \times N_2$ be a submodule of M. Then*

(i) $srad_M(N_1 \times M_2) = srad_{M_1}(N_1) \times M_2$
(ii) $srad_M(M_1 \times N_2) = M_1 \times srad_{M_2}(N_2)$

Proof. Clear by Proposition 3.2.

□

Proposition 3.3. *([6], Proposition 2.12) Let $N = N_1 \times N_2$ be a submodule of M. Then $\langle E_M(N) \rangle = \langle E_{M_1}(N_1) \rangle \times \langle E_{M_2}(N_2) \rangle$.*

Theorem 3.1. *M s.t.s.r.f. if and only if M_i s.t.s.r.f. for all $i = 1, 2$.*

Proof. Assume M s.t.s.r.f.. Take a submodule N_1 of M_1. Then $N_1 \times M_2$ s.t.s.r.f., so that $srad_{M_1}(N_1) \times M_2 = \langle E_{M_1}(N_1) \rangle \times \langle E_{M_2}(M_2) \rangle$. Now, let $x \in srad_{M_1}(N_1)$. Then $(x, m) \in srad_{M_1}(N_1) \times M_2$ and hence $x \in \langle E_{M_1}(N_1) \rangle$. Similiarly it can be shown that $srad_{M_2}(N_2) = \langle E_{M_2}(N_2) \rangle$.

Conversely assume that M_1 and M_2 s.t.s.r.f.. Take any submodule $N_1 \times N_2$ of $M_1 \times M_2$. Then

$$\begin{aligned}srad_M(N_1 \times N_2) &= srad_{M_1}(N_1) \times srad_{M_2}(N_2) \\ &= \langle E_{M_1}(N_1) \rangle \times \langle E_{M_2}(N_2) \rangle \\ &= \langle E_M(N_1 \times N_2) \rangle\end{aligned}$$

Thus, $M = M_1 \times M_2$ s.t.s.r.f.. □

References

[1] J. Jenkins and P. F. Smith, On the prime radical of a module over a commutative ring, Comm. in Algebra. Vol:20, No.12 (1992), 3593 – 3602.

[2] A. Azizi and A. Nikseresht, On radical formula in modules, Glasgow. Math. J. Vol:53, No.3 (2011), 657 – 668.

[3] A. Parkash, Arithmetical rings satisfy the radical formula, Journal of Commutative Algebra. Vol:4, No.2 (2012), 293 – 296.

[4] E. Ylmaz and S. Klarslan Cansu, Baer's lower nilradical and classical prime submodules, Bul. Iran Math. Soc., to appear.

[5] M. Alkan and Y. Tra, On prime submodules, Rocky Mountain Journal of Mathematics, Vol:37, No.3 (2007), 709 – 722.

[6] S. Atani and F. K. Saraei, Modules which satisfy the radical formula, Int. J. Contemp. Math. Sci. Vol:2, No.1 (2007), 13 – 18.

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