Konuralp Journal of Mathematics
Volume 2 No. 2 pp. 42–52 (2014) ©KJM

UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING TWO FINITE SETS IN $\mathbb{C}$ WITH FINITE WEIGHT

ABHIJIT BANERJEE AND GOUTAM HALDAR

Abstract. With the aid of the notion of weighted sharing of sets of meromorphic functions we improve some previous results concerning a particular range set.

1. Introduction, Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \quad (r \longrightarrow \infty, r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z)-a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a\in S}\{z : f(z)-a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values.

In connection with the famous "Gross Question" {see [8]} in the uniqueness literature Gross and Yang [9] (see also [16]) made a vital contribution by introducing the new idea of unique range set for meromophic function (URSM in brief). We recall that "Gross's Question" was the first one which deal with the uniqueness of

2000 *Mathematics Subject Classification.* 30D35.
Key words and phrases. eromorphic function, Uniqueness, Shared Set, Weighted sharing.
The first author is thankful to DST-PURSE programme for financial assistance.

two functions that share sets of distinct elements instead of values. Initially Gross and Yang proved that if f and g are two non-constant entire functions and S_1, S_2 and S_3 are three distinct finite sets such that $f^{-1}(S_i) = g^{-1}(S_i)$ for $i = 1, 2, 3,$ then $f \equiv g$. In [8] Gross posed the following question:

Question A. *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

In 2003, the following question was asked by Lin and Yi [18] which is also pertinent with that of Gross.

Question B. *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2$ must be identical ?*

During the last two decades, the main investigations on two set sharing problems of entire and meromorphic functions have been oriented on the basis of this two questions. Gradually the research in this direction has somehow been shifted to give explicitly a set S with n elements and make n as small as possible such that any two meromorphic functions f and g that share the value ∞ and the set S must be equal {cf.[1]-[7], [11], [15], [17]-[18], [20]-[23]}.

In this connection, we recall the following theorem of Yi [20].

Theorem A. [20] *Let $S = \{z : z^n + az^{n-m} + b = 0\}$ where n and m are two positive integers such that $m \geq 2$, $n \geq 2m + 7$ with n and m having no common factor, a and b be two nonzero constants such that $z^n + az^{n-m} + b = 0$ has no multiple root. If f and g are two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

In the same paper Yi [19] also asked the following question:
What can be said if $m = 1$ in *Theorem A*?
To answer this question Yi [20] proved the following theorem.

Theorem B. [20] *Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n(\geq 9)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g be two non-constant meromorphic functions such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then either $f \equiv g$ or $f \equiv \frac{-ah(h^{n-1}-1)}{h^n-1}$ and $g \equiv \frac{-a(h^{n-1}-1)}{h^n-1}$, where h is a non-constant meromorphic function.*

In 2001 the idea of gradation of sharing of values and sets known as weighted sharing has been introduced in [13, 14] which measures how close a shared value is to being shared IM or to being shared CM. We now give the definition.

Definition 1.1. [13, 14] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [13] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\cup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

The notion of weighted sharing of set has immense applications to deal with the *Questions A* and *B*. In particular there are many refinements and improvements of *Theorem B* {[2]-[4], [15]} using this notion. But in all the papers, to serve the purpose, the variations over different deficiency conditions have been taken under considerations.

In 1996, Yi [21] proved that the set S as defined in *Theorem A* is an URSM when $m \geq 2$ and $n \geq 2m+9$. Clearly in that case $S = \{z : z^{13} + z^{11} + 1 = 0\}$ is a URSM. So it would be natural to explore the analogous situation in the direction of *Question B*, corresponding to the set S as defined in *Theorem B* such that the uniqueness of meromorphic functions only depends on the sharing of the range sets in $\mathbb{C}$. The purpose of the paper is to find a suitable range set, together with S as defined in *Theorem B*, such that for the uniqueness of two non-constant meromorphic functions sharing two sets with finite weight, the conditions over deficiencies will no longer required. Following two theorems are the main results of the paper which will improve all the subsequent improvements of *Theorem B* in some sense.

Theorem 1.1. *Let* $S_1 = \{0, -a\frac{n-1}{n}\}$, $S_2 = \{z : z^n + az^{n-1} + b = 0\}$ *where* $n(\geq 7)$ *be an integer and* a *and* b *be two nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no multiple root. If* $E_f(S_1, 2) = E_g(S_1, 2)$, *and* $E_f(S_2, 3) = E_g(S_2, 3)$, *then* $f \equiv g$.

Theorem 1.2. *Let* S_i, $i = 1, 2$ *be given as in* Theorem 1.1 *where* $n(\geq 8)$ *be an integer. If* $E_f(S_1, m) = E_g(S_1, m)$, $E_f(S_2, p) = E_g(S_2, p)$, *then* $f \equiv g$, *where* $\frac{7}{2m} + \frac{m+1}{m(2p+1)} < 2$, *with* $\frac{7}{2m} + \frac{1}{m(2p+1)} > 1$.

Corollary 1.1. Theorem 1.2 *holds for the following pairs of least values of* p *and* m*: (i) p=1, m=3; (ii) p=3, m=2.*

Though for the standard definitions and notations of the value distribution theory we refer to [10], we now explain some notations which are used in the paper.

Definition 1.3. [12] For $a \in \mathbb{C} \cup \{\infty\}$we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Definition 1.4. [14] We denote by $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$.

Definition 1.5. [13, 14] Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and in particular if f and g share (a, p) then $\overline{N}_*(r, a; f, g) \leq \overline{N}(r, a; f \mid\geq p+1) = \overline{N}(r, a; g \mid\geq p+1)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$ as follows

$$F=\frac{f^{n-1}(f+a)}{-b},\quad G=\frac{g^{n-1}(g+a)}{-b}, \tag{2.1}$$

where a, b two nonzero constants defined as in *Theorem B*. Henceforth we shall denote by H and Φ the following two functions

$$H=\left(\frac{F^{''}}{F^{'}}-\frac{2F^{'}}{F-1}\right)-\left(\frac{G^{''}}{G^{'}}-\frac{2G^{'}}{G-1}\right) \tag{2.2}$$

and

$$\Phi=\frac{F^{'}}{F-1}-\frac{G^{'}}{G-1}. \tag{2.3}$$

Lemma 2.1. *([14], Lemma 1) Let F, G be two non-constant meromorphic functions sharing $(1,1)$ and $H\not\equiv 0$. Then*

$$N(r,1;F\mid=1)=N(r,1;G\mid=1)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.2. *Let S_1 and S_2 be defined as in* Theorem 1.1 *and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_f(S_1,p)=E_g(S_1,p)$, $E_f(S_2,0)=E_g(S_2,0)$, where $0\leq p<\infty$ and $H\not\equiv 0$ then*

$$\begin{aligned}N(r,H)\quad\leq\quad &\overline{N}(r,0;f\mid\geq p+1)+\overline{N}\left(r,-a\frac{n-1}{n};f\mid\geq p+1\right)+\overline{N}_*(r,1;F,G)\\ &+\overline{N}(r,\infty;f)++\overline{N}(r,\infty;g)+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'}),\end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ is the reduced counting function of those zeros of $f^{'}$ which are not the zeros of $f\left(f-a\frac{n-1}{n}\right)(F-1)$ and $\overline{N}_0(r,0;g^{'})$ is similarly defined.

Proof. We note that $F^{'}=\frac{f^{n-2}(nf+a(n-1))f^{'}}{-b}$, $G^{'}=\frac{g^{n-2}(ng+a(n-1))g^{'}}{-b}$ and

$$F^{''}=\frac{f^{n-2}(nf+a(n-1))f^{''}+f^{n-3}(n(n-1)f+a(n-1)(n-2))f^{'2}}{-b},$$

$$G^{''}=\frac{g^{n-2}(ng+a(n-1))g^{''}+g^{n-3}(n(n-1)g+a(n-1)(n-2))g^{'2}}{-b}.$$

So

$$\begin{aligned}H\quad=\quad&\frac{(n-1)(nf+a(n-2))f^{'}}{f(nf+a(n-1))}-\frac{(n-1)(ng+a(n-2))g^{'}}{g(ng+a(n-1))}\\ &+\frac{f^{''}}{f^{'}}-\frac{g^{''}}{g^{'}}-\left(\frac{2F^{'}}{F-1}-\frac{2G^{'}}{G-1}\right).\end{aligned}$$

Since $E_f(S_1,0)=E_g(S_1,0)$ it follows that if z_0 is a 0-point of f (g) then either $g(z_0)=0$ $(f(z_0)=0)$ or $g(z_0)=-a\frac{n-1}{n}$ $(f(z_0)=-a\frac{n-1}{n})$. Clearly F and G share $(1,0)$. Since H has only simple poles, the lemma can easily be proved by simple calculations. □

Lemma 2.3. [5] *Let f and g be two meromorphic functions sharing $(1,m)$, where $1 \leq m < \infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\\ \leq\ & \frac{1}{2}\left[N(r,1;f)+N(r,1;g)\right]\end{aligned}$$

Lemma 2.4. [19] *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{k=0}^{n}a_kf^k}{\sum\limits_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$ Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.5. *Let S_1 and S_2 be defined as in* Theorem 1.1 *with $n \geq 3$ and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_f(S_1,p)=E_g(S_1,p)$, $E_f(S_2,m)=E_g(S_2,m)$, $0 \leq p < \infty$ and $\Phi \not\equiv 0$ then*

$$\begin{aligned}&(2p+1)\left\{\overline{N}\left(r,0;f\mid\geq p+1\right)+\overline{N}\left(r,-a\frac{n-1}{n};f\mid\geq p+1\right)\right\}\\ \leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Proof. By the given condition clearly F and G share $(1,m)$. Also we see that

$$\Phi=\frac{f^{n-2}\left(nf+a(n-1)\right)f^{'}}{-b(F-1)}-\frac{f^{n-2}\left(nf+a(n-1)\right)f^{'}}{-b(G-1)}.$$

Let z_0 be a zero or a $-a\frac{n-1}{n}$- point of f with multiplicity r. Since $E_f(S_1,p)=E_g(S_1,p)$ then that would be a zero of Φ of multiplicity *min* $\{(n-2)r+r-1,\ r+r-1\}$ i.e., of multiplicity *min* $\{(n-1)r-1,\ 2r-1\}$ if $r \leq p$ and a zero of multiplicity at least *min*$\{(n-2)(p+1)+p,\ p+1+p\}$ i.e., a zero of multiplicity at least *min*$\{(n-1)p+(n-2),\ 2p+1\}$ if $r > p$. So using *Lemma 2.4* by a simple calculation we can write

$$\begin{aligned}&\min\{(n-1)p+(n-2),(2p+1)\}\left\{\overline{N}(r,0;f\mid\geq p+1)\right.\\ &\left.+\overline{N}(r,-a\frac{n-1}{n};f\mid\geq p+1)\right\}\\ \leq\ & N(r,0;\Phi)\\ \leq\ & T(r,\Phi)\\ \leq\ & N(r,\infty;\Phi)+S(r,F)+S(r,G)\\ \leq\ & \overline{N}_*(r,1;F,G)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f)+S(r,g).\end{aligned}$$

□

Lemma 2.6. *Let S_1, S_2 be defined as in* Theorem 1.1 *and F, G be given by (2.1). If for two non-constant meromorphic functions f and g $E_f(S_1,p) = E_g(S_1,p)$, $E_f(S_2,m) = E_g(S_2,m)$, where $0 \le p < \infty$, $2 \le m < \infty$ and $H \not\equiv 0$, then*

$$\begin{aligned}
& (n+1)\left\{T(r,f)+T(r,g)\right\} \\
\le\ & 2\left\{\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)\right\}+\overline{N}(r,0;f\mid\ge p+1) \\
& +\overline{N}\left(r,-a\frac{n-1}{n};f\mid\ge p+1\right)+2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\} \\
& +\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned}
(2.4)\qquad & (n+1)\{T(r,f)+T(r,g)\} \\
\le\ & \overline{N}(r,1;F)+\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G) \\
& +\overline{N}(r,0;g)+\overline{N}\left(r,-a\frac{n}{n-1};g\right)+\overline{N}(r,\infty;g)-N_0(r,0;f^{'}) \\
& -N_0(r,0;g^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Using *Lemmas 2.1*, *2.2*, *2.3* and *2.4* we note that

$$\begin{aligned}
(2.5)\qquad & \overline{N}(r,1;F)+\overline{N}(r,1;G) \\
\le\ & \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+N(r,1;F\mid=1)-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G) \\
\le\ & \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+\overline{N}(r,0;f\mid\ge p+1) \\
& +\overline{N}\left(r,-a\frac{n-1}{n};f\mid\ge p+1\right)+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G) \\
& +\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Using (2.5) in (2.4) and noting that $\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right) = \overline{N}(r,0;g)+\overline{N}\left(r,-a\frac{n-1}{n};g\right)$ the lemma follows. □

Lemma 2.7. *Let f, g be two non-constant meromorphic functions such that $E_f(\{0,-a\frac{n-1}{n}\},0) = E_g(\{0,-a\frac{n-1}{n}\},0)$ then $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$ implies $f \equiv g$, where $n\ (\ge 2)$ is an integer and a is a nonzero finite constant.*

Proof. Let

$$f^{n-1}(f+a) \equiv g^{n-1}(g+a)$$

and suppose $f \not\equiv g$. We consider two cases:

Case I Let $y = \frac{g}{f}$ be a constant. Then from (2.6) it follows that $y \ne 1$, $y^{n-1} \ne 1$, $y^n \ne 1$ and $f \equiv -a\frac{y^{n-1}-1}{y^n-1}$, a constant, which is impossible.

Case II Let $y = \frac{g}{f}$ be non-constant. Suppose none of 0 and $-a\frac{n-1}{n}$ is an exceptional value of Picard (e.v.P.) of f and g. Then from (2.6) we see that if z_0 is a 0 $(-a\frac{n-1}{n})$-point of f then that must be a 0 $(-a\frac{n-1}{n})$-point of g. That is f, g share $(0,\infty)$ (∞,∞). So y has no zero and pole. Again from (2.6) we observe that

$$f(y^n-1) \equiv -a(\ \ y^{n-1}-1).$$

Clearly $y \not\equiv 1$. So eliminating this common factor we are left with

$$f(y-\alpha_1)(y-\alpha_2)\ldots(y-\alpha_{n-1}) \equiv -a(y-\beta_1)(y-\beta_2)\ldots(y-\beta_{n-2}),$$

where $\alpha_j = exp\ (\frac{2j\pi i}{n})$ for $j = 1,2,\ldots,n-1$ and $\beta_k = exp\ (\frac{2k\pi i}{n-1})$ for $k = 1,2,\ldots,n-2$. Clearly none of the α_j's coincides with β_k's. First we observe that y can not omit any of the $2n-3$ distinct values α_j or β_k for $j = 1,2,\ldots,n-1$ and $k = 1,2,\ldots,n-2$, since otherwise y will have more than two Picard exceptional value, a contradiction.

So if z_0 is a point such that $y(z_0) = \alpha_j$, then we have $(y(z_0)-\beta_1)(y(z_0)-\beta_2)\ldots(y(z_0)-\beta_{n-2}) \equiv 0$, a contradiction. On the otherhand if z_1 is a point such that $y(z_1) = \frac{g(z_1)}{f(z_1)} = \beta_k \neq 0$, $k = 1,2,\ldots,n-2$, then we must have $f(z_1) = 0$ which is impossible as f and g share $(0,\infty)$.

If 0 is an e.v.P. or 0 and $-a\frac{n-1}{n}$ both are e.v.P. of f and g then by the same argument as above we can obtain a contradiction.

If $-a\frac{n-1}{n}$ is an e.v.P. of f and g, then we have from (2.7) that

$$\left(f + a\frac{(n-1)}{n}\right)\{n(y^n-1)\} \equiv a\{(n-1)y^n - ny^{n-1}+1\}.$$

If we assume $p(z) = (n-1)z^n - nz^{n-1} + 1$, then $p(0) \neq 0$ and $p(1) = p^{'}(1) = 0$. From above we see that $p(y)$ has $n-1$ distinct zeros none of which coincides with α_j, $j = 1,2,\ldots,n-1$. Then again by the same argument as above we have at last left with a point say z_2 such that $f(z_2) = -a\frac{n-1}{n}$, a contradiction.

Hence $f \equiv g$ and this proves the lemma. □

Lemma 2.8. *Let f, g be two non-constant meromorphic functions such that $E_f(\{0,-a\frac{n-1}{n}\},0) = E_g(\{0,-a\frac{n-1}{n}\},0)$ and suppose n (≥ 3) be an integer. Then*

$$f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2,$$

where a, b are finite nonzero constants.

Proof. If possible, let us suppose

$$f^{n-1}(f+a)g^{n-1}(g+a) \equiv b^2. \tag{2.6}$$

Let z_0 be a zero of f (g). Then z_0 must be either a 0-point or a $-a\frac{n-1}{n}$ point of g (f), which is impossible from (2.6). It follows that f (g) has no zero.

Next let z_0 be a zero of $f+a$ with multiplicity p. Then z_0 is a pole of g with multiplicity q such that $p = (n-1)q + q = nq \geq n$.

Since the poles of f are the zeros of $g+a$ only, we get

$$\overline{N}(r,\infty;f) \leq \overline{N}(r,-a;g) \leq \frac{1}{n}T(r,g).$$

By the second fundamental theorem we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,-a;f) + S(r,f) \\ &\leq \frac{1}{n}\ N(r,-a;f) + \frac{1}{n}T(r,g) + S(r,f) \\ &\leq \frac{1}{n}T(r,f) + \frac{1}{n}T(r,g) + S(r,f). \end{aligned}$$

i.e.,

$$(1-\frac{1}{n})\ T(r,f) \leq \frac{1}{n}\ T(r,g) + S(r,f).$$

Similarly

$$(1-\frac{1}{n})\,T(r,g)\leq\frac{1}{n}\,T(r,f)+S(r,g)$$

Adding (2.9) and (2.10) we get

$$(1-\frac{2}{n})\,\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

a contradiction for $n\geq 3$. This proves the lemma. □

Lemma 2.9. *Let F, G be given by (2.1) and they share $(1,m)$. Also let $\omega_1,\omega_2\ldots\omega_n$ are the members of the set $S_2=\{z:z^n+az^{n-1}+b=0\}$, where a, b are nonzero constants such that $z^n+az^{n-1}+b=0$ has no repeated root and n (≥ 3) is an integer. Then*

$$\overline{N}_*(r,1;F,G)\leq\frac{1}{m}\left[\overline{N}(r,0;f)+\overline{N}(r,-a\frac{n-1}{n};f)\right]+S(r,f).$$

Proof. First we note that since S_2 has distinct elements, $-a\frac{n-1}{n}$ can not be a member of S_2. So

$$\begin{aligned}
&\overline{N}_*(r,1;F,G)\\
\leq\;&\overline{N}(r,1;F\mid\geq m+1)\\
\leq\;&\frac{1}{m}\,\big(N(r,1;F)-\overline{N}(r,1;F)\big)\\
\leq\;&\frac{1}{m}\left[\sum_{j=1}^{n}\big(N(r,\omega_j;f)-\overline{N}(r,\omega_j;f)\big)\right]\\
\leq\;&\frac{1}{m}\left[N\left(r,0;f^{'}\mid f\neq 0,\;-a\frac{n-1}{n}\right)\right]]\\
\leq\;&\frac{1}{m}\left[\overline{N}\left(r,\infty;\frac{f(f+a\frac{n-1}{n})}{f'}\right)\right]\\
\leq\;&\frac{1}{m}\left[N\left(r,\infty;\frac{f^{'}}{f(f+a\frac{n-1}{n})}\right)\right]+S(r,f)\\
\leq\;&\frac{1}{m}\left[\overline{N}(r,0;f)+\overline{N}(r,-a\frac{n-1}{n};f)\right]+S(r,f)
\end{aligned}$$

□

Lemma 2.10. [2] *Let F, G be given by (2.1) where $n\geq 7$ is an integer. If $H\equiv 0$ then either $f^{n-1}(f+a)g^{n-1}(g+a)\equiv b^2$ or $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$*

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1). Then F and G share $(1,3)$. We consider the following cases.

Case 1. Suppose that $\Phi\not\equiv 0$.

Subcase 1.1 Let $H\not\equiv 0$. Then using *Lemma 2.6* for $m=3$ and $p=2$, *Lemma 2.5* for $p=0$, *Lemma 2.4* and *Lemma 2.9* for $m=3$ we obtain

$$\begin{aligned}
(3.1)\quad & (n+1)\left\{T(r,f)+T(r,g)\right\} \\
\leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)\right\}+2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\} \\
& +\overline{N}(r,0;f\mid\geq 3)+\overline{N}\left(r,-a\frac{n-1}{n};f\mid\geq 3\right)+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right] \\
& -\frac{3}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \\
\leq\ & (4+\frac{1}{5})\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right] \\
& +\frac{7}{60}\left\{\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)+\overline{N}(r,0;g)+\overline{N}\left(r,-a\frac{n-1}{n};g\right)\right\} \\
& +S(r,f)+S(r,g) \\
\leq\ & (\frac{n}{2}+4+\frac{13}{30})\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).
\end{aligned}$$

(3.1) gives a contradiction for $n\geq 7$.

Subcase 1.2 Let $H\equiv 0$. Now the conclusion of the theorem can be obtain from *Lemmas 2.10*, *2.8* and *2.7*.

Case 2. Suppose that $\Phi\equiv 0$. On integration we get $(F-1)\equiv A(G-1)$ for some non zero constant A. So in view of *Lemma 2.4* we have

$$T(r,f)=T(r,g)+O(1). \tag{3.2}$$

Since by the given condition of the theorem $E_f(S_1,0)=E_g(S_1,0)$ we consider the following cases.

Subcase 2.1. Let us first assume f and g share $(0,0)$ and $(-a\frac{n-1}{n},0)$. If one of 0 or $-a\frac{n-1}{n}$ is an e.v.P. of both f and g, then we get $A=1$ and we have $F\equiv G$, which in view of *Lemma 2.7* implies $f\equiv g$. Let both 0 and $-a\frac{n-1}{n}$ are e.v.P. of f as well as g then noting that here $F\equiv AG+(1-A)$, suppose $A\neq 1$. Using *Lemma 2.4*, (3.2) and the second fundamental theorem we get

$$\begin{aligned}
& nT(r,f) \\
\leq\ & \overline{N}(r,0;F)+\overline{N}(r,1-A;F)+\overline{N}(r,\infty;F)+S(r,F) \\
\leq\ & \overline{N}(r,0;f)+\overline{N}(r,-a;f)+\overline{N}(r,0;G)+\overline{N}(r,\infty;f)+S(r,f) \\
\leq\ & 2T(r,f)+T(r,g)+S(r,f) \\
\leq\ & 3T(r,f)+S(r,f),
\end{aligned}$$

which implies a contradiction since $n\geq 7$.

Subcase 2.2. Next suppose that there is at least one point z_0 such that $f(z_0)=0$ and $g(z_0)=-a\frac{n-1}{n}$. At the point z_0, we have $F(z_0)=0$ and $G(z_0)=\beta$ (say). So $A=\frac{1}{1-\beta}$. Putting this values we obtain from above

$$F\equiv\frac{1}{1-\beta}G+\frac{\beta}{\beta-1}.$$

If $\beta\neq 0$ then again noting that $\overline{N}(r,\frac{\beta}{\beta-1};F)=\overline{N}(r,0;G)$, we can again get a contradiction as above when $n\geq 7$. □

Proof of Theorem 1.2. Let F, G be given by (2.1). Then F and G share $(1,3)$. We consider the following cases.
Case 1. Suppose that $\Phi \not\equiv 0$.
Subcase 1.1. Let $H \not\equiv 0$. Then using *Lemma 2.6*, *Lemma 2.5* for $p=0$, *Lemma 2.4* and *Lemma 2.9* we obtain

$$
\begin{aligned}
(3.3)\quad & (n+1)\left\{T(r,f)+T(r,g)\right\} \\
\leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)\right\}+2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\} \\
& +\overline{N}(r,0;f\mid\geq p+1)+\overline{N}\left(r,-a\frac{n-1}{n};f\mid\geq p+1\right)+\frac{1}{2}\left[N(r,1;F)\right. \\
& \left.+N(r,1;G)\right]+\left(\frac{3}{2}-m\right)\overline{N}_{*}(r,1;F,G)+S(r,f)+S(r,g) \\
\leq\ & \left(4+\frac{1}{2p+1}\right)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right] \\
& +\left(\frac{7}{4m}+\frac{1}{2m(2p+1)}-\frac{1}{2}\right)\left\{\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)+\overline{N}(r,0;g)\right. \\
& \left.+\overline{N}\left(r,-a\frac{n-1}{n};g\right)\right\}+S(r,f)+S(r,g) \\
\leq\ & \left(\frac{n}{2}+3+\frac{7}{2m}+\frac{m+1}{m(2p+1)}\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).
\end{aligned}
$$

Since $\frac{7}{2m}+\frac{m+1}{m(2p+1)}<2$, with $\frac{7}{2m}+\frac{1}{m(2p+1)}>1$ and $n\geq 8$, (3.2) gives a contradiction.
We now omit the rest of the proof since the same is similar to that of *Theorem 1.1.* □

References

[1] A. Banerjee, *On the uniqueness of meromorphic functions that share two sets*, Georgian Math., Vol:15, No.1 (2008), 21-38.
[2] A. Banerjee, *Uniqueness of meromorphic functions sharing two sets with finite weight*, Portugal. Math. (N.S.), Vol:65, No.1 (2008), 81-93.
[3] A. Banerjee, *Some further results ona question of Yi*, Publ. De Inst. Math., Vol:92, No.6 (2012), 177-187.
[4] A. Banerjee, and S. Mukherjee, *Uniqueness of meromorphic functions sharing two or three sets*, Hokkaido Math. J., Vol:37, No.3 (2008), 507-530.
[5] A. Banerjee and P. Bhattacharajee, *Uniqueness and set sharing of derivatives of meromorphic functions*, Math. Slovaca, Vol:61, No.2 (2011), pp. 197-214.
[6] M. Fang and H.Guo, *On meromorphic functions sharing two values*, Analysis, Vol:17 (1997), 355-366.
[7] M. Fang and I. Lahiri, *Unique range set for certain meromorphic functions*, Indian J. Math., Vol:45, No.2 (2003), 141-150.
[8] F. Gross, *Factorization of meromorphic functions and some open problems*, Proc. Conf. Univ. Kentucky, Leixngton, Ky(1976); Lecture Notes in Math., 599 (1977), 51-69, Springer(Berlin).
[9] F. Gross and C. C. Yang, *On preimage and range sets of meromorphic functions*, Proc. Japan Acad., Vol:58 (1982), 17-20.
[10] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).
[11] I. Lahiri, *The range set of meromorphic derivatives*, Northeast. Math. J., Vol:14, No.3 (1998), 353-360.

[12] I. Lahiri, *Value distribution of certain differential polynomials*, Int. J. Math. Math. Sci., Vol:28, No.2 (2001), 83-91.
[13] I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., Vol:161 (2001), 193-206.
[14] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., Vol:46 (2001), 241-253.
[15] I. Lahiri, *On a question of Hong Xun Yi*, Arch. Math. (Brno), Vol:38 (2002), 119-128.
[16] P. Li and C. C. Yang, *Some further results on the unique range sets for meromorphic functions*, Kodai Math. J.,Vol:18 (1995), 437-450.
[17] P. Li and C. C. Yang, *On the unique range sets for meromorphic functions*, Proc. Amer. Math. Soc., Vol:124 (1996) 177-185.
[18] W. C. Lin and H. X. Yi, *Some further results on meromorphic functions that share two sets*, Kyungpook Math. J., Vol:43 (2003), 73-85.
[19] A. Z. Mohon'ko, *On the Nevanlinna characteristics of some meromorphic functions*, Theory of Functions. Funct. Anal. Appl., Vol:14 (1971), 83-87.
[20] H. X. Yi, *Unicity theorems for meromorphic or entire functions II*, Bull. Austral. Math. Soc., Vol:52 (1995), 215-224.
[21] H. X. Yi, *Unicity theorems for meromorphic or entire functions III*, Bull. Austral. Math. Soc.,Vol:53 (1996), 71-82.
[22] H. X. Yi and W. C. Lin, *Uniqueness of meromorphic functions and a question of Gross*, Kyungpook Math. J, Vol:46 (2006), 437-444.
[23] H. X. Yi and W. R. Lü, *Meromorphic functions that share two sets II*, Acta Math. Sci. Ser.B Engl. Ed., Vol:24 No. 1 (2004), 83-90.

Department of Mathematics, University of Kalyani, West Bengal 741235, India.
E-mail address: abanerjee_kal@yahoo.co.in, abanerjeekal@gmail.com