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ON PURE $\mathcal{LA}$-SEMIHYPERGROUPS

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Abstract. We generalize the existing theory of an associative structure [6] by studying it in a non-associative hyper-structure called an LA-semihypergroup. The results obtained will also generalize the results on LA-semigroup without hyper theory. As an application of our results we characterize $(0,2)$-hyperideals of an LA-semihypergroup H and prove that A is a $(0,2)$-hyperideal of H if and only if A is a left hyperideal of some left hyperideal of H. We also show that an LA-semihypergroup H is 0-$(0,2)$-bisimple if and only if H is right 0-simple. Finally we give the connection of ordered and hyper theories of an LA-semigroup.

1. Introduction

Hyperstructure theory was introduced in 1934, when F. Marty [9] defined hypergroups, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science and they are studied in many countries of the world. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. A lot of papers and several books have been written on hyperstructure theory, see [2]. Many authors studied different aspects of semihypergroups, for instance, Corsini et al. [1], Davvaz et al. [3], Hila et al. [5] and Leoreanu [8].

A left almost semigroup ($\mathcal{LA}$-semigroup) is a groupoid $\mathcal{S}$ whose elements satisfy the following left invertive law $(ab)c = (cb)a$ for all $a, b, c \in \mathcal{S}$.This concept was first given by Kazim and Naseeruddin in 1972 [7]. In an $\mathcal{LA}$-semigroup, the medial law [7] $(ab)(cd) = (ac)(bd)$ holds, $\forall\ a, b, c, d \in \mathcal{S}$. An $\mathcal{LA}$-semigroup may or may not contain a left identity. The left identity of an $\mathcal{LA}$-semigroup allow us to introduce the inverses of elements in an $\mathcal{LA}$-semigroup. If an $\mathcal{LA}$-semigroup contains a left

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identity, then it is unique [10]. In an $\mathcal{LA}$-semigroup $\mathcal{S}$ with left identity, the paramedial law $(ab)(cd) = (dc)(ba)$ holds, $\forall\ a, b, c, d \in \mathcal{S}$. By using medial law with left identity, we get $a(bc) = b(ac)$ for all $a, b, c \in \mathcal{S}$.

An $\mathcal{LA}$-semigroup is a non-associative and non-commutative algebraic structure mid way between a groupoid and a commutative semigroup. This structure is closely related to a commutative semigroup; indeed if an $\mathcal{LA}$-semigroup contains a right identity, then it becomes a commutative semigroup [10]. The connection between a commutative inverse semigroup and an $\mathcal{LA}$-semigroup was established by Yousafzai et al. in [18] as follows: a commutative inverse semigroup $(\mathcal{S}, .)$ becomes an $\mathcal{LA}$-semigroup $(\mathcal{S}, *)$ under $a * b = ba^{-1}r^{-1}$ for all $a, b, r \in \mathcal{S}$. An $\mathcal{LA}$-semigroup $\mathcal{S}$ with a left identity becomes a semigroup under the binary operation "$\circ_e$" defined as follows: $x \circ_e y = (xe)y$ for all $x, y \in \mathcal{S}$ [18]. There are lot of results which have been added to the theory of an $\mathcal{LA}$-semigroup by Mushtaq, Kamran, Holgate, Jezek, Protic, Madad, Yousafzai and many other researchers. An $\mathcal{LA}$-semigroup is a generalization of a semigroup [10] and it has vast applications in semigroups, as well as in other branches of mathematics. Yaqoob et al. [11, 12] and Yousafzai et al. [15, 16, 17] studied some aspects of fuzzy $\mathcal{LA}$-semigroups and fuzzy $\mathcal{AG}$-groupoids.

Recently, Hila et al. introduced the notion of $\mathcal{LA}$-semihypergroups [4]. They investigated several properties of hyperideals of $\mathcal{LA}$-semihypergroup and defined the topological space and study the topological structure of $\mathcal{LA}$-semihypergroups using hyperideal theory. In [13], Yaqoob et al. have characterized intra-regular $\mathcal{LA}$-semihypergroups by using the properties of their left and right hyperideals, and investigated some useful conditions for an $\mathcal{LA}$-semihypergroup to become an intra-regular $\mathcal{LA}$-semihypergroup. This non-associative hyper structure has been further explored by Yousafzai et al. in [19] and [21]. Yaqoob and Gulistan [14] defined partial order relations on $\mathcal{LA}$-semihypergroups.

In this paper, we discuss 0-minimal hyperideals and $(0, 2)$-hyperideals. We characterize an $\mathcal{LA}$-semihypergroup in terms of $(1, 2)$-hyperideals and show that a $(1, 2)$-hyperideal of a pure $\mathcal{LA}$-semihypergroup is a left hyperideal of some bi-hyperideal. We give the necessary and sufficient condition for an $\mathcal{LA}$-semihypergroup to become right 0-simple.

2. Preliminaries and examples

A map $\circ : \mathcal{H} \times \mathcal{H} \to \mathcal{P}^*(\mathcal{H})$ is called *hyperoperation* or *join operation* on the set $\mathcal{H}$, where $\mathcal{H}$ is a non-empty set and $\mathcal{P}^*(\mathcal{H}) = \mathcal{P}(\mathcal{H})\backslash\{\emptyset\}$ denotes the set of all non-empty subsets of $\mathcal{H}$. A hypergroupoid is a set $\mathcal{H}$ together with a (binary) hyperoperation.

Let A and B be two non-empty subsets of $\mathcal{H}$, then we denote

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, \quad a \circ A = \{a\} \circ A \ \text{ and } a \circ B = \{a\} \circ B.$$

A hypergroupoid $(\mathcal{H}, \circ)$ is called an $\mathcal{LA}$-semihypergroup [4] if $(x \circ y) \circ z = (z \circ y) \circ x$ holds for all $x, y, z \in \mathcal{H}$. The law is called a left invertive law. A hypergroupoid $(\mathcal{H}, \circ)$ is called a right almost semihypergroup ($\mathcal{RA}$-semihypergroup) if $x \circ (y \circ z) = z \circ (y \circ x)$ hold for all $x, y, z \in \mathcal{H}$. The law is called a right invertive law. A hypergroupoid $(\mathcal{H}, \circ)$ is called an almost semihypergroup ($\mathcal{A}$-semihypergroup) if it is both an $\mathcal{LA}$-semihypergroup and an $\mathcal{RA}$-semihypergroup.

Every $\mathcal{LA}$-semihypergroup satisfies the law $(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w)$ for all $w, x, y, z \in \mathcal{H}$. This law is known as medial law (cf. [4]).

Let $\mathcal{H}$ be an $\mathcal{LA}$-semihypergroup [13], then an element $e \in \mathcal{H}$ is called

(i) a left identity (resp. pure left identity) if for all $a \in \mathcal{H}$, $a \in e \circ a$ (resp. $a = e \circ a$),

(ii) a right identity (resp. pure right identity) if for all $a \in \mathcal{H}$, $a \in a \circ e$ (resp. $a = a \circ e$),

(iii) an identity (resp. pure identity) if for all $a \in \mathcal{H}$, $a \in e \circ a \cap a \circ e$ (resp. $a = e \circ a \cap a \circ e$).

We have shown in [19] that unlike an $\mathcal{LA}$-semigroup, an $\mathcal{LA}$-semihypergroup may have a right identity or an identity. This fact can lead us to the following major remark.

Remark 2.1. The right identity of an $\mathcal{LA}$-semihypergroup need not to be a left identity in general. An $\mathcal{LA}$-semihypergroup may have a left identity or a right identity or an identity. Moreover, an $\mathcal{LA}$-semihypergroup with a right identity need not to be associative.

However an $\mathcal{LA}$-semihypergroup with pure right identity becomes a commutative semigroup with identity [19].

An $\mathcal{LA}$-semihypergroup with pure left identity e is called a *pure $\mathcal{LA}$-semihypergroup.* A pure $\mathcal{LA}$-semihypergroup $(\mathcal{H}, \circ)$ satisfy the following laws for all $w, x, y, z \in \mathcal{H}$:

$$(x \circ y) \circ (z \circ w) = (w \circ z) \circ (y \circ x),$$

called a paramedial law, and

$$x \circ (y \circ z) = y \circ (x \circ z).$$

Example 2.1. Let $(\mathcal{H}, \circ)$ be an $\mathcal{LA}$-semihypergroup with pure left identity e. Define a binary hyperoperation $\hat{o}$ (e-sandwich hyperoperation) as follows:

$$a \ \hat{o} \ b = (a \circ e) \circ b \text{ for all } a, b \in \mathcal{H}.$$

Then $(\mathcal{H}, \hat{o})$ becomes a commutative semihypergroup with pure identity.

Example 2.2. An $\mathcal{A}$-semihypergroup $\mathcal{H}$ with pure left identity becomes an abelian hypergroup.

If there is an element 0 of an $\mathcal{LA}$-semihypergroup $(H, \circ)$ such that $x \circ 0 = 0 \circ x = x$ $\forall \ x \in H$, we call 0 a *zero element* of H.

Example 2.3. Let us consider the following table for $H = \{a, b, c, d, e\}$ with a pure left identity d. It is easy to see that $(H, \circ)$ is a pure unitary $\mathcal{LA}$-semigroup with a zero element a.

$\circ$	a	b	c	d	e
a	a	a	a	a	a
b	a	$\{a,e\}$	$\{a,e\}$	$\{a,c\}$	$\{a,e\}$
c	a	$\{a,e\}$	$\{a,e\}$	$\{a,b\}$	$\{a,e\}$
d	a	b	c	d	e
e	a	$\{a,e\}$	$\{a,e\}$	$\{a,e\}$	$\{a,e\}$

A subset A of an $\mathcal{LA}$-semihypergroup H is called a *left* (*right*) hyperideal of H if $H \circ A \subseteq A$ ($A \circ H \subseteq A$), and is called a *hyperideal* of H if it is both left and right hyperideal of H. A subset A of an $\mathcal{LA}$-semihypergroup H is called an $\mathcal{LA}$-*subsemihypergroup* of H if $A^2 \subseteq A$. A hyperideal A of an $\mathcal{LA}$-semihypergroup H with zero is said to be 0-*minimal* if $A \neq \{0\}$ and $\{0\}$ is the only hyperideal of H properly contained in A. An $\mathcal{LA}$-semihypergroup H with zero is said to be 0-$(0,2)$-*bisimple* if $H^2 \neq \{0\}$ and $\{0\}$ is the only proper $(0,2)$-bi-hyperideal of H.

3. 0-minimal $(0,2)$-bi-hyperideals in pure $\mathcal{LA}$-semihypergroups

If H is a pure $\mathcal{LA}$-semihypergroup, then it is easy to see that $H \circ H = H$, $H \circ A^2 = A^2 \circ H$ and $A \subseteq H \circ A \ \forall\ A \subseteq H$. Note that every right hyperideal of a pure $\mathcal{LA}$-semihypergroup H is a left hyperideal of H but the converse is not true in general. Example 2.3 shows that there exists a subset $\{a, b, e\}$ of H which is a left hyperideal of H but not a right hyperideal of H. It is easy to see that $H \circ A$ and $H \circ A^2$ are the left and right hyperideals of a pure $\mathcal{LA}$-semihypergroup H. Thus $H \circ A^2$ is a hyperideal of a pure $\mathcal{LA}$-semihypergroup H.

Lemma 3.1. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then A is a $(0,2)$-hyperideal of H if and only if A is a hyperideal of some left hyperideal of H.*

Proof. Let A be a $(0,2)$-hyperideal of H, then

$$(H \circ A) \circ A = (A \circ A) \circ H = H \circ A^2 \subseteq A,$$

and

$$A \circ (H \circ A) = H \circ (A \circ A) = (H \circ H) \circ (A \circ A) = H \circ A^2 \subseteq A.$$

Hence A is a hyperideal of a left hyperideal $H \circ A$ of H.

Conversely, assume that A is a left hyperideal of a left hyperideal L of H, then

$$H \circ A^2 = (A \circ A) \circ H = (H \circ A) \circ A \subseteq (H \circ L) \circ A \subseteq L \circ A \subseteq A,$$

and clearly A is an $\mathcal{LA}$-subsemihypergroup of H, therefore A is a $(0,2)$-hyperideal of H. □

Corollary 3.1. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then A is a $(0,2)$-hyperideal of H if and only if A is a left hyperideal of some left hyperideal of H.*

Lemma 3.2. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then A is a $(0,2)$-bi-hyperideal of H if and only if A is a hyperideal of some right hyperideal of H.*

Proof. Let A be a $(0,2)$-bi-hyperideal of H, then

$$(H \circ A^2) \circ A = (A^2 \circ H) \circ A = (A \circ H) \circ A^2 \subseteq H \circ A^2 \subseteq A,$$

and

$$\begin{aligned} A \circ (H \circ A^2) &= (H \circ H) \circ (A \circ A^2) = (A^2 \circ A) \circ (H \circ H) \\ &= (H \circ A) \circ A^2 \subseteq H \circ A^2 \subseteq A. \end{aligned}$$

Hence A is a hyperideal of some right hyperideal $H \circ A^2$ of H.

Conversely, assume that A is a hyperideal of a right hyperideal R of H, then

$$\begin{aligned} H \circ A^2 &= A \circ (H \circ A) = A \circ ((H \circ H) \circ A) = A \circ ((A \circ H) \circ H) \\ &\subseteq A \circ ((R \circ H) \circ R) \subseteq A \circ R \subseteq A, \end{aligned}$$

and

$$(A \circ H) \circ A \subseteq (R \circ H) \circ A \subseteq R \circ A \subseteq A,$$

which shows that A is a $(0,2)$-hyperideal of H. □

Theorem 3.1. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then the following statements are equivalent.*

(i) *A is a $(1,2)$-hyperideal of H;*
(ii) *A is a left hyperideal of some bi-hyperideal of H;*
(iii) *A is a bi-hyperideal of some hyperideal of H;*
(iv) *A is a $(0,2)$-hyperideal of some right hyperideal of H;*
(v) *A is a left hyperideal of some $(0,2)$-hyperideal of H.*

Proof. $(i) \Longrightarrow (ii)$. It is easy to see that $(H \circ A^2) \circ H$ is a bi-hyperideal of H. Let A be a $(1,2)$-hyperideal of H, then

$$\begin{aligned}\{(H \circ A^2) \circ H\} \circ A &= \{(H \circ A^2) \circ (H \circ H)\} \circ A = \{(H \circ H) \circ (A^2 \circ H)\} \circ A \\ &= \{H \circ (A^2 \circ H)\} \circ A = (A^2 \circ H) \circ A \\ &= (A \circ H) \circ A^2 \subseteq A,\end{aligned}$$

which shows that A is a left hyperideal of a bi-hyperideal $(H \circ A^2) \circ H$ of H.

$(ii) \Longrightarrow (iii)$. Let A be a left hyperideal of a bi-hyperideal B of H, then

$$\begin{aligned}\{A \circ (H \circ A^2)\} \circ A &= \{H \circ (A \circ A^2)\} \circ A \\ &\subseteq [H \circ \{(H \circ A) \circ (A \circ A)\}] \circ A \\ &= [H \circ \{(A \circ A) \circ (A \circ H)\}] \circ A \\ &= [(A \circ A) \circ \{H \circ (A \circ H)\}] \circ A \\ &= [\{(H \circ (A \circ H)) \circ A\} \circ A] \circ A \\ &= [\{(A \circ H) \circ A\} \circ A] \circ A \\ &\subseteq [\{(B \circ H) \circ B\} \circ A] \circ A \\ &\subseteq (B \circ A) \circ A \subseteq A,\end{aligned}$$

which shows that A is a bi-hyperideal of a hyperideal $H \circ A^2$ of H.

$(iii) \Longrightarrow (iv)$. Let A be a bi-hyperideal of a hyperideal I of H, then

$$\begin{aligned}(H \circ A^2) \circ A^2 &= \{A^2 \circ (A \circ A)\} \circ H = \{A \circ (A^2 \circ A)\} \circ H \\ &\subseteq [A \circ \{(A \circ I) \circ A\}] \circ H = (A \circ A) \circ H \\ &= (H \circ A) \circ A \subseteq (H \circ I) \circ H \subseteq I,\end{aligned}$$

which shows that A is a $(0,2)$-hyperideal of a right hyperideal $H \circ A^2$ of H.

$(iv) \Longrightarrow (v)$. It is easy to see that $H \circ A^3$ is a $(0,2)$-hyperideal of H. Let A be a $(0,2)$-hyperideal of a right hyperideal R of H, then

$$\begin{aligned}A \circ (H \circ A^3) &= A \circ \{(H \circ H) \circ (A^2 \circ A)\} \\ &= A \circ \{(A \circ A^2) \circ H\} \\ &\subseteq A \circ [\{(H \circ A) \circ (A \circ A)\} \circ H] \\ &= A \circ [\{(A \circ A) \circ (A \circ H)\} \circ H] \\ &= (A \circ A) \circ [\{A \circ (A \circ H)\} \circ H] \\ &= [H \circ \{A \circ (A \circ H)\}] \circ A^2 \\ &= [A \circ \{H \circ (A \circ H)\}] \circ A^2 \\ &\subseteq (R \circ H) \circ A^2 \subseteq R \circ A^2 \subseteq A,\end{aligned}$$

which shows that A is a left hyperideal of a $(0,2)$-hyperideal $H \circ A^3$ of H.

$(v) \Longrightarrow (i)$. Let A be a left hyperideal of a $(0,2)$-hyperideal O of H, then

$$\begin{aligned}(A \circ H) \circ A^2 &= \{(A \circ A) \circ (H \circ H)\} \circ A = (H \circ A^2) \circ A \\ &\subseteq (H \circ O^2) \circ A \subseteq O \circ A \subseteq A,\end{aligned}$$

which shows that A is a $(1,2)$-hyperideal of H. □

Lemma 3.3. *Let H be a pure $\mathcal{LA}$-semihypergroup and A be an idempotent subset of H. Then A is a $(1,2)$-hyperideal of H if and only if there exist a left hyperideal L and a right hyperideal R of H such that $R \circ L \subseteq A \subseteq R \cap L$.*

Proof. Assume that A is a $(1,2)$-hyperideal of H such that A is idempotent. Setting $L = H \circ A$ and $R = H \circ A^2$, then

$$\begin{aligned}R \circ L &= (H \circ A^2) \circ (H \circ A) \\ &= (A^2 \circ H) \circ (H \circ A) \\ &= \{(H \circ A) \circ (H \circ H)\} \circ A^2 \\ &= \{(H \circ H) \circ (A \circ H)\} \circ A^2 \\ &= [H \circ \{(A \circ A) \circ (H \circ H)\}] \circ A^2 \\ &= [H \circ \{(H \circ H) \circ (A \circ A)\}] \circ A^2 \\ &= [H \circ [A \circ \{(H \circ H) \circ A\}]] \circ A^2 \\ &= [A \circ \{H \circ (H \circ A)\}] \circ A^2 \\ &\subseteq (A \circ H) \circ A^2 \subseteq A.\end{aligned}$$

It is clear that $A \subseteq R \cap L$.

Conversely, let R be a right hyperideal and L be a left hyperideal of H such that $R \circ L \subseteq A \subseteq R \cap L$, then

$$(A \circ H) \circ A^2 = (A \circ H) \circ (A \circ A) \subseteq (R \circ H) \circ (H \circ L) \subseteq R \circ L \subseteq A.$$

□

Assume that H is a pure unitary $\mathcal{LA}$-semihypergroup with zero. Then it is easy to see that every left (right) hyperideal of H is a $(0,2)$-hyperideal of H. Hence if O is a 0-minimal $(0,2)$-hyperideal of H and A is a left (right) hyperideal of H contained in O, then either $A = \{0\}$ or $A = O$.

Lemma 3.4. *Let H be a pure $\mathcal{LA}$-semihypergroup with zero. Assume that A is a 0-minimal hyperideal of H and O is an $\mathcal{LA}$-subsemihypergroup of A. Then O is a $(0,2)$-hyperideal of H contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

Proof. Let O be a $(0,2)$-hyperideal of H contained in a 0-minimal hyperideal A of H. Then $H \circ O^2 \subseteq O \subseteq A$. Since $H \circ O^2$ is a hyperideal of H, therefore by minimality of A, $H \circ O^2 = \{0\}$ or $H \circ O^2 = A$. If $H \circ O^2 = A$, then $A = H \circ O^2 \subseteq O$ and therefore $O = A$. Let $H \circ O^2 = \{0\}$, then

$$O^2 \circ H = H \circ O^2 = \{0\} \subseteq O^2,$$

which shows that O^2 is a right hyperideal of H, and hence a hyperideal of H contained in A, therefore by minimality of A, we have $O^2 = \{0\}$ or $O^2 = A$. Now if $O^2 = A$, then $O = A$.

Conversely, let $O^2 = \{0\}$, then

$$H \circ O^2 = O^2 \circ H = \{0\} \circ H = \{0\} = O^2.$$

Now if $O = A$, then

$$H \circ O^2 = (H \circ H) \circ (O \circ O) = (H \circ A) \circ (H \circ A) \subseteq A = O,$$

which shows that O is a $(0,2)$-hyperideal of H contained in A. □

Corollary 3.2. *Let H be a pure $\mathcal{LA}$-semihypergroup with zero. Assume that A is a 0-minimal left hyperideal of H and O is an $\mathcal{LA}$-subsemihypergroup of A. Then O is a $(0,2)$-hyperideal of H contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

Lemma 3.5. *Let H be a pure $\mathcal{LA}$-semihypergroup with zero and O be a 0-minimal $(0,2)$-hyperideal of H. Then $O^2 = \{0\}$ or O is a 0-minimal right (left) hyperideal of H.*

Proof. Let O be a 0-minimal $(0,2)$-hyperideal of H, then

$$\begin{aligned} H \circ (O^2)^2 &= (H \circ H) \circ (O^2 \circ O^2) = (O^2 \circ O^2) \circ H \\ &= (H \circ O^2) \circ O^2 \subseteq O \circ O^2 \subseteq O^2, \end{aligned}$$

which shows that O^2 is a $(0,2)$-hyperideal of H contained in O, therefore by minimality of O, $O^2 = \{0\}$ or $O^2 = O$. Suppose that $O^2 = O$, then $O \circ H = (O \circ O) \circ (H \circ H) = H \circ O^2 \subseteq O$, which shows that O is a right hyperideal of H. Let R be a right hyperideal of H contained in O, then $R^2 \circ H = (R \circ R) \circ H \subseteq (R \circ H) \circ H \subseteq R$. Thus R is a $(0,2)$-hyperideal of H contained in O, and again by minimality of O, $R = \{0\}$ or $R = O$. □

The following Corollary follows from Lemma 3.4 and Corollary 3.2.

Corollary 3.3. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then O is a minimal $(0,2)$-hyperideal of H if and only if O is a minimal left hyperideal of H.*

Theorem 3.2. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then A is a minimal $(2,1)$-hyperideal of H if and only if A is a minimal bi-hyperideal of H.*

Proof. Let A be a minimal $(2,1)$-hyperideal of H. Then

$$\begin{aligned} [\{(A^2 \circ H) \circ A\}^2 \circ H] \circ \{(A^2 \circ H) \circ A\} &= [[\{(A^2 \circ H) \circ A\} \circ \{(A^2 \circ H) \circ A\}] \circ H] \\ &\quad \circ\{(A^2 \circ H) \circ A\} \\ &\subseteq [[\{(A \circ H) \circ A\} \circ \{(A \circ H) \circ A\}] \circ H] \\ &\quad \circ\{(A \circ H) \circ A\} \\ &= [[\{(A \circ H) \circ (A \circ H)\} \circ (A \circ A)] \circ H] \\ &\quad \circ\{(A \circ H) \circ A\} \\ &= [\{(A^2 \circ H) \circ (A \circ A)\} \circ H] \circ \{(A \circ H) \circ A\} \\ &\subseteq [\{(A \circ H) \circ (A \circ H)\} \circ H] \circ \{(A \circ H) \circ A\} \\ &= \{(A^2 \circ H) \circ H\} \circ \{(A \circ H) \circ A\} \\ &\subseteq \{(A \circ H) \circ H\} \circ \{(A \circ H) \circ A\} \\ &= \{(A \circ H) \circ (A \circ H)\}(H \circ A) \\ &= (A^2 \circ H) \circ (H \circ A) = (A \circ H) \circ (H \circ A^2) \\ &= \{(H \circ A^2) \circ H\} \circ A = \{(A^2 \circ H) \circ H\} \circ A \\ &= \{(H \circ H) \circ (A \circ A)\} \circ A = (A^2 \circ H) \circ A, \end{aligned}$$

and similarly we can show that $\{(A^2\circ H)\circ A\}^2 \subseteq (A^2\circ H)\circ A$. Thus $(A^2\circ H)\circ A$ is a $(2,1)$-hyperideal of H contained in A, therefore by minimality of A, $(A^2\circ H)\circ A = A$. Now

$$\begin{aligned}(A\circ H)\circ A &= (A\circ H)\circ\{(A^2\circ H)\circ A\} = [\{(A^2\circ H)\circ A\}\circ H]\circ A\\ &= \{(H\circ A)\circ(A^2\circ H)\}\circ A = [A^2\circ\{(H\circ A)\circ H\}]\circ A\\ &\subseteq (A^2\circ H)\circ A = A,\end{aligned}$$

It follows that A is a bi-hyperideal of H. Suppose that there exists a bi-hyperideal B of H contained in A, then $(B^2\circ H)\circ B \subseteq (B\circ H)\circ B \subseteq B$, so B is a $(2,1)$-hyperideal of H contained in A, therefore $B = A$.

Conversely, assume that A is a minimal bi-hyperideal of H, then it is easy to see that A is a $(2,1)$-hyperideal of H. Let C be a $(2,1)$-hyperideal of H contained in A, then

$$\begin{aligned}[\{(C^2\circ H)\circ C\}\circ H]\circ\{(C^2\circ H)\circ C\} &= \{(H\circ C)\circ(C^2\circ H)\}\circ\{(C^2\circ H)\circ C\}\\ &= \{(H\circ C^2)\circ(C\circ H)\}\circ\{(C^2\circ H)\circ C\}\\ &= [C\circ\{(H\circ C^2)\circ H\}]\circ\{(C^2\circ H)\circ C\}\\ &= [\{(C^2\circ H)\circ C\}\circ\{(H\circ C^2)\\ &\quad \circ(H\circ H)\}]\circ C\\ &= [\{(C^2\circ H)\circ C\}\circ\{H\circ(C^2\circ H)\}]\circ C\\ &= [\{(C^2\circ H)\circ C\}\circ(C^2\circ H)]\circ C\\ &= [C^2\circ[\{(C^2\circ H)\circ C\}\circ H]]\circ C\\ &\subseteq (C^2\circ H)\circ C.\end{aligned}$$

This shows that $(C^2\circ H)\circ C$ is a bi-hyperideal of H, and by minimality of A, $(C^2\circ H)\circ C = A$. Thus $A = (C^2\circ H)\circ C \subseteq C$, and therefore A is a minimal $(2,1)$-hyperideal of H. □

Theorem 3.3. *Let A be a 0-minimal $(0,2)$-bi-hyperideal of a pure $\mathcal{LA}$-semihypergroup H with zero. Then exactly one of the following cases occurs*:

(*i*) $A=\{0,a\}$, $a^2=0$;
(*ii*) $\forall\ a\in A\backslash\{0\}$, $H\circ\circ a^2 = A$.

Proof. Assume that A is a 0-minimal $(0,2)$-bi-hyperideal of H. Let $a\in A\backslash\{0\}$, then $H\circ a^2\subseteq A$. Also $H\circ a^2$ is a $(0,2)$-bi-hyperideal of H, therefore $H\circ a^2=\{0\}$ or $H\circ a^2 = A$.

Let $H\circ a^2=\{0\}$. Since $a^2\subseteq A$, we have either $a^2=a$ or $a^2=0$ or $a^2\subseteq A\backslash\{0,a\}$. If $a^2=a$, then $a^3=a^2\circ a=a$, which is impossible because $a^3\subseteq a^2\circ H = H\circ a^2 = \{0\}$. Let $a^2\subseteq A\backslash\{0,a\}$, we have

$$H\circ[\{0,a^2\}\{0,a^2\}] = (H\circ H)\circ(a^2\circ a^2) = (H\circ a^2)\circ(H\circ a^2) = \{0\}\subseteq\{0,a^2\},$$

and

$$[\{0,a^2\}H]\{0,a^2\} = \{0,a^2H\}\{0,a^2\} = a^2H\cdot a^2\subseteq Ha^2 = \{0\}\subseteq\{0,a^2\}.$$

Therefore $\{0,a^2\}$ is a $(0,2)$-bi-hyperideal of H contained in A. We observe that $\{0,a^2\}\neq\{0\}$ and $\{0,a^2\}\neq A$. This is a contradiction to the fact that A is a 0-minimal $(0,2)$-bi-hyperideal of H. Therefore $a^2=0$ and $A=\{0,a\}$.

If $H\circ a^2\neq\{0\}$, then $H\circ a^2 = A$. □

Corollary 3.4. *Let A be a 0-minimal $(0,2)$-bi-hyperideal of a pure $\mathcal{LA}$-semihypergroup H with zero such that $A^2 \neq 0$. Then $A = H \circ a^2$ for every $a \in A \backslash \{0\}$.*

Lemma 3.6. *Let H be a pure $\mathcal{LA}$-semihypergroup. Then every right hyperideal of H is a $(0,2)$-bi-hyperideal of H.*

Proof. Assume that A is a right hyperideal of H, then

$$H \circ A^2 = (A \circ A) \circ (H \circ H) = (A \circ H) \circ (A \circ H) \subseteq A \circ A \subseteq A \circ H \subseteq A, \ (A \circ H) \circ A \subseteq A,$$

and clearly $A^2 \subseteq A$, therefore A is a $(0,2)$-bi-hyperideal of H. □

The converse of Lemma 3.6 is not true in general. Example 2.3 shows that there exists a $(0,2)$-bi-hyperideal $A = \{a, c, e\}$ of H which is not a right hyperideal of H.

Theorem 3.4. *Let H be a pure $\mathcal{LA}$-semihypergroup with zero. Then $H \circ a^2 = H$ $\forall\ a \in H \backslash \{0\}$ if and only if H is 0-$(0,2)$-bisimple if and only if H is right 0-simple.*

Proof. Assume that $H \circ a^2 = H$ for every $a \in H \backslash \{0\}$. Let A be a $(0,2)$-bi-hyperideal of H such that $A \neq \{0\}$. Let $a \in A \backslash \{0\}$, then $H = H \circ a^2 \subseteq H \circ A^2 \subseteq A$. Therefore $H = A$. Since $H = H \circ a^2 \subseteq H \circ H = H^2$, we have $H^2 = H \neq \{0\}$. Thus H is 0-$(0,2)$-bisimple. The converse statement follows from Corollary 3.4.

Let R be a right hyperideal of 0-$(0,2)$-bisimple H. Then by Lemma 3.6, R is a $(0,2)$-bi-hyperideal of H and so $R = \{0\}$ or $R = H$.

Conversely, assume that H is right 0-simple. Let $a \in H \backslash \{0\}$, then $H \circ a^2 = H$. Hence H is 0-$(0,2)$-bisimple. □

Theorem 3.5. *Let A be a 0-minimal $(0,2)$-bi-hyperideal of a pure $\mathcal{LA}$-semihypergroup H with zero. Then either $A^2 = \{0\}$ or A is right 0-simple.*

Proof. Assume that A is 0-minimal $(0,2)$-bi-hyperideal of H such that $A^2 \neq \{0\}$. Then by using Corollary 3.4, $H \circ a^2 = A$ for every $a \in A \backslash \{0\}$. Since $a^2 \subseteq A \backslash \{0\}$ for every $a \in A \backslash \{0\}$, we have $a^4 = (a^2)^2 \subseteq A \backslash \{0\}$ for every $a \in A \backslash \{0\}$. Let $a \in A \backslash \{0\}$, then

$$\begin{aligned} \{(A \circ a^2) \circ H\} \circ (A \circ a^2) &= (a^2 \circ A) \circ \{H \circ (A \circ a^2)\} = [\{H \circ (A \circ a^2)\} \circ A] \circ a^2 \\ &\subseteq \{(H \circ A) \circ A\} \circ a^2 = \{(A \circ A) \circ (H \circ H)\} \circ a^2 \\ &= (H \circ A^2) \circ a^2 \subseteq A \circ a^2, \end{aligned}$$

and

$$\begin{aligned} H \circ (A \circ a^2)^2 &= H \circ \{(A \circ a^2) \circ (A \circ a^2)\} = H \circ \{(a^2 \circ A) \circ (a^2 \circ A)\} \\ &= H \circ [a^2 \circ \{(a^2 \circ A) \circ A\}] = (a \circ a) \circ [H \circ \{(a^2 \circ A) \circ A\}] \\ &= [\{(a^2 \circ A) \circ A\} \circ H] \circ a^2 \subseteq \{(A \circ A) \circ (H \circ H)\} \circ a^2 \\ &= (H \circ A^2) \circ a^2 \subseteq A \circ a^2, \end{aligned}$$

which shows that $A \circ a^2$ is a $(0,2)$-bi-hyperideal of H contained in A. Hence $A \circ a^2 = \{0\}$ or $A \circ a^2 = A$. Since $a^4 \subseteq A \circ a^2$ and $a^4 \subseteq A \backslash \{0\}$, we get $A \circ a^2 = A$. Thus by using Theorem 3.4, A is right 0-simple. □

4. Construction of $\mathcal{LA}$-semihypergroups

Let $(H,\cdot,\leq)$ be any ordered $\mathcal{LA}$-semigroup [20]. Define a hyperoperation $\circ$ on H by:

$$x\circ y=\{z\in H : z\leq xy\}=(xy] \text{ for all } x,y\in H.$$

Then for all $a,b,c\in H$, we claim that $(a\circ b)\circ c=((ab)c]$. Let $x\in(a\circ b)\circ c$, then $x\in y\circ c$ for some $y\in a\circ b$, which shows that $x\leq yc$ and $y\leq ab$. Hence $x\leq(ab)c$, so $(a\circ b)\circ c\subseteq((ab)c]$. Let $x\in((ab)c]$, then $x\leq(ab)c$. Thus $x\in ab\circ c\subseteq \bigcup\limits_{y\in a\circ b} y\circ c=(a\circ b)\circ c$. Consequently $(a\circ b)\circ c=((ab)c]$. Similarly we can show that $(c\circ b)\circ a=((cb)a]$, which shows that $(a\circ b)\circ c=(c\circ b)\circ a$ for all $a,b,c\in H$. Thus $(H,\circ)$ becomes an $\mathcal{LA}$-semihypergroup.

Let us consider an ordered $\mathcal{LA}$-semigroup $H=\{a,b,c\}$ in the following multiplication table and ordered below:

$\cdot$	a	b	c
a	b	c	a
b	a	b	c
c	c	a	b

$$\leq:=\{(a,a),(b,b),(c,c),(a,b),(b,c),(a,c)\}.$$

The hyperoperation $\circ$ is defined in the following table.

$\circ$	a	b	c
a	$\{a,b\}$	$\{a,b,c\}$	a
b	a	$\{a,b\}$	$\{a,b,c\}$
c	$\{a,b,c\}$	a	$\{a,b\}$

Then $(H,\circ)$ is an $\mathcal{LA}$-semihypergroup because $(a\circ b)\circ c=(c\circ b)\circ a$ for all $a,b,c\in H$ and $(c\circ b)\circ c\neq c\circ(b\circ c)$ for $b,c\in H$.

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