



ON THE DIOPHANTINE EQUATION $3^x + 5^y + 7^z = w^2$

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ABSTRACT. We exhaust all solutions of the Diophantine equation

$$3^x + 5^y + 7^z = w^2$$

in non-negative integers using elementary methods.

1. INTRODUCTION

One of the many interesting equations that are being studied by number theorists is the class of Diophantine equations. Diophantine equations are usually polynomial equations in two or more variables and mathematicians are searching only for integer solutions. The simplest type of such is the so-called linear Diophantine equation, which is of the form

$$(1.1) \quad ax + by = 1,$$

where x and y are unknowns while a and b are constants. An equation that can not be transformed into (1.1) is usually referred to as nonlinear diophantine equation. Pell's equation is an example, which is of the form $x^2 - by^2 = 1$, where b is not a perfect square integer, and we are searching for the integer values of x and y that satisfy the equation. Another example is the equation

$$(1.2) \quad x^n + y^n = z^n,$$

which has infinitely many integer solutions (x, y, z) if $n = 2$ but no integer solutions whenever $n > 2$. Then there is the class of the so-called exponential Diophantine equation which is formed when one or more exponents serve as unknowns as well. An example is the equation

$$(1.3) \quad a^x - b^y = 1,$$

where a, b, x , and y are all positive integers greater than 1. In 1844, Charles Catalan conjectured that the only solution to (1.3) is the 4-tuple $(a, b, x, y) = (3, 2, 2, 3)$ and

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this was finally proven in 2002 by Preda Mihăilescu (cf. [6]). In 2007, D. Acu [1] showed that the Diophantine equation $2^x + 5^y = z^2$ has exactly two solutions in non-negative integers. In 2011, A. Suvarnamani, A. Singta, and S. Chotchaisthit [2] used P. Mihăilescu's theorem (Catalan's conjecture) to show that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solution in non-negative integers. In [3], S. Chotchaisthit studied the Diophantine equation $4^x + p^y = z^2$ in non-negative integers, and in [4], he obtained a complete solution to the Diophantine equation $2^x + 11^y = z^2$ in non-negative integers. In fact, the latter equation has already been studied by A. D. Nicoară and C. E. Pumnea [7] without the use of an auxiliary result (Mihăilescu's Theorem). Banyat Sroysang also studied several exponential Diophantine equations of $a^x + b^y = z^2$ type (cf. [11, 12, 13, 14, 15, 16]). In [16], Sroysang was asking for the set of all solutions (x, y, z, w) for the Diophantine equation

$$(1.4) \quad 3^x + 5^y + 7^z = w^2$$

in non-negative integers. As a response, we present in this paper that the only non-negative integer solutions (x, y, z, w) to (1.4) are $(0, 0, 1, 3)$, $(1, 1, 0, 3)$ and $(3, 1, 2, 9)$. Related exponential Diophantine equations in the form $p^x \pm q^y \pm r^z = c$ where p, q, r are primes, x, y, z are non-negative integers, and c an integer have been studied in [5] and [10]. Particularly, J. Leitner [5] solved the equation $3^a + 5^b - 7^c = 1$ for non-negative integers a, b, c and the equation $y^2 = 3^a + 2^b + 1$ for non-negative integers a, b and integer y . R. Scott and R. Styer [10] studied, among other things, the Diophantine equation $p^x \pm q^y \pm 2^z = 0$ for primes p and q and integer c in positive integers x, y , and z . They used elementary methods to show that, with a few explicitly listed exceptions, there are at most two solutions (x, y) to $|p^x \pm q^y| = c$ and at most two solutions (x, y, z) to $p^x \pm q^y \pm 2^z = 0$ in positive integers. In the following section we shall use elementary methods to prove our main result.

2. MAIN RESULT

Theorem $(0, 0, 1, 3)$, $(1, 1, 0, 3)$ and $(3, 1, 2, 9)$ are the only solutions (x, y, z, w) to the exponential Diophantine equation (1.4) in non-negative integers.

The proof considers separate cases where at least one of the three exponents is zero or where all of them are strictly positive. Throughout the discussion, \mathbb{N} and \mathbb{N}_0 denote the set of positive and non-negative integers, respectively.

2.1. The case $\min\{x, y, z\} = 0$. We consider the following cases:

$$(2.1) \quad 3^x + 5^y = w^2 - 1,$$

$$(2.2) \quad 3^x + 7^z = w^2 - 1,$$

$$(2.3) \quad 5^y + 7^z = w^2 - 1.$$

We prove the following lemmas.

LEMMA 1: $(1, 1, 3)$ is the unique solution to equation (2.1) in \mathbb{N}_0 .

Note that by considering only the case when $w > 1$ is sufficient to study the problem since the left hand side (LHS) of (1.4) is greater than or equal to three. So we proceed as follows.

Proof of Lemma 1. Let $x, y, w \in \mathbb{N}_0$ with $w > 1$ and suppose that $3^x + 5^y = w^2 - 1$ has a solution in \mathbb{N}_0 . First, we let $x = 0$. So, we have $2 + 5^y = w^2$ which is impossible because $2 + 5^y \equiv 3 \pmod{4}$ whereas $w^2 \equiv 0, 1 \pmod{4}$. If $y = 0$, then we obtain $3^x + 2 = w^2$ which is also impossible since $3^x + 2 \equiv 2 \pmod{3}$. This leaves to consider $\min\{x, y\} > 0$. Note that $3^x \equiv 1 \pmod{4}$ when x is even and $3^x \equiv 3 \pmod{4}$ when x is odd. Also, note that $5^y \equiv 1 \pmod{4}$ for any $y \in \mathbb{N}_0$ so, x is odd, i.e. $x = 2r + 1$ for some $r \in \mathbb{N}_0$. Furthermore, since $w^2 \equiv 0, 1, 4 \pmod{8}$, $3^{2r+1} + 1 \equiv 4 \pmod{8}$ and, $5^y \equiv 1 \pmod{8}$ when y is odd and $5^y \equiv 5 \pmod{8}$ when y is even, we conclude that y is odd, i.e. $y = 2s + 1$ for some $s \in \mathbb{N}_0$. We also conclude that w is odd.

Since x and y are odd, we can now express (2.1) as $3^{2r+1} + 5^{2s+1} = w^2 - 1$. Writing w as $w = 2m + 1$ for some $m \in \mathbb{N}$, we further express (2.1) as

$$3^{2r+1} + 5^{2s+1} = 4m^2 + 4m = 4m(m + 1).$$

Dividing this equation by 8, we obtain

$$(2.4) \quad \frac{3^{2r+1} + 5^{2s+1}}{8} = \frac{m(m + 1)}{2}.$$

Notice that the LHS of (2.4) is a triangular number. It is easy to see that the equation is true for $r = s = 0$, i.e. $(3^1 + 5^1)/8 = (1)(2)/2$ giving us $(x, y, w) = (1, 1, 3)$ as a solution to (2.1). Now suppose that there is another solution (r', s') such that $\min\{r', s'\} > 0$. Hence, $w^2 \equiv 0, 1, 4, 7 \pmod{9}$, $3^{2r'+1} + 1 \equiv 1 \pmod{9}$, and $5^{2s'+1} \equiv 1, 4, 7 \pmod{9}$. It follows that $3^{2r'+1} + 5^{2s'+1} + 1 \equiv 2, 5, 8 \not\equiv w^2 \pmod{9}$. Thus, $(r, s, w) = (0, 0, 3)$ is the only solution to $3^{2r+1} + 5^{2s+1} = w^2 - 1$ and this completes the proof of the lemma. \square

Note: In equation (2.4), the triangular number is 8 times a perfect square plus 1. So (2.4) has the form $1 + p^{\alpha_1} + \dots + p^{\alpha_k} = A^2$ where p is prime and A, a_1, \dots, a_k are positive integers. This equation has already been studied by Rotkiewicz and Złotkowski in 1987 (cf. [9]). Also, Scott and Styer already proved that the equation $3^{2r+1} + 5^{2s+1} = 8 \times c$, where c is a triangular number, has at most 2 solutions (cf. Theorem 7 of [10]).

LEMMA 2: $(0, 1, 3)$ is the unique solution to equation (2.2) in \mathbb{N}_0 .

Proof of Lemma 2. Consider equation (2.2) modulo 4. Observe that $w^2 - 1 \equiv 0, 3 \pmod{4}$ and $7^z \equiv 1 \pmod{4}$ if z is even and $7^z \equiv 3 \pmod{4}$ if z is odd. Hence, x and z in (2.2) are of different parity. Suppose x is odd and z is even. Then, $3^x \equiv 0 \pmod{3}$ and $7^z \equiv 1 \pmod{3}$. It follows that $3^x + 7^z + 1 \equiv 2 \not\equiv w^2 \pmod{3}$ because $w^2 \equiv 0, 1 \pmod{3}$. Thus, the equation (2.2) has no solution. Suppose now that $x = 2r$ and $z = 2t + 1$ for some $r, t \in \mathbb{N}_0$. We can express (2.2) as

$$(2.5) \quad 8(7^{2t} - 7^{2t-1} + \dots + 1) = (w + 3^r)(w - 3^r).$$

For $t = 0$ we can distribute the factor 8 as $8 = (w + 3^r)(w - 3^r)$ with either $w + 3^r = 8, w - 3^r = 1$ or $w + 3^r = 4, w - 3^r = 2$ (but not $w + 3^r = 2$ and $w - 3^r = 4$, etc because $w + 3^r > w - 3^r$). If $w + 3^r = 8$ and $w - 3^r = 1$, then by addition, $2w = 9$ which is clearly impossible. If $w + 3^r = 4$ and $w - 3^r = 2$, then $w = 3, r = 0$ and here we obtain $(x, y, z, w) = (0, 0, 1, 3)$ which is a solution to (1.4). Now to deal with the two conditions; namely,

- i) $t > 0, 2|(w + 3^r)$, and $4|(w - 3^r)$, and
- ii) $t > 0, 4|(w + 3^r)$, and $2|(w - 3^r)$,

it suffices to assume that $r > 0$. We treat these two cases at once. Note that for $r > 0, 3^{2r} \equiv 0 \pmod{3}$ and $7^z \equiv 1 \pmod{3}$. Hence, we see that $3^x + 7^z + 1 \equiv 2 \not\equiv w^2$ since $w^2 \equiv 0, 1 \pmod{3}$. The conclusion follows. \square

LEMMA 3: $(0, 1, 3)$ is the unique solution to equation (2.3) in \mathbb{N}_0 .

Proof of Lemma 3. Consider now equation (2.3) modulo 4. Noting that w is odd we have $w^2 - 1 \equiv 0, 3 \pmod{4}$ which implies that z is odd. Since $w^2 \equiv 1 \pmod{8}$, $7^{2t+1} + 1 \equiv 7 + 1 \equiv 0 \pmod{8}$, and $5^y \equiv 1 \pmod{8}$ when y is even and $5^y \equiv 5 \pmod{8}$ when y is odd, we conclude that y is even.

If $y = 2s$ and $z = 2t+1$ for some $s, t \in \mathbb{N}_0$, then we obtain $8(7^{2t} - 7^{2t-1} + \dots + 1) = (w + 5^s)(w - 5^s)$. Note that $w + 5^s > w - 5^s$. So for $t = 0$, we can only distribute 8 as factors of $(w + 5^s)(w - 5^s)$ as follows: $w + 5^s = 8$ and $w - 5^s = 1$ or $w + 5^s = 4$ and $w - 5^s = 2$. The first pair of equations is clearly impossible since, by addition, $2w = 9$. However, the second pair of equations gives us $2w = (w + 5^s) + (w - 5^s) = 6$, or equivalently, $w = 3$. Here we obtain $(x, y, z, w) = (0, 0, 1, 3)$ as a solution to (1.4) and so it follows that $(0, 1, 3)$ is a solution to (2.3).

For $t > 0$, we consider the following cases:

- i) $4|(w + 5^s)$ and $2|(w - 5^s)$; and
- ii) $2|(w + 5^s)$ and $4|(w - 5^s)$.

We treat these cases at once and we may assume (WLOG) that $s > 0$.

For $\min\{s, t\} > 0$ we have,

$$(5^{2s} - 1) + (7^{2t} + 1) = w^2 - 1 \quad \Leftrightarrow \quad 8 \left[\frac{(5^{2s} - 1) + (7^{2t} + 1)}{8} \right] = (w + 1)(w - 1).$$

Since w is odd then $w + 1$ and $w - 1$ were both even then the LHS of the latter equation is 8 times an odd integer. Hence, if $4|(w \pm 1)$ and $2|(w \mp 1)$ then $w \pm 1 = 4(2k + 1)$ and $w \mp 1 = 2(2l + 1)$ for some $k, l \in \mathbb{N}$. Subtracting these two equations yields $2 = 2[2(2k + 1) - (2l + 1)]$ or equivalently, $1 = 2(2k + 1) - (2l + 1)$ which implies that $k = l = 0$. This contradicts our assumption that $k, l \in \mathbb{N}$. Thus, $5^{2s} + 7^{2s+1} = w^2 - 1$ has no solution for $\min\{s, t\} > 0$ which completes the proof. \square

2.2. The case $\min\{x, y, z\} > 0$. Let $\min\{x, y, z\} > 0$. We first determine a possible parity of x, y, z so that equation (1.4) has a solution in positive integers. Taking modulo 4 of both sides of (1.4) we see that x and z must be of different parity and w is odd. If we take modulo 3 of (1.4) both sides, then y must be odd. Lastly, taking modulo 8 of (1.4) both sides we conclude that x is odd and z is even. Hence, (1.4) is only possible in positive integers provided x is odd, y is odd, z is even, and w is odd.

Let $w = 2m + 1$ where $m \in \mathbb{N}$. Suppose that

$$(2.6) \quad 3^{2r+1} + 5^{2s+1} + 7^{2t} = w^2,$$

for some $r, s, t \in \mathbb{N}_0$. Since $w^2 \equiv 0 \pmod{3}$ and $w^2 \equiv 1 \pmod{8}$ then $w = 24n - 15$ for some $n \in \mathbb{N}$. The least possible value of w would be 9. Letting $w = 9$ we see that $3^{2r+1} + 5^{2s+1} = 9^2 - 7^{2t} = (9 + 7^t)(9 - 7^t)$ in which follows that $t = 0$ or $z = 2$. Now we have $3^{2r+1} + 5^{2s+1} = 32$. Since $\min\{r, s\} > 0$ and, $2r + 1 \leq 3$ and $2s + 1 \leq 2$ it follows that $r = 1$ and $s = 0$ giving us $(1, 0, 1, 9)$ as a solution to (2.6)

or equivalently, $(3, 1, 2, 9)$ as a solution to (1.4). Now suppose that there is another solution to (2.6) with $s > 0$ then we can express (2.6) as

$$(2.7) \quad 4[(3^{2r} - 3^{2r-1} + \dots + 1) + (5^{2s} + 5^{2s-1} + \dots + 1)] = (w + 7^t)(w - 7^t).$$

Note that the term inside $[\cdot]$ is even and $w + 7^t$ and $w - 7^t$ were both even. In addition, $w = 8B + 1$ and $7^t = (8 - 1)^t = 8A \pm 1$ for some $A, B \in \mathbb{N}$. So $w^2 - 7^{2t} = (8B + 1 + 8A \pm 1)(8B + 1 - (8A \pm 1)) = 16(A + B)(4(B - A) + 1)$. Since $A + B$ is even then the LHS of (2.7) is 32 times an odd integer. Hence, we can rewrite equation (2.7) as

$$(2.8) \quad 32 \left[\frac{(3^{2r} - 3^{2r-1} + \dots + 1) + (5^{2s} + 5^{2s-1} + \dots + 1)}{8} \right] = (w + 7^t)(w - 7^t).$$

Now we consider the following possibilities: (i) $(3^{2r} - 3^{2r-1} + \dots + 1) + (5^{2s} + 5^{2s-1} + \dots + 1) = 8$ where $16|(w + 7^s)$ or $16|(w - 7^s)$; and (ii) $8|(w + 7^t)$ and $4|(w - 7^t)$ or $8|(w - 7^t)$ and $4|(w + 7^t)$, which are the ways of distributing the factor 16 of the LHS across the two factors of the RHS. If $16|(w + 7^t)$ or $16|(w - 7^t)$, then we have $(3^{2r} - 3^{2r-1} + \dots + 1) + (5^{2s} + 5^{2s-1} + \dots + 1) = 8$ which is impossible since $s > 0$. If $8|(w \pm 7^t)$ and $4|(w \mp 7^t)$, then the requirements of $w \pm 7^t$ being both odd mean $2m + 1 + 7^t = 8(2k + 1)$ and $2m + 1 - 7^t = 4(2l + 1)$ for some $k, l \in \mathbb{N}$. Subtracting these two equations yields $2 \cdot 7^t = 4(2(2k + 1) - (2l + 1))$ or equivalently $7^t = 2(2(2k + 1) - (2l + 1))$ which is clearly impossible. Therefore the only solutions in \mathbb{N}_0 to the exponential Diophantine equation (1.4) are $(0, 0, 1, 3)$, $(1, 1, 0, 3)$ and $(3, 1, 2, 9)$.

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