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HERMITE-HADAMARD'S INEQUALITIES FOR PREQUASIINVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

IMDAT ISCAN

Abstract. In this paper, we extend some estimates of the right hand side of Hermite-Hadamard type inequality for prequasiinvex functions via fractional integrals.

1. Introduction and Preliminaries

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This doubly inequality is known in the literature as Hermite-Hadamard integral inequality for convex mapping. For several recent results concerning the inequality (1.1) we refer the interested reader to [2, 3, 4, 5, 7, 17, 18].

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if inequality

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\},$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Clearly, any convex function is quasi-convex function. Furthemore there exist quasi-convex functions which are not convex (see [7]).

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.1. Let $f \in L[a, b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of oder $\alpha > 0$ with $a \geq 0$ are defined by

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$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent result connected with fractional integral inequalities see ([13, 16, 18, 19]).

In [13], Ozdemir and Yıldız proved the Hadamard inequality for quasi-convex functions via Riemann-Liouville fractional integrals as follows:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$, be positive function with $0 \le a < b$ and $f \in L[a,b]$. If f is a quasi-convex function on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \le \max\{f(a), f(b)\} \tag{1.2}$$

with $\alpha > 0$.

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is quasi-convex on $[a,b]$, $\alpha > 0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ \le\ & \frac{b-a}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right) \max\{|f'(a)|, |f'(b)|\}. \end{aligned} \tag{1.3}$$

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$, and $q > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ \le\ & \frac{b-a}{2(\alpha p+1)^{\frac{1}{p}}} \left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} \end{aligned} \tag{1.4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0,1]$.

Theorem 1.4. *Let $f : [a,b] \to \mathbb{R}$, be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$, and $q \ge 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ \le\ & \frac{b-a}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right) \left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} \end{aligned} \tag{1.5}$$

with $\alpha > 0$.

In recent years several extentions and generalizations have been considered for classical convexity. A significant generalization of convex functions is the invex functions introduced by Hanson in [6]. Weir and Mond [20] introduced the concept of preinvex functions and applied it to the establisment of the sufficient optimality conditions and duality in nonlinear programming. Pini [15] introduced the concept of prequasiinvex function as a generalization of invex functions. Later, Mohan and Neogy [9] obtained some properties of generalized preinvex functions. Noor ([11]-[12])has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions. In recent papers Yang et al. in [22] studied prequasiinvex functions and semistrictly prequasiinvex functions, Barani et al. in [4] presented some generalizations of the right hand side of a Hermite-Hadamard type inequality for prequasiinvex functions and Park in [14] established generalized Simpson-like and Hermite-Hadamard-like type integral inequalities for functions whose second derivatives in absolutely value at certain powers are preinvex and prequasiinvex.

In this paper we generalized the results in [13] for prequasiinvex functions. Now we recall some notions in invexity analysis which will be used throught the paper (see [1, 21] and references therein)

Let $f : A\to \mathbb{R}$ and $\eta : A\times A \to \mathbb{R}$,where A is a nonempty set in $\mathbb{R}^n$, be continuous functions.

Definition 1.2. The set $A \subseteq \mathbb{R}^n$ is said to be invex with respect to $\eta(.,.)$, if for every $x, y \in A$ and $t \in [0, 1]$,

$$x + t\eta(y, x) \in A.$$

The invex set A is also called a $\eta-$connected set.

It is obvious that every convex set is invex with respect to $\eta(y, x) = y - x$, but there exist invex sets which are not convex [1].

Definition 1.3. The function f on the invex set A is said to be preinvex with respect to η if

$$f(x + t\eta(y, x)) \leq (1 - t) f(x) + tf(y),\ \forall x, y \in A,\ t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

Definition 1.4. The function f on the invex set A is said to be prequasiinvex with respect to η if

$$f(x + t\eta(y, x)) \leq \max\{f(x), f(y)\},\ \forall x, y \in A,\ t \in [0, 1].$$

Every quasi-convex function is a prequasinvex with respect to $\eta(y, x) = y - x$, but the converse does not holds (see example 1.1 in [22])

We also need the following assumption regarding the function η which is due to Mohan and Neogy [9]:

Condition C: Let $A \subseteq \mathbb{R}^n$ be an invex subset with respect to $\eta : A \times A \to \mathbb{R}$. For any $x, y \in A$ and any $t \in [0, 1]$,

$$\begin{aligned} \eta(y, y + t\eta(x, y)) &= -t\eta(x, y) \\ \eta(x, y + t\eta(x, y)) &= (1 - t)\eta(x, y). \end{aligned}$$

Note that for every $x, y \in A$ and every $t \in [0, 1]$ from condition C, we have

$$\eta\left(y+t_2\eta(x,y),y+t_1\eta(x,y)\right)=(t_2-t_1)\eta(x,y). \tag{1.6}$$

In [4] Barani et al. proved the Hermite-Hadamard type inequality for prequasiinvex as follows:

Theorem 1.5. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta : A\times A\to\mathbb{R}$. Suppose that $f : A\to\mathbb{R}$ is a differentiable function. If $|f'|$ is prequasiinvex on A then, for every $a,b\in A$ the following inequalities holds*

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|\\ \le\ &\frac{|\eta(b,a)|}{4}\max\left\{|f'(a)|,|f'(b)|\right\}.\end{aligned} \tag{1.7}$$

Theorem 1.6. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta : A\times A\to\mathbb{R}$. Suppose that $f : A\to\mathbb{R}$ is a differentiable function. Assume that $p\in\mathbb{R}$ with $p>1$. If $|f'|^{\frac{p}{p-1}}$ is preinvex on A then, for every $a,b\in A$ the following inequalities holds*

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|\\ \le\ &\frac{|\eta(b,a)|}{2(p+1)^{\frac{1}{p}}}\left(\max\left\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}.\end{aligned} \tag{1.8}$$

In order to prove our main results we need the following lemma:

Lemma 1.1. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta : A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$. Suppose that $f : A\to\mathbb{R}$ is a differentiable function. If f' is preinvex function on A and $f'\in L\left[a,a+\eta(b,a)\right]$ then, the following equality holds:*

$$\begin{aligned}&\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\\ =\ &\frac{\eta(b,a)}{2}\int_0^1\left[t^{\alpha}-(1-t)^{\alpha}\right]f'\left(a+t\eta(b,a)\right)dt\end{aligned} \tag{1.9}$$

By using partial integration in right hand of (1.9) equality, the proof is obvious (see [8]).

In this paper, using Lemma 1.1 we obtained new inequalities related to the right side of Hermite-Hadamard inequalities for prequasiinvex functions via fractional integrals.

2. Main Results

Theorem 2.1. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta : A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$. If $f : \left[a,a+\eta(b,a)\right]\to(0,\infty)$ is a prequasiinvex function, $f\in L\left[a,a+\eta(b,a)\right]$ and η satisfies condition C then, the following*

inequalities for fractional integrals holds:

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right] \\ \leq & \max\{f(a),f(a+\eta(b,a)\}\leq\max\{f(a),f(b)\} \end{aligned} \tag{2.1}$$

Proof. Since $a,b\in A$ and A is an invex set with respect to η, for every $t\in[0,1]$, we have $a+t\eta(b,a)\in A$. By prequasiinvexity of f and inequality (1.6) for every $t\in[0,1]$ we get

$$\begin{aligned} f(a+t\eta(b,a)) &= f(a+\eta(b,a)+(1-t)\eta(a,a+\eta(b,a))) \\ &\leq \max\{f(a),f(a+\eta(b,a)\} \end{aligned} \tag{2.2}$$

and similarly

$$\begin{aligned} f(a+(1-t)\eta(b,a)) &= f(a+\eta(b,a)+t\eta(a,a+\eta(b,a))) \\ &\leq \max\{f(a),f(a+\eta(b,a)\}. \end{aligned}$$

By adding these inequalities we have

$$f(a+t\eta(b,a))+f(a+(1-t)\eta(b,a))\leq 2\max\{f(a),f(a+\eta(b,a)\} \tag{2.3}$$

Then multiplying both (2.3) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1}f(a+t\eta(b,a))\,dt+\int_0^1 t^{\alpha-1}f(a+(1-t)\eta(b,a))\,dt\leq 2\max\{f(a),f(a+\eta(b,a)\}\int_0^1 t^{\alpha-1}dt.$$

i.e.

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\leq\frac{2\max\{f(a),f(a+\eta(b,a)\}}{\alpha}.$$

Using the mapping η satisfies condition C the proof is completed. □

Remark 2.1. In Theorem 2.1, if we take $\eta(b,a)=b-a$, then inequality (2.1) become inequality (1.2) of Theorem 1.1.

Theorem 2.2. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L[a,a+\eta(b,a)]$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $|f'|$ is prequasiinvex function on $[a,a+\eta(b,a)]$ then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$\begin{aligned} & \left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\ \leq & \frac{\eta(b,a)}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\{|f'(a)|,|f'(b)|\}. \end{aligned} \tag{2.4}$$

Proof. Using Lemma 1.1 and the prequasiinvexity of $|f'|$ we get

$$\begin{aligned} &\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\ \leq\ & \frac{\eta(b,a)}{2}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|dt \\ \leq\ & \frac{\eta(b,a)}{2}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt \\ \leq\ & \frac{\eta(b,a)}{2}\left\{\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt\right\} \\ =\ & \eta(b,a)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\left(\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt\right) \\ =\ & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}, \end{aligned}$$

which completes the proof. □

Remark 2.2. a) In Theorem 2.2, if we take $\eta(b,a)=b-a$, then inequality (2.4) become inequality (1.3) of Theorem 1.2

b) In Theorem2.2, if we take $\alpha=1$, then inequality (2.4) become inequality (1.7) of Theorem 1.5.

c) In Theorem2.2, assume that η satisfies condition C.Using inequality (2.2) we get

$$\begin{aligned} &\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\ \leq\ & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|f'(a)\right|,\left|f'(a+\eta(b,a))\right|\right\} \end{aligned}$$

Theorem 2.3. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L\left[a,a+\eta(b,a)\right]$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $\left|f'\right|^{q}$ is prequasiinvex function on $\left[a,a+\eta(b,a)\right]$ for some fixed $q\geq1$ then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\ \leq\ & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}} \end{aligned} \tag{2.5}$$

where $\alpha>0$.

Proof. From Lemma1.1 and using Power-mean inequality with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{\eta(b,a)}{2}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|dt \\
\leq\ & \frac{\eta(b,a)}{2}\left(\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|dt &= \int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]dt \\
&= \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right).
\end{aligned}$$

Since $|f'|^{q}$ is prequasiinvex function on $[a,a+\eta(b,a)]$, we obtain

$$\left|f'\left(a+t\eta(b,a)\right)\right|^{q}\leq\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\},\quad t\in[0,1]$$

and

$$\begin{aligned}
\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|^{q}dt &\leq \int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}dt \\
&= \max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}.\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|dt \\
&= \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}
\end{aligned}$$

from here we obtain inequality (2.5). This completes the proof. □

Remark 2.3. a) In Theorem2.3, if we take $\eta(b,a)=b-a$ then inequality (2.5)become inequality (1.5) Theorem1.4.

b) In Theorem2.3, assume that η satisfies condition C. Using inequality (2.2) we get

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(a+\eta(b,a))\right|^{q}\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 2.4. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\rightarrow\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L\left[a,a+\eta(b,a)\right]$. Suppose that $f:A\rightarrow\mathbb{R}$ is a differentiable function. If $|f'|^{q}$ is prequasiinvex function on $[a,a+\eta(b,a)]$ for*

some fixed $q>1$ then the following inequality holds:

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right| \tag{2.6}$$
$$\leq \frac{\eta(b,a)}{2\left(\alpha p+1\right)^{\frac{1}{p}}}\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(a+\eta(b,a))\right|^{q}\right\}\right)^{\frac{1}{q}}.$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha\in[0,1]$.

Proof. From Lemma1.1 and using Hölder inequality with properties of modulus, we have

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{\eta(b,a)}{2}\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|dt$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(a+t\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}.$$

We know that for $\alpha\in[0,1]$ and $\forall t_1,t_2\in[0,1]$,

$$\left|t_1^{\alpha}-t_2^{\alpha}\right|\leq\left|t_1-t_2\right|^{\alpha},$$

therefore

$$\begin{aligned}\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|^{p}dt &\leq \int_0^1\left|1-2t\right|^{\alpha p}dt\\ &= \int_0^{\frac{1}{2}}\left[1-2t\right]^{\alpha p}dt+\int_{\frac{1}{2}}^1\left[2t-1\right]^{\alpha p}dt\\ &= \frac{1}{\alpha p+1}.\end{aligned}$$

Since $|f'|^q$ is prequasiinvex on $[a,a+\eta(b,a)]$, we have inequality (2.6). This completes the proof. □

Remark 2.4. a) In Theorem 2.4, if we take $\eta(b,a)=b-a$ then inequality (2.6) become inequality (1.4) of Theorem 1.3.

b) In Theorem 2.4, if we take $\alpha=1$then inequality (2.6) become inequality (1.8) of Theorem 1.6.

c) In Theorem 2.4, assume that η satisfies condition C. Using inequality (2.2) we get

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{\eta(b,a)}{2\left(\alpha p+1\right)^{\frac{1}{p}}}\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(a+\eta(b,a))\right|^{q}\right\}\right)^{\frac{1}{q}}.$$

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Giresun University, Science and Art Faculty, Department of Mathematics, Giresun-TURKEY

E-mail address: imdat.iscan@giresun.edu.tr