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AREA FORMULAS FOR A TRIANGLE IN THE $m-$PLANE

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Abstract. In this paper, we give three area formulas for a triangle in the m-plane in terms of the $m-$distance. The two of them are $m-$version of the standart area formula for a triangle in the Euclidean plane, and the third one is a $m-$version of the well-known Heron′s formula.

1. Introduction

If one want to measure the distance between two points on a plane, then one can use frequently Euclidean distance which is defined as the length of segment between these points. Although it is the most popular distance function, it is not practical when we measure the distance which we actually move in the real world. So taxicab distance and Chinese checkers distance were introduced. Taxicab and Chinese checkers distance functions are similar to moving with a car or Chinese chess in the real world. Later, Tian [16] introduced α-distance function which includes the taxicab and Chinese checkers metrics as special cases. Then, some authors developed and studied on these topics (see [7], [8], [10]). In [5] Colakoğlu and Kaya gave a new distance function in the real plane which includes alpha, Chinese checkers, taxicab distances as special cases. The distance function is called m-distance. If $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ are two points in $\mathbb{R}^2$, then for each real numbers u, v and m such that $u \geq v \geq 0 \neq u$, the distance function

$$d_m : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$$

defined by

$$d_m(P,Q) = (u\Delta_{PQ} + v\delta_{PQ}) \,/\, \left(\sqrt{1+m^2}\right)$$

where $\Delta_{PQ} = \max\{|(x_1-x_2)+m(y_1-y_2)|, |m(x_1-x_2)-(y_1-y_2)|\}$ and $\delta_{PQ} = \min\{|(x_1-x_2)+m(y_1-y_2)|, |m(x_1-x_2)-(y_1-y_2)|\}$. Obviously, there are infinitely many different distance function depending on values u, v and m. But we suppose that values u and v are initially determined and fixed unless otherwise stated.

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According to m-distance function, the $m-$distance between points P and Q is constant u multiple of the Euclidean length of one of the shortest paths from P to Q composed of line segments each parallel to one of lines with slope m, $-1/m$, $\left[m(u^2 - v^2) + 2uv\right] / \left[(u^2 - v^2) - 2uvm\right]$ or $\left[m(u^2 - v^2) - 2uv\right] / \left[(u^2 - v^2) + 2uvm\right]$. See Figure 1.

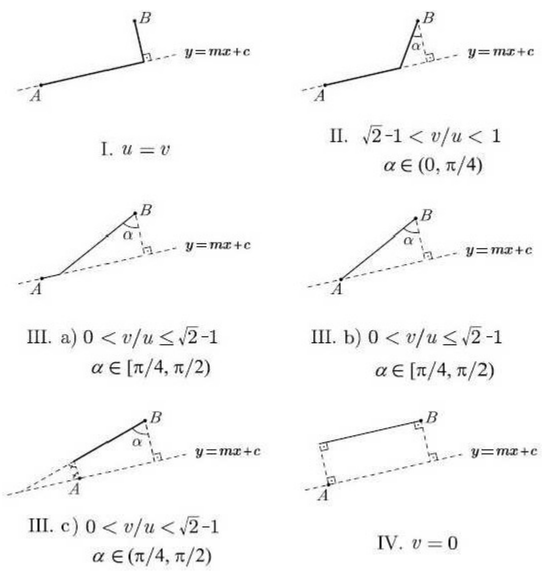

Figure 1

In this paper, we give area formulas for a triangle in the m-plane in terms of the $m-$distance. In this study, we use the usual Euclidean area notion. One can easily see that in the m-plane, there are triangles whose -lengths of corresponding sides are the same, while areas of these triangles are different (see Figure 2).This fact arises a natural question: How can one compute the area of a triangle in the m-plane? It is obvious that every formula to compute the area of a triangle depends on some parameters, and using different parameters gives different formulas. Here we give three formulas to compute the area of a triangle in the m-plane, using different parameters.

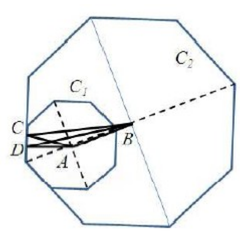

Figure 2

Let the line AB be parallel to the line $y = mx$, let C_1 be a $m-$circle with center A and radius b, C_2 a $m-$circle with center B and radius $b + c$, and C and D two points in $C_1 \cap C_2$. For different C and D such that C and D are not symmetric to the line AB, $Area(ABC) \neq Area(ABD)$, while $d_m(A, C) = d_m(A, D)$ and $d_m(B, C) = d_m(B, D)$.

2. Area of a Triangle in the m-plane

It is well-known that if ABC is a triangle with the area $\mathcal{A}$ in the Euclidean plane, and H is the point of orthogonal projection of the point A on the line BC, then standard area formula for the triangle ABC is $\mathcal{A} = \mathbf{ah}/2$, where $\mathbf{a} = d_E(B, C)$ and $\mathbf{h} = d_E(A, H)$ or $\mathbf{h} = d_E(A, BC)$ (see Figure 3). In this section, we give two m-versions of standard area formula in terms of m-distance. Clearly, a m-version of standard area formula for triangle ABC would be an equation that relates the two m-distances a and h, where $a = d_m(B, C)$, $h = d_m(A, H)$ or $h = d_m(A, BC)$ and area $\mathcal{A}$ of triangle ABC. Here, we give two m-versions of the area formula that depend on one parameter, namely, the slope of the base segment, in addition to the other parameters. Note that the real numbers u, v and m are fixed.

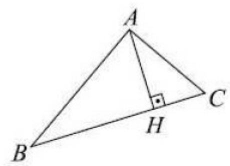

Figure 3

The following equation, which relates the Euclidean distance to the m-distance between two points in the Cartesian coordinate plane, plays an important role in the first m-version of the area formula. Following two proposition are given without proofs. One can see [2] for proofs.

Proposition 2.1. *For any two points P and Q in the Cartesian plane that do not lie on a vertical line, if n is the slope of the line through P and Q, then*

$$d_E(P, Q) = \rho(n) d_m(P, Q)$$

where $\rho(n) = \dfrac{\sqrt{(1+n^2)(1+m^2)}}{u \max\{|1+mn|, |m-n|\} + v \min\{|1+mn|, |m-n|\}}$.
If P and Q lie on a vertical line, then by definition,

$$d_E(P, Q) = \frac{\sqrt{1+m^2}}{u \max\{1, |m|\} + v \min\{1, |m|\}} \; d_m(P, Q).$$

If P and Q lie on the lines $y = mx$ or $y = \frac{-1}{m}x$, then

$$d_E(P, Q) = \frac{1}{u} d_m(P, Q).$$

Another useful fact that can be verified by direct calculation is:

Proposition 2.2. *For any real number $n \neq 0$*

$$\rho(n) = \rho(-1/n).$$

We first note by Proposition 1 and Proposition 2 that the m-distance between two points is invariant under all translations. If $b/a \neq \sqrt{2} - 1$ in m-plane, the rotations of $\pi/2$, π and $3\pi/2$ radians around a point, and the reflections about the lines parallel to $y = nx + c$ such that $n \in \left\{ m, \dfrac{-1}{m}, \dfrac{m-1}{1+m}, \dfrac{1+m}{1-m} \right\}$ preserve the $m-$distance. If $b/a = \sqrt{2} - 1$ in the $m-$plane, the rotations of $\pi/4$, $\pi/2$, $3\pi/4$, π, $5\pi/4$, $3\pi/2$ and $7\pi/4$ radians around a point, and the reflections about the lines parallel to $y = nx + c$

such that $n \in \left\{ m, \dfrac{-1}{m}, \dfrac{m-1}{1+m}, \dfrac{1+m}{1-m}, \dfrac{(1-\sqrt{2})m-1}{(1-\sqrt{2})+m}, \dfrac{(1+\sqrt{2})m-1}{(1+\sqrt{2})+m}, \dfrac{(1-\sqrt{2})m+1}{(1-\sqrt{2})-m}, \dfrac{(1+\sqrt{2})m+1}{(1+\sqrt{2})-m} \right\}$ preserve the $m-$distance (see [5]).

The following theorem gives a m-version of the well-known Euclidean area formula of a triangle:

Theorem 2.1. *Let ABC be a triangle with the area $\mathcal{A}$ in the m-plane, H be orthogonal projection (in the Euclidean sense) of the point A on the line BC, n be the slope of the line BC, and let $a = d_m(B, C)$ and $h = d_m(A, H)$.*
(i) If BC is parallel to one of the lines $y = mx$ or $y = \frac{-1}{m}x$, then

$$\mathcal{A} = \frac{1}{u^2}\frac{ah}{2}.$$

(ii) If BC is parallel to a coordinate axis, then

$$\mathcal{A} = [\rho(n)]^2 \frac{ah}{2},$$

where $\rho(n) = \dfrac{\sqrt{1+n^2}}{u \max\{|n|, 1\} + v \min\{|n|, 1\}}$.
(iii) f BC is not parallel to any one of coordinate axes or the lines $y = mx$ or $y = \frac{-1}{m}x$, then

$$\mathcal{A} = [\rho(n)]^2 \frac{ah}{2}$$

where $\rho(n) = \dfrac{\sqrt{(1+n^2)(1+m^2)}}{u \max\{|1+mn|, |m-n|\} + v \min\{|1+mn|, |m-n|\}}$.

Proof. Let $\mathbf{a} = d_E(B, C)$ and $\mathbf{h} = d_E(A, H)$. Then $\mathcal{A} = \dfrac{\mathbf{ah}}{2}$.
(i) If BC is parallel to one of the lines $y = mx$ or $y = \frac{-1}{m}x$, then obviously $\mathbf{a} = \frac{1}{u}a$ and $\mathbf{h} = \frac{1}{u}h$. Hence $\mathcal{A} = \frac{1}{u^2}\frac{ah}{2}$.
(ii) If BC not be parallel to any one of the lines $y = mx$ or $y = \frac{-1}{m}x$, and let the slpe of the line BC be n. Then the slope of the line AH is $\frac{-1}{n}$. By proposition 1 and Proposition 2, $\mathbf{a} = \rho(n)a$, $\mathbf{h} = \rho(n)h$, hence $\mathcal{A} = [\rho(n)]^2 \frac{ah}{2}$. □

In the m-plane, m-distance from a point P to a line l is defined by

$$d_m(P, l) = \min_{Q \in l}\{d_m(P, Q)\}$$

as in the Euclidean plane. It is well-known that in the Euclidean plane, Euclidean distance from a point $P = (x_0, y_0)$ to a line $l : ax + by + c = 0$ can be calculated by the following formula:

$$d_E(P, l) = |ax_0 + by_0 + c| \diagup (a^2 + b^2)^{1/2}. \tag{2.1}$$

In Proposition 4 we give a similar formula for $d_m(P, l)$, using m-circles (see [2]). One can see by calculation that if $0 < v/u < 1$, then the unit $m-$circle is an octagon with vertices $A_1 = (\frac{1}{uk}, \frac{m}{uk})$, $A_2 = (\frac{1-m}{(u+v)k}, \frac{1+m}{(u+v)k})$, $A_3 = (\frac{-m}{uk}, \frac{1}{uk})$, $A_4 = (\frac{-1-m}{(u+v)k}, \frac{1-m}{(u+v)k})$, $A_5 = (\frac{-1}{uk}, \frac{-m}{uk})$, $A_6 = (\frac{m-1}{(u+v)k}, \frac{-1-m}{(u+v)k})$, $A_7 = (\frac{m}{uk}, \frac{-1}{uk})$, $A_8 = (\frac{1+m}{(u+v)k}, \frac{m-1}{(u+v)k})$, where $k = \sqrt{1+m^2}$. If $u = v$ or $v = 0$, then unit $m-$circle is a square with vertices A_1, A_3, A_5, A_7 or A_2, A_4, A_6, A_8, respectively (See figure 4).

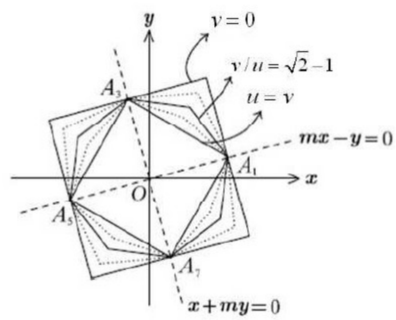

Figure 4

The next proposition introduced $m-$distance from a point P to a line l. For the proof of proposition, one can see in [2].

Proposition 2.3. *Given a point $P = (x_0, y_0)$, and a line $l : ax + by + c = 0$ in the m-plane. Then the m-distance from the point P to the line l can be calculated by the following formula:*

(2.2)
$$d_m(P,l) = \begin{cases} \dfrac{|ax_0 + by_0 + c|\sqrt{1+m^2}}{\max\left\{\frac{|a+bm|}{u}, \frac{|am-b|}{u}\right\}} & , u = v \\ \dfrac{|ax_0 + by_0 + c|\sqrt{1+m^2}}{\max\left\{\frac{|a(1-m)+b(1+m)|}{u}, \frac{|a(1+m)-b(1-m)|}{u}\right\}} & , v = 0 \\ \dfrac{|ax_0 + by_0 + c|\sqrt{1+m^2}}{\max\left\{\frac{|a+bm|}{u}, \frac{|am-b|}{u}, \frac{|a(1-m)+b(1+m)|}{u+v}, \frac{|a(1+m)-b(1-m)|}{u+v}\right\}} & , 0 < v/u < 1. \end{cases}$$

The following equation, which relates the Euclidean distance to the m-distance from a point to a line in the Cartesian coordinate plane, plays an important role in the second m-version of the area formula.

Proposition 2.4. *Given a point P, and a line l in the Cartesian plane that is not a vertical line, if n is the slope of the line l, then*

$$d_E(P,l) = \tau(n)d_\alpha(P,l) \tag{2.3}$$

$$\text{where } \tau(n)=\begin{cases}\dfrac{\max\left\{\dfrac{|1+nm|}{u},\dfrac{|m-n|}{u}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ u=v\\ \dfrac{\max\left\{\dfrac{|n(1-m)-(1+m)|}{u},\dfrac{|n(1+m)+(1-m)|}{u}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ v=0\\ \dfrac{\max\left\{\dfrac{|1+nm|}{u},\dfrac{|m-n|}{u},\dfrac{|n(1-m)-(1+m)|}{u+v},\dfrac{|n(1+m)+(1-m)|}{u+v}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ 0<v/u<1\end{cases}$$

If l is a vertical line, then

$$\tau(n)=\begin{cases}\dfrac{\max\left\{\dfrac{|m|}{u},\dfrac{1}{u}\right\}}{\sqrt{1+m^2}} & ,\ u=v\\ \dfrac{\max\left\{\dfrac{|1-m|}{u},\dfrac{|1+m|}{u}\right\}}{\sqrt{1+m^2}} & ,\ v=0\\ \dfrac{\max\left\{\dfrac{|m|}{u},\dfrac{1}{u},\dfrac{|1-m|}{u+v},\dfrac{|1+m|}{u+v}\right\}}{\sqrt{1+m^2}} & ,\ 0<v/u<1\end{cases}.$$

Proof. Let $P=(x_0,y_0)$ be a point, and $l: ax+by+c=0$ be a line in the Cartesian plane. If l is not a vertical line, then $b\neq 0$ and $n=-\frac{a}{b}$. Using n in equation 2.1 and equation 2.2, one gets $d_E(P,l)=|ax_0+by_0+c| \big/ |b|\,(1+n^2)^{1/2}$ and

$$d_m(P,l)=\begin{cases}\dfrac{|ax_0+by_0+c|\sqrt{1+m^2}}{|b|\max\left\{\frac{|1+nm|}{u},\frac{|m-n|}{u}\right\}} & ,u=v\\ \dfrac{|ax_0+by_0+c|\sqrt{1+m^2}}{|b|\max\left\{\frac{|n(1-m)-(1+m)|}{u},\frac{|n(1+m)+(1-m)|}{u}\right\}} & ,v=0\\ \dfrac{|ax_0+by_0+c|\sqrt{1+m^2}}{|b|\max\left\{\frac{|1+nm|}{u},\frac{|m-n|}{u},\frac{|n(1-m)-(1+m)|}{u+v},\frac{|n(1+m)+(1-m)|}{u+v}\right\}} & ,0<v/u<1.\end{cases}$$

Hence, $d_E(P,l)=\tau(n)d_m(P,l)$ where

$$\tau(n)=\begin{cases}\dfrac{\max\left\{\dfrac{|1+nm|}{u},\dfrac{|m-n|}{u}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ u=v\\ \dfrac{\max\left\{\dfrac{|n(1-m)-(1+m)|}{u},\dfrac{|n(1+m)+(1-m)|}{u}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ v=0\\ \dfrac{\max\left\{\dfrac{|1+nm|}{u},\dfrac{|m-n|}{u},\dfrac{|n(1-m)-(1+m)|}{u+v},\dfrac{|n(1+m)+(1-m)|}{u+v}\right\}}{\sqrt{(1+n^2)(1+m^2)}} & ,\ 0<v/u<1.\end{cases}$$

If l is a vertical line, then $b = 0$ and $a \neq 0$. Therefore, $d_E(P,l) = |ax_0 + c| \diagup |a|$ and

$$d_m(P,l) = \begin{cases} \dfrac{|ax_0+c|\sqrt{1+m^2}}{|a|\max\left\{\frac{1}{u}, \frac{|m|}{u}\right\}} & , u = v \\ \dfrac{|ax_0+c|\sqrt{1+m^2}}{|a|\max\left\{\frac{|1-m|}{u}, \frac{|1+m|}{u}\right\}} & , v = 0 \\ \dfrac{|ax_0+c|\sqrt{1+m^2}}{|a|\max\left\{\frac{1}{u}, \frac{|m|}{u}, \frac{|1-m|}{u}, \frac{|1+m|}{u}\right\}} & , 0 < v/u < 1 \end{cases},$$

hence

$$\tau(n) = \begin{cases} \dfrac{\max\left\{\dfrac{|m|}{u}, \dfrac{1}{u}\right\}}{\sqrt{1+m^2}} & , \ u = v \\ \dfrac{\max\left\{\dfrac{|1-m|}{u}, \dfrac{|1+m|}{u}\right\}}{\sqrt{1+m^2}} & , \ v = 0 \\ \dfrac{\max\left\{\dfrac{|m|}{u}, \dfrac{1}{u}, \dfrac{|1-m|}{u+v}, \dfrac{|1+m|}{u+v}\right\}}{\sqrt{1+m^2}} & , \ 0 < v/u < 1. \end{cases}$$

□

The following theorem gives another α-version of the well-known Euclidean area formula of a triangle:

Theorem 2.2. *Let ABC be a triangle with area $\mathcal{A}$ in the m-plane, n be the slope of the line BC, and let $a = d_\alpha(B,C)$ and $h = d_\alpha(A,BC)$. Then the area of ABC is*

$$\mathcal{A} = \sigma(n)ah/2$$

(i) If BC is parallel to lines $y = mx$ or $y = \frac{-1}{m}x$, then

$$\sigma(n) = \begin{cases} 1/u^2 & , \ u = v \ \text{or} \ v = 0 \\ 1/u(u+v) & , \ 0 < v/u < 1 \end{cases}$$

(ii) If BC is not parallel to any one of the lines $y = mx$ or $y = \frac{-1}{m}x$, then

$$\sigma(n) = \begin{cases} \dfrac{\max\left\{\dfrac{|1+nm|}{u}, \dfrac{|m-n|}{u}\right\}}{u\max\{|1+nm|, \ |m-n|\} + u\min\{|1+nm|, \ |m-n|\}} & , \ u = v \\ \dfrac{\max\left\{\dfrac{|n(1-m)-(1+m)|}{u}, \dfrac{|n(1+m)+(1-m)|}{u}\right\}}{u\max\{|1+nm|, \ |m-n|\}} & , \ v = 0 \\ \dfrac{\max\left\{\dfrac{|1+nm|}{u}, \dfrac{|m-n|}{u}, \dfrac{|n(1-m)-(1+m)|}{u+v}, \dfrac{|n(1+m)+(1-m)|}{u+v}\right\}}{u\max\{|1+nm|, \ |m-n|\} + v\min\{|1+nm|, \ |m-n|\}} & , \ 0 < v/u < 1. \end{cases}$$

Proof. Let $\mathbf{a} = d_E(B,C)$ and $\mathbf{h} = d_E(A,BC)$. Then, $\mathcal{A} = \mathbf{ah}/2$.
(i) If BC is parallel to lines $y = mx$ or $y = \frac{-1}{m}x$, then clearly $a = \frac{1}{u}\mathbf{a}$ and $h = \tau(n)\mathbf{h}$,

where

$$\tau(n)=\begin{cases}\dfrac{1}{u} & ,\ u=v \text{ or } v=0\\ \dfrac{1}{\max\{u,u+v\}} & ,\ 0<v/u<1.\end{cases}$$

Hence, $\mathcal{A}=\sigma(n)ah/2$.
(ii) Let BC not be parallel to any one of the coordinate axes, and let the slope of the line BC be n. Then, by Proposition 1 and Proposition 5, $\mathbf{a}=\rho(n)a$, $\mathbf{h}=\tau(n)h$, hence $\mathcal{A}=\rho(n)\tau(n)ah/2$. Since $\rho(n)\tau(n)=\sigma(n)$, we get $\mathcal{A}=\sigma(n)ah/2$. □

3. m Version of Heron's Formula

It is well-known that if ABC is a triangle with the area $\mathcal{A}$ in the Euclidean plane, and $\mathbf{a}=d_E(B,C)$, $\mathbf{b}=d_E(A,C)$, $\mathbf{c}=d_E(A,B)$, and $\mathbf{p}=(\mathbf{a}+\mathbf{b}+\mathbf{c})/2$, then $\mathcal{A}=[\mathbf{p}(\mathbf{p}-\mathbf{a})(\mathbf{p}-\mathbf{b})(\mathbf{p}-\mathbf{c})]^{1/2}$, which is known as *Heron's formula.* In this section, we give an m-version of this formula in terms of m-distance. Clearly, an m-version of Heron's formula for triangle ABC would be an equation that relates the three m-distances a, b and c, where $a=d_\alpha(B,C)$, $b=d_\alpha(A,C)$, $c=d_\alpha(A,B)$, and the area $\mathcal{A}$ of triangle ABC. Here, we give an m-version of Heron's formula that depend on three new parameters in addition to a, b, c and $\mathcal{A}$.

We need following two definitions which is revised according to given in [15] and [13] respectively, to give an m-version of Heron's formula:

Definition 3.1. Let ABC be any triangle in the m-plane. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to lines $y=mx$ or $y=\frac{-1}{m}x$. A line l is called a **base line** of ABC if and only if
(1) l passes through a vertex,
(2) l is parallel to lines $y=mx$ or $y=\frac{-1}{m}x$,
(3) l intersects the opposite side (as a line segment) of the vertex in (1).

Clearly, at least one of vertices of the triangle always has one or two base lines. Such a vertex of the triangle is called a **basic vertex**. A **base segment** is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

Definition 3.2. A line with slope n is called a **steep line**, a **gradual line** and a **separator** if $n>\frac{1+m}{1-m}$ or $n<\frac{-1+m}{1+m}$ or $n\to\infty$, $\frac{-1+m}{1+m}<n<\frac{1+m}{1-m}$ and $n=m$ or $n=\frac{-1}{m}$ for $0\le m\le 1$, respectively.

The following theorem gives an α-version of Heron's formula:

Theorem 3.1. *Let ABC be a triangle with area $\mathcal{A}$ in the m-plane, such that C is a basic vertex, $a=d_m(B,C)$, $b=d_m(A,C)$ and $c=d_m(A,B)$. Let D be the intersection point of a base line and AB, the opposite side of the basic vertex C. Let H_1 and H_2 be orthogonal projections (in the Euclidean sense) of A and B on the base line CD, respectively. Then,*

$$\mathcal{A}=\begin{cases}\frac{l}{2u}\left[\sqrt{1+m^2}(2p-c)-v(l_1+l_2)\right] & ;\ \textit{if } C_1 \textit{ is valid}\\ \frac{l}{2v}\left[\sqrt{1+m^2}(2p-c)-u(l_1+l_2)\right] & ;\ \textit{if } C_2 \textit{ is valid}\\ \frac{l}{2uv}\left[\sqrt{1+m^2}\left(2p-c+(v-1)\,b+(u-1)\,a\right)-\left(v^2l_1+u^2l_2\right)\right] & ;\ \textit{if } C_3 \textit{ is valid}\\ \frac{l}{2uv}\left[\sqrt{1+m^2}\left(2p-c+(u-1)\,b+(v-1)\,a\right)-\left(u^2l_1+v^2l_2\right)\right] & ;\ \textit{if } C_4 \textit{ is valid}\end{cases}$$

where $p=(a+b+c)/2$, $l=d_m(C,D)$, $l_1=d_m(C,H_1)$, $l_2=d_m(C,H_2)$,

C_1 : *lines AC and BC are not gradual and base line CD is horizontal, or lines AC and BC are not steep and base line CD is vertical,*

C_2 : *lines AC and BC are not steep and base line CD is horizontal, or lines AC and BC are not gradual and base line CD is vertical,*

C_3 : *line AC is not gradual, line BC is not steep and base line CD is horizontal, or line AC is not steep, line BC is not gradual and base line CD is vertical,*

C_4 : *line AC is not steep, line BC is not gradual and base line CD is horizontal, or line AC is not gradual, line BC is not steep and base line CD is vertical.*

Proof. Let ABC be a triangle with area $\mathcal{A}$ in the m-plane, such that C is a basic vertex, $a = d_m(B, C)$, $b = d_m(A, C)$ and $c = d_m(A, B)$. Let D be the intersection point of a base line and AB, the opposite side of the basic vertex C. Let H_1 and H_2 be orthogonal projections of A and B on the base line CD, respectively. And let $p = (a + b + c)/2$, $l = d_m(C, D)$, $l_1 = d_m(C, H_1)$, $l_2 = d_m(C, H_2)$, $h_1 = d_m(A, H_1)$, $h_2 = d_m(B, H_2)$. The m-distance between two points is invariant under all translations. If $b/a \neq \sqrt{2} - 1$ in m-plane, the rotations of $\pi/2$, π and $3\pi/2$ radians around a point, and the reflections about the lines parallel to $y = nx + c$ such that $n \in \left\{ m, \frac{-1}{m}, \frac{m-1}{1+m}, \frac{1+m}{1-m} \right\}$ preserve the $m-$distance. If $b/a = \sqrt{2} - 1$ in the $m-$plane, the rotations of $\pi/4$, $\pi/2$, $3\pi/4$, π, $5\pi/4$, $3\pi/2$ and $7\pi/4$ radians around a point, and the reflections about the lines parallel to $y = nx + c$ such that
$$n \in \left\{ m, \frac{-1}{m}, \frac{m-1}{1+m}, \frac{1+m}{1-m}, \frac{(1-\sqrt{2})m-1}{(1-\sqrt{2})+m}, \frac{(1+\sqrt{2})m-1}{(1+\sqrt{2})+m}, \frac{(1-\sqrt{2})m+1}{(1-\sqrt{2})-m}, \frac{(1+\sqrt{2})m+1}{(1+\sqrt{2})-m} \right\}$$
preserve the $m-$distance. Therefore Figure 5 represent all triangles for which C_1 holds, Figure 6 represent all triangles for which C_2 holds, Figure 7 represent all triangles for which C_3 holds, and finally Figure 8 represent all triangles for which C_4 holds.

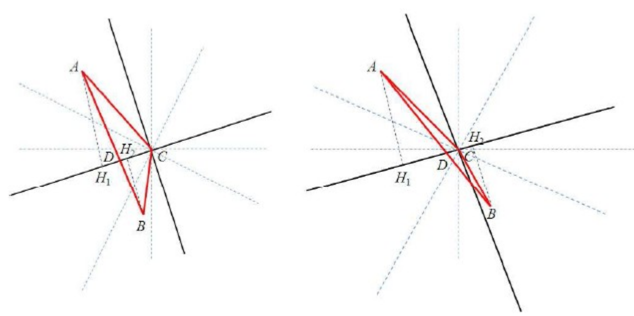

Figure 5

In Figure 5, $a = (uh_2 + vl_2) / \sqrt{1 + m^2}$ and $b = (uh_1 + vl_1) / \sqrt{1 + m^2}$ by m-distance definition. Since $A(ABC) = A(ADC) + A(BDC) = \frac{l}{2}(h_1 + h_2)$, using h_1 and h_2

values, one gets $\mathcal{A} = \dfrac{l}{2u}\left[\sqrt{1+m^2}\,(2p-c) - v(l_1+l_2)\right]$.

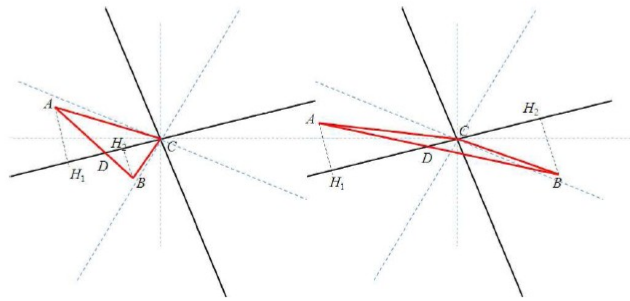

Figure 6

In Figure 6, $a = (ul_2 + vh_2)/\sqrt{1+m^2}$ and $b = (ul_1 + vh_1)/\sqrt{1+m^2}$ by m-distance definition. Since $A(ABC) = A(ADC) + A(BDC) = \frac{l}{2}(h_1+h_2)$, using h_1 and h_2 values, one gets $\mathcal{A} = \dfrac{l}{2v}\left[\sqrt{1+m^2}\,(2p-c) - u(l_1+l_2)\right]$.

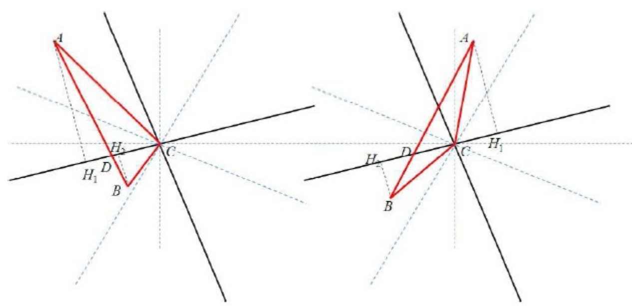

Figure 7

In Figure 7, $a = (ul_2 + vh_2)/\sqrt{1+m^2}$ and $b = (uh_1 + vl_1)/\sqrt{1+m^2}$ by m-distance definition. Since $A(ABC) = A(ADC) + A(BDC) = \frac{l}{2}(h_1+h_2)$, using h_1 and h_2 values, one gets $\mathcal{A} = \dfrac{l}{2uv}\left[\sqrt{1+m^2}\,(2p-c+(v-1)b+(u-1)\,a) - v^2 l_1 - u^2 l_2\right]$.

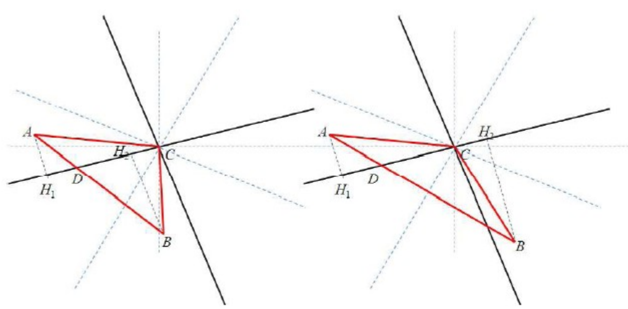

Figure 8

In Figure 8, $a = (uh_2 + vl_2)/\sqrt{1+m^2}$ and $b = (ul_1 + vh_1)/\sqrt{1+m^2}$ by m-distance definition. Since $A(ABC) = A(ADC)+A(BDC) = \frac{l}{2}(h_1+h_2)$, using h_1 and h_2 values, one gets $\mathcal{A} = \dfrac{l}{2uv}\left[\sqrt{1+m^2}\,(2p-c+(u-1)b+(v-1)\,a) - u^2 l_1 - v^2 l_2\right]$. $\square$

Since well-known taxicab, Chinese Checker and $\alpha-$distances are special cases of m-distance for $m = 0$ and $u = v$, $v/u = \sqrt{2} - 1$ and $0 < v/u < 1$, respectively, Theorem 3, Theorem 6 and Theorem 7 give also taxicab, Chinese Checker and $\alpha-$versions of area formulas for a triangle, when $m = 0$ and $u = v$, $v/u = \sqrt{2} - 1$ and $0 < v/u < 1$, respectively, (see [12], [15], [11] and [6]).

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