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FIXED POINT THEOREMS IN CONVEX PARTIAL METRIC SPACES

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Abstract. Partial metric spaces were introduced by S. G. Matthews [1] as a part of the study of denotational semantics of dataflow networks, the author introduced and studied the concept of partial metric space, and obtained a Banach type fixed point theorem on complete partial metric spaces. In this paper, we study some fixed point theorems for self-mappings satisfying certain contraction principles on a convex complete partial metric space, these theorem generalize previously obtained results in convex metric space.

1. Introduction

In 1970, Takahashi [2] introduced the notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. A convex metric space is a generalized space. For example, every normed space and cone Banach space is a convex metric space and convex complete metric space, Subsequently, Beg [3], Beg and Abbas [4, 5], Chang, Kim and Jin [6], Ciric [7], Shimizu and Takahashi [8], Tian [9], Ding [10], and many others studied fixed point theorems in convex metric spaces.

There exist many generalizations of the concept of metric spaces in the literature. In particular, Matthews [1] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, showing that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification.

After that, fixed point results in partial metric spaces were studied by many other authors. Refs. [11,12] are some works in this line of research. The existence of several connections between partial metrics and topological aspects of domain theory have been pointed out in, e.g., [13–18].

The purpose of this paper is to study the existence of a fixed point for self-mappings defined on a nonempty closed convex subset of a convex complete partial metric space that satisfies certain conditions, and knowing that "the partial metric space is a generalization of a metric space" from [13], our result improves and

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extends M. Moosaei result in [19] from a convex complete metric space to a convex complete partial metric space.

2. **Preliminaries**

Definition 2.1. Let X be a nonempty set and let p: $X \times X \to \mathbb{R}^+$ satisfy

$(A1)$ $0 \leq p(x,x) \leq p(x,y)$ (nonnegativity and small self-distances),

$(A2)$ $x = y \Longleftrightarrow p(x,x) = p(y,y) = p(x,y)$ (indistancy implies equality),

$(A3)$ $p(x,y) = p(y,x)$ (symmetry), and

$(A4)$ $p(x,y) \leq p(x,z) + p(y,z) - p(z,z)$ (triangularity).

for all x,y and z $\in X$. then tha pair (X,p) is called a partial matric space and p is called a partial metric on X.

Remark 2.1. It is clear that, if $p(x,y) = 0$, then from $(A1)$ and $(A2)$, $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls

$\{B_p(x,\varepsilon), x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + B_p(x,\varepsilon)\}$ for all $x \in X$ and $\varepsilon > 0$.

For a partial metric p on X, the function $d_p : X \times X \to \mathbb{R}^+$ defined as

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

satisfies the conditions of a metric on X; therefore it is a (usual) metric on X.

Example 2.1. Let $max(a,b)$ be the maximum of any two nonnegative real numbers a and b; then max is a partial metric over $\mathbb{R}^+ = [0,\infty)$.

Example 2.2. If $X := \{[a,b] \,/\, a,b \in \mathbb{R},\ a \leq b\}$ then

$$p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$$

defines a partial metric p on X.

Example 2.3. If (X,d) is a metric space and $c \geq 0$ is arbitrary, then

$$p(x,y) = d(x,y) + c$$

defines a partial metric on X and the corresponding metric is $d_p(x,y) = 2d(x,y)$.

Definition 2.2. Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = lim_{n\to+\infty}p(x,x_n)$,

(ii) $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $lim_{n,m\to+\infty}p(x_n,x_m)$.

Definition 2.3. A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, such that $p(x,x) = lim_{n,m\to+\infty}p(x_n,x_m)$.

Remark 2.2. It is easy to see that every closed subset of a complete partial metric space is complete.

Theorem 2.1. [1]. *Let f be a mapping of a complete partial metric space (X, p) into itself such that there is a real number c with $0 \leq c < 1$, satisfying for all $x, y \in X$:*

$$p(fx, fy) < cp(x, y).$$

Then f has a unique fixed point.

Definition 2.4. Let (X, p) be a partial metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$p(u, W(x, y, \lambda)) \leq \lambda p(u, x) + (1 - \lambda)p(u, y)$$

A metric space (X, p) together with a convex structure W is called a convex partial metric space, which is denoted by (X, p, W).

Example 2.4. Let $(X, |.|)$ is a metric space, then

$$p(x, y) = \frac{|x - y| + x + y}{2}$$

defines a partial metric p on X and it can shown that (X, p) is a convex partial metric space.

Definition 2.5. Let (X, p, W) be a convex partial metric space. A nonempty subset C of X is said to be convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times I$.

Definition 2.6. Let $f : X \to X$. A point $x \in X$ is called a fixed point of f if $f(x) = x$.

F(f), C(f, g), and F(f, g) denote the set of all fixed points of f, coincidence points of the pair (f, g), and common fixed points of the pair (f, g), respectively.

Theorem 2.2. [19] . *Let C be a nonempty closed convex subset of a convex complete metric space (X, d, W) and f be a self-mapping of C. If there exist a, b, c, k such that*

$$2b - |c| \leq k < 2(a + b + c) - |c|,$$

$$ad(x, f(x)) + bd(y, f(y)) + cd(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in C$, then f has at least one fixed point.

3. **Main result**

The following theorem improves and extends Theorem 3.2 in **[19].**

Theorem 3.1. *Let C be a nonempty closed convex subset of a* ***convex complete partial metric space*** (X, p, W) *and f be a self-mapping of C. If there exist k such that*

$$0 \leq k < \frac{1}{4},$$

$$p(x, f(y)) + p(f(x), f(y)) \leq kp(y, f(x)) \tag{3.1}$$

for all $x, y \in C$, then f has at least one fixed point.

Proof. From definition 4 and by using $(A1)$ and $(A3)$,we have

$$p(x, W(x, y, \frac{1}{2})) \leq \frac{1}{2}p(x, x) + \frac{1}{2}p(x, y), \tag{3.2}$$

$$p(y, W(x, y, \frac{1}{2})) \leq \frac{1}{2}p(y, y) + \frac{1}{2}p(x, y). \tag{3.3}$$

so, we find

$$p(x, W(x, y, \frac{1}{2})) + p(y, W(x, y, \frac{1}{2})) \leq 2p(x, y). \tag{3.4}$$

Suppose $x_0 \in C$ is arbitrary. We define a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$x_n = W(x_{n-1}, f(x_{n-1}, \frac{1}{2}), \ n = 1, ... \tag{3.5}$$

As C is convex, $x_n \in C$ for all $n \in \mathbb{N}$. From (3.5) and (3.6) , we have

$$p(x_n, x_{n+1}) + p(x_{n+1}, f(x_n)) \leq 2p(x_n, f(x_n)). \tag{3.6}$$

By using $(A4)$ and $(A1)$,we have

$$\begin{aligned} p(x_n, f(x_n)) &\leq p(x_n, f(x_{n-1})) + p(f(x_n), f(x_{n-1})) - p(f(x_{n-1}), f(x_{n-1})) \\ &\leq p(x_n, f(x_{n-1})) + p(f(x_n), f(x_{n-1})) \end{aligned}$$

then, we get

$$2p(x_n, f(x_n)) - 2p(x_n, f(x_{n-1})) \leq 2p(f(x_n), f(x_{n-1})), \tag{3.7}$$

from (3.7) and (3.8) , we obtain

$$p(x_n, x_{n+1}) + p(x_{n+1}, f(x_n)) - 2p(x_n, f(x_{n-1})) \leq 2p(f(x_n), f(x_{n-1})). \tag{3.8}$$

For all $n \in \mathbb{N}$. Now by substituting x with x_n and y with x_{n-1} in (3.2) , we get

$$p(x_n, f(x_{n-1})) + p(f(x_n), f(x_{n-1}) \leq kp(x_{n-1}, f(x_n)),$$

for all $n \in \mathbb{N}$. Therefore, from (3.9) , it follows that

$$p(x_n, x_{n+1}) + p(x_{n+1}, f(x_n)) \leq 2kp(x_{n-1}, f(x_n)), \tag{3.9}$$

from $(A4)$ and $(A1)$, we have

$$p(x_{n-1}, f(x_n) \leq p(x_n, f(x_n)) + p(x_{n-1}, x_n),$$

by this inequality and (3.10) , we get

$$p(x_n, f(x_n)) \leq p(x_n, x_{n+1}) + p(x_{n+1}, f(x_n)) \leq 2kp(x_n, f(x_n)) + 2kp(x_{n-1}, x_n),$$

so, we obtain

$$(1 - 2k)\, p(x_n, f(x_n)) \leq 2kp(x_{n-1}, x_n). \tag{3.10}$$

Now, from (3.3) and (3.6) , we have

$$p(x_n, f(x_n)) \geq 2p(x_n, x_{n+1}) - p(x_n, x_n)$$

for all $n \in \mathbb{N}$. from (3.11) , it follows that

$$(2 - 4k)\, p(x_n, x_{n+1}) \leq 2kp(x_{n-1}, x_n) + (1 - 2k)\, p(x_n, x_n)$$

by using $(A1)$, we obtain

$$p(x_n, x_{n+1}) \leq \frac{1}{2 - 4k} p(x_{n-1}, x_n). \tag{3.11}$$

for all $n \in \mathbb{N}$. from (3.11), $\frac{1}{2-4k} \in [0,1)$, and hence, $\{x_n\}_{n=1}^{\infty}$ is a contraction sequence in C. Therefore, it is a cauchy sequence. Since C is a closed subset of a complete space, there exists $v \in C$ such that $lim_{n\to+\infty} x_n = v$.

Therefore, by using (3.4) and (3.6), we have

$$p(x_n, f(x_{n-1})) \leq \frac{1}{2} p(f(x_{n-1}), f(x_{n-1})) + \frac{1}{2} p(x_{n-1}, f(x_{n-1})),$$

we put $lim_{n\to+\infty} f(x_{n-1}) = \alpha$, letting $n \to +\infty$ in the above inequality, it follows that

$$p(v,\alpha) \leq \frac{1}{2} p(\alpha,\alpha) + \frac{1}{2} p(v,\alpha) \Longrightarrow p(v,\alpha) \leq p(\alpha,\alpha) \tag{3.12}$$

and from $(A1)$, we have

$$p(\alpha,\alpha) \leq p(v,\alpha). \tag{3.13}$$

Then from (3.13) and (3.14), we obtain $(v,\alpha) = p(\alpha,\alpha)$, so $lim_{n\to+\infty} f(x_n) = v$. Now by substituting x with v and y with x_n in (3.2), we obtain

$$p(v, f(x_n)) + p(f(v), f(x_n) \leq kp(x_n, f(v))$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in the above inequality, it follows that

$$p(v,v) + (1-k)\, p(v, f(v)) \leq 0.$$

Since $(1-k)$ is positive from (3.1) and from $(A2)$, we get

$$p(v,v) = p(v, f(v)) = 0,$$

then from $(A2)$, this implies $f(v) = v$, and the proof of the theorem is complete. □

Corollary 3.1. *From theorem 3, we deduce that for all x, $y \in C$, then F(f) is a nonempty set.*

Corollary 3.2. *Let $(X,\ d, W)$ be a convex complete partial metric space and C be a nonempty subset of X. Suppose that f, g are self-mappings of C, and there exist a, b, c, k such that*

$$0 \leq k < \frac{1}{4}, \tag{3.14}$$

$$p(g(x), f(y)) + p(f(x), f(y)) \leq kp(g(y), f(x)) \tag{3.15}$$

for all x, $y \in C$. If g has the property

$$g(W(x,y,\lambda)) = W(g(x), g(y), \lambda) \text{ for each } x,y \in C \text{ and } \lambda \in I = [0,1].$$

and F(g) is a nonempty closed subset of C, then F(f,g) is nonempty.

References

[1] Matthews, SG: Partial metric topology. In: Proc. 8th Summer Conference on General Topology and Applications. Ann. New York Acad. Sci., vol. 728, pp. 183-197 (1994).

[2] Takahashi, T: A convexity in metric spaces and nonexpansive mapping I. Kodai Math. Sem. Rep. 22, 142-149(1970).

[3] Beg, I: An iteration scheme for asymptotically nonexpansive mappings on uniformly convex metric spaces. Nonlinear Analysis Forum. 6:1, 27-34(2001).

[4] Beg, I, Abbas, M: Common fixed points and best approximation in convex metric spaces. Soochow Journal of Mathematics. 33:4, 729-738(2007).

[5] Beg, I, Abbas, M: Fixed-point theorem for weakly inward multivalued maps on a convex metric space. Demonstratio Mathematica. 39:1, 149-160(2006).

[6] Chang, SS, Kim, JK, Jin, DS: Iterative sequences with errors for asymptotically quasi nonexpansive mappings in convex metric spaces. Arch. Inequal. Appl. 2, 365-374(2004).
[7] Ciric, L: On some discontinuous fixed point theorems in convex metric spaces. Czech. Math. J. 43:188, 319-326(1993).
[8] Shimizu, T, Takahashi, W: Fixed point theorems in certain convex metric spaces. Math. Japon. 37, 855-859(1992).
[9] Tian, YX: Convergence of an Ishikawa type iterative scheme for asymptotically quasi nonexpansive mappings. Computers and Maths. with Applications. 49, 1905-1912(2005).
[10] Ding, XP: Iteration processes for nonlinear mappings in convex metric spaces. J. Math. Anal. Appl. 132, 114-122(1998).
[11] I. Altun, F. Sola, H. Simsek, Generalized contractions on partial metric spaces, Topology Appl. 157 (18) (2010) 2778–2785.
[12] S. Romaguera, A kirk type characterization of completeness for partial metric spaces, Fixed Point Theory Appl. (2010) doi:10.1155/2010/493298. 6
[13] M. Bukatin, R. Kopperman, S. Matthews, H. Pajoohesh, Partial metric spaces, Amer. Math. Monthly 116 (2009) 708–718.
[14] A.F. Rabarison, in: Hans-Peter A. Künzi (Ed.), Partial Metrics, African Institute for Mathematical Sciences, 2007, Supervised.
[15] M.A. Bukatin, S.Yu. Shorina, Partial metrics and co-continuous valuations, in: M. Nivat, et al. (Eds.), Foundations of Software Science and Computation Structure, in: Lecture Notes in Computer Science, vol. 1378, Springer, 1998, pp. 125–139.
[16] S.G. Matthews, An extensional treatment of lazy data flow deadlock, Theoret. Comput. Sci. 151 (1995) 195–205.
[17] S. Romaguera, M. Schellekens, Duality and quasi-normability for complexity spaces, Appl. Gen. Topol. 3 (2002) 91–112.
[18] M.P. Schellekens, The correspondence between partial metrics and semivaluations, Theoret. Comput. Sci. 315 (2004) 135–149.
[19] M. Moosaei, Fixed Point Theorems in Convex Metric Spaces, Fixed Point Theory and Applications 2012, 2012:164 doi:10.1186/1687-1812-2012-164 Published: 25 September 2012.

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