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AN L^p HEISENBERG-PAULI-WEYL UNCERTAINTY PRINCIPLE FOR THE DUNKL TRANSFORM

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Abstract. In this paper, we give a generalization of the Heisenberg-Pauli-Weyl uncertainty inequality for the Dunkl transform on $\mathbb{R}^d$ in L^p-norm.

1. Introduction and preliminaries

In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle .,. \rangle$ and norm $|y| := \sqrt{\langle y, y\rangle}$. For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha y := y - \frac{2\langle \alpha, y\rangle}{|\alpha|^2}\alpha.$$

A finite set $\Re \subset \mathbb{R}^d \backslash \{0\}$ is called a root system, if $\Re \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \Re = \Re$ for all $\alpha \in \Re$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in \Re$. For a root system $\Re$, the reflections σ_α, $\alpha \in \Re$, generate a finite group $G \subset O(d)$, the reflection group associated with $\Re$. All reflections in G, correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \backslash \bigcup_{\alpha\in\Re} H_\alpha$, we fix the positive subsystem $\Re_+ := \{\alpha \in \Re : \langle \alpha, \beta\rangle > 0\}$. Then for each $\alpha \in \Re$ either $\alpha \in \Re_+$ or $-\alpha \in \Re_+$.

Let $k : \Re \to \mathbb{C}$ be a multiplicity function on $\Re$ (that is, a function which is constant on the orbits under the action of G). As an abbreviation, we introduce the index $\gamma = \gamma_k := \sum_{\alpha\in\Re_+} k(\alpha)$.

Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in \Re$. Moreover, let w_k denote the weight function $w_k(y) := \prod_{\alpha\in\Re_+} |\langle \alpha, y\rangle|^{2k(\alpha)}$, for all $y \in \mathbb{R}^d$, which is G-invariant and homogeneous of degree 2γ.

The Dunkl operators $\mathcal{D}_j$; $j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given, for a function f of class C^1 on $\mathbb{R}^d$,

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by

$$\mathcal{D}_j f(y) := \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha\in\Re_+} k(\alpha)\alpha_j \frac{f(y) - f(\sigma_\alpha y)}{\langle \alpha, y\rangle}.$$

For $y \in \mathbb{R}^d$, the initial problem $\mathcal{D}_j u(.,y)(x) = y_j u(x,y)$, $j = 1,...,d$, with $u(0,y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x,y)$ and called Dunkl kernel [4, 7]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$. In our case, $|E_k(-ix,y)| \leq 1$, for all $x, y \in \mathbb{R}^d$.

Let c_k be the Mehta-type constant given by $c_k := (\int_{\mathbb{R}^d} e^{-|y|^2/2} w_k(y)\mathrm{d}y)^{-1}$. We denote by μ_k the measure on $\mathbb{R}^d$ given by $\mathrm{d}\mu_k(y) := c_k w_k(y)\mathrm{d}y$; and by $L^p(\mu_k)$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^p(\mu_k)} := \Big(\int_{\mathbb{R}^d} |f(y)|^p \mathrm{d}\mu_k(y)\Big)^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_{L^\infty(\mu_k)} := \operatorname*{ess\,sup}_{y\in\mathbb{R}^d} |f(y)| < \infty.$$

If $f \in L^1(\mu_k)$ with $f(x) = F(|x|)$, then

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_k(x) = \frac{1}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})} \int_0^\infty F(t)t^{2\gamma+d-1}\mathrm{d}t. \tag{1.1}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and was introduced by Dunkl in [5], where already many basic properties were established. Dunkl's results were completed and extended later by de Jeu [7]. The Dunkl transform of a function f in $L^1(\mu_k)$, is

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix,y) f(y)\mathrm{d}\mu_k(y), \quad x \in \mathbb{R}^d.$$

Some of the properties of Dunkl transform $\mathcal{F}_k$ are collected bellow (see [5, 7]).

(a) $L^1 - L^\infty$-boundedness. For all $f \in L^1(\mu_k)$, $\mathcal{F}_k(f) \in L^\infty(\mu_k)$ and

$$\|\mathcal{F}_k(f)\|_{L^\infty(\mu_k)} \leq \|f\|_{L^1(\mu_k)}. \tag{1.2}$$

(b) Inversion theorem. Let $f \in L^1(\mu_k)$, such that $\mathcal{F}_k(f) \in L^1(\mu_k)$. Then

$$f(x) = \mathcal{F}_k\Big(\mathcal{F}_k(f)\Big)(-x), \quad \text{a.e.} \quad x \in \mathbb{R}^d.$$

(c) Plancherel theorem. The Dunkl transform $\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2(\mu_k)$ onto itself. In particular,

$$\|f\|_{L^2(\mu_k)} = \|\mathcal{F}_k(f)\|_{L^2(\mu_k)}. \tag{1.3}$$

Using relations (1.2) and (1.3) with Marcinkiewicz's interpolation theorem [10, 11], we deduce that for every $1 \leq p \leq 2$, and for every $f \in L^p(\mu_k)$, the function $\mathcal{F}_k(f)$ belongs to the space $L^q(\mu_k)$, $q = p/(p-1)$, and

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \|f\|_{L^p(\mu_k)}. \tag{1.4}$$

Many uncertainty principles have already been proved for the Dunkl transform, namely by Rösler [8] and Shimeno [9] who established the Heisenberg-Pauli-Weyl inequality for the Dunkl transform, by showing that for every $f \in L^2(\mu_k)$,

$$\|f\|^2_{L^2(\mu_k)} \leq \frac{2}{2\gamma+d} \| \, |x|f\|_{L^2(\mu_k)} \| \, |y|\mathcal{F}_k(f)\|_{L^2(\mu_k)}. \tag{1.5}$$

Building on the techniques of Ciatti et al. [1] we show a general form of the Heisenberg-Pauli-Weyl inequality for the Dunkl transform $\mathcal{F}_k$. More precisely, we prove that for all $f \in L^p(\mu_k)$, $1 < p \leq 2$, $q = p/(p-1)$ and $0 < a < (2\gamma + d)/q$, $b > 0$,

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq C(a,b) \| \, |x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+b}} \| \, |y|^b \mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+b}}, \tag{1.6}$$

where $C(a,b)$ is a positive constant. This inequality generalizes the Heisenberg-Pauli-Weyl inequality given by (1.5); and in the case $k = 0$ and $q = 2$, this inequality is due to Cowling-Price [2] and Hirschman [6].

We shall use the Heisenberg-Pauli-Weyl principle (1.6); and building on the techniques of Donoho and Stark [3], we show a continuous-time principle for the L^p theory, when $1 < p \leq 2$.

This paper is organized as follows. In Section 2 we list some basic properties of the Dunkl transform $\mathcal{F}_k$. In Section 3 we prove a general form of the Heisenberg-Pauli-Weyl inequality for $\mathcal{F}_k$. The last section is devoted to Donoho-Stark's uncertainty principle for the Dunkl transform $\mathcal{F}_k$ in the L^p theory, when $1 < p \leq 2$.

2. L^p Heisenberg-Pauli-Weyl inequality

In this section, we extend the Heisenberg-Pauli-Weyl uncertainty principle (1.5) to more general case. We need to use the method of Ciatti et al. [1], which is the counterpart in the Euclidean case. We begin by the following lemma.

Lemma 2.1. *Let $1 < p \leq 2$, $q = p/(p-1)$ and $0 < a < (2\gamma + d)/q$. Then for all $f \in L^p(\mu_k)$ and $t > 0$,*

$$\|e^{-t|y|^2} \mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \Big(1 + \frac{a_k}{(2q)^{(\gamma+\frac{d}{2})\frac{1}{q}}}\Big) t^{-a/2} \| \, |x|^a f\|_{L^p(\mu_k)}, \tag{2.1}$$

where

$$a_k = \Big[(2\gamma + d - qa) 2^{\gamma+\frac{d}{2}-1} \Gamma(\gamma + \frac{d}{2})\Big]^{-1/q}.$$

Proof. Inequality (2.1) holds if $\| \, |x|^a f\|_{L^p(\mu_k)} = \infty$. Assume that $\| \, |x|^a f\|_{L^p(\mu_k)} < \infty$. For $r > 0$, let $B_r = \{x : \ |x| < r\}$ and $B_r^c = \mathbb{R}^d \backslash B_r$. Denote by χ_{B_r} and $\chi_{B_r^c}$ the characteristic functions. Let $f \in L^p(\mu_k)$, $1 < p \leq 2$ and let $q = p/(p-1)$. Since $|(f\chi_{B_r^c})(x)| \leq r^{-a}|x|^a|f(x)|$, then by (1.4),

$$\begin{aligned} \|e^{-t|y|^2} \mathcal{F}_k(f\chi_{B_r^c})\|_{L^q(\mu_k)} &\leq \|e^{-t|y|^2}\|_{L^\infty(\mu_k)} \|\mathcal{F}_k(f\chi_{B_r^c})\|_{L^q(\mu_k)} \\ &\leq \|f\chi_{B_r^c}\|_{L^p(\mu_k)} \leq r^{-a} \| \, |x|^a f\|_{L^p(\mu_k)}. \end{aligned}$$

On the other hand, by (1.2) and Hölder's inequality,

$$\begin{aligned} \|e^{-t|y|^2} \mathcal{F}_k(f\chi_{B_r})\|_{L^q(\mu_k)} &\leq \|e^{-t|y|^2}\|_{L^q(\mu_k)} \|\mathcal{F}_k(f\chi_{B_r})\|_{L^\infty(\mu_k)} \\ &\leq \|e^{-t|y|^2}\|_{L^q(\mu_k)} \|f\chi_{B_r}\|_{L^1(\mu_k)} \\ &\leq \|e^{-t|y|^2}\|_{L^q(\mu_k)} \| \, |x|^{-a}\chi_{B_r}\|_{L^q(\mu_k)} \| \, |x|^a f\|_{L^p(\mu_k)}. \end{aligned}$$

By (1.1), we have $\|e^{-t|y|^2}\|_{L^q(\mu_k)} = \frac{1}{(2q)^{(\gamma+\frac{d}{2})\frac{1}{q}}} t^{-(\gamma+\frac{d}{2})\frac{1}{q}}$ and $\| \, |x|^{-a}\chi_{B_r}\|_{L^q_k} = a_k r^{-a+(2\gamma+d)/q}$. Hence,

$$\|e^{-t|y|^2} \mathcal{F}_k(f\chi_{B_r})\|_{L^q_k} \leq \frac{a_k}{(2q)^{(\gamma+\frac{d}{2})\frac{1}{q}}} r^{-a+(2\gamma+d)/q} t^{-(\gamma+\frac{d}{2})\frac{1}{q}} \| \, |x|^a f\|_{L^p(\mu_k)},$$

and

$$\begin{aligned}\|e^{-t|y|^2}\mathcal{F}_k(f)\|_{L^q(\mu_k)} &\leq \|e^{-t|y|^2}\mathcal{F}_k(f\chi_{B_r})\|_{L^q(\mu_k)} + \|e^{-t|y|^2}\mathcal{F}_k(f\chi_{B_r^c})\|_{L^q(\mu_k)} \\ &\leq r^{-a}\Big(1+\frac{a_k}{(2q)^{(\gamma+\frac{d}{2})\frac{1}{q}}}r^{(2\gamma+d)/q}t^{-(\gamma+\frac{d}{2})\frac{1}{q}}\Big)\|\,|x|^a f\|_{L^p(\mu_k)}.\end{aligned}$$

Choosing $r = t^{1/2}$, we obtain (2.1). □

Theorem 2.1. *Let $1 < p \leq 2$, $q = p/(p-1)$, $0 < a < (2\gamma+d)/q$ and $b > 0$, then for all $f \in L^p(\mu_k)$,*

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq C(a,b)\|\,|x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+b}}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+b}}, \tag{2.2}$$

where $C(a,b)$ is a positive constant.

Proof. Let $f \in L^p(\mu_k)$, $1 < p \leq 2$, such that $\|\,|x|^a f\|_{L^p(\mu_k)} + \|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)} < \infty$. Assume that $0 < a < (2\gamma+d)/q$ and $b \leq 2$. By Lemma 2.1, for all $t > 0$,

$$\begin{aligned}\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} &\leq \|e^{-t|y|^2}\mathcal{F}_k(f)\|_{L^q(\mu_k)} + \|(1-e^{-t|y|^2})\mathcal{F}_k(f)\|_{L^q(\mu_k)} \\ &\leq \Big(1+\frac{a_k}{(2q)^{(\gamma+\frac{d}{2})\frac{1}{q}}}\Big)t^{-a/2}\|\,|x|^a f\|_{L^p(\mu_k)} + \|(1-e^{-t|y|^2})\mathcal{F}_k(f)\|_{L^q(\mu_k)}.\end{aligned}$$

On the other hand,

$$\|(1-e^{-t|y|^2})\mathcal{F}_k(f)\|_{L^q(\mu_k)} = t^{b/2}\|(t|y|^2)^{-b/2}(1-e^{-t|y|^2})|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

Since $(1-e^{-t})t^{-b/2}$ is bounded for $t \geq 0$ if $b \leq 2$. Hence,

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq C\left(t^{-a/2}\|\,|x|^a f\|_{L^p(\mu_k)} + t^{b/2}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}\right).$$

We choose $t = \left(\frac{a}{b}\frac{\|\,|x|^a f\|_{L^p(\mu_k)}}{\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}}\right)^{\frac{2}{a+b}}$, we obtain the result

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq C\|\,|x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+b}}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+b}}, \quad \text{for all } b \leq 2. \tag{2.3}$$

If $b > 2$. For $u \geq 0$, $u \leq 1 + u^b$ which for $u = \frac{|y|}{\varepsilon}$ gives the inequality $\frac{|y|}{\varepsilon} \leq 1 + \left(\frac{|y|}{\varepsilon}\right)^b$, for all $\varepsilon > 0$. It follows that

$$\|\,|y|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \varepsilon\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} + \varepsilon^{1-b}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

We choose $\varepsilon = (b-1)^{1/b}\left(\frac{\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}}{\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}}\right)^{1/b}$, we get

$$\|\,|y|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \frac{b}{b-1}(b-1)^{1/b}\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{b-1}{b}}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{1/b} \tag{2.4}$$

Then, by (2.3) and (2.4) we obtain

$$\begin{aligned}\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} &\leq C\|\,|x|^a f\|_{L^p(\mu_k)}^{\frac{1}{a+1}}\|\,|y|\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+1}} \\ &\leq C\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a(b-1)}{b(a+1)}}\||x|^a f\|_{L^p(\mu_k)}^{\frac{1}{a+1}}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{b(a+1)}}.\end{aligned}$$

Thus,

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a+b}{b(a+1)}} \leq C\|\,|x|^a f\|_{L^p(\mu_k)}^{\frac{1}{a+1}}\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{b(a+1)}},$$

which gives the result for $b > 2$. □

Remark 2.1. When $q = 2$, by (1.3) we obtain

$$\|f\|_{L^2(\mu_k)} \le C(a,b)\| \, |x|^a f\|_{L^2(\mu_k)}^{\frac{b}{a+b}} \| \, |y|^b \mathcal{F}_k(f)\|_{L^2(\mu_k)}^{\frac{a}{a+b}},$$

which is the general case of the inequality (1.5) proved by Rösler [8] and Shimeno [9].

Now, we give application of the L^p Heisenberg-Pauli-Weyl inequality to the Donoho-Stark uncertainty principle.

Let E be measurable subset of $\mathbb{R}^d$. We introduce the partial sum operator S_E by

$$\mathcal{F}_k(S_E f) = \mathcal{F}_k(f)\chi_E \tag{2.5}$$

Let $b > 0$. We say that a function $f \in L^p(\mu_k)$, $1 \le p \le 2$, is $|y|^b\mathcal{F}_k(f)$ is ε-concentrated to E in $L^q(\mu_k)$-norm, $q = p/(p-1)$, if there is a function $h(y)$ vanishing outside E with $\| \, |y|^b\mathcal{F}_k(f) - h\|_{L^q(\mu_k)} \le \varepsilon \| \, |y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}$.

From (2.5) it follows that $|y|^b\mathcal{F}_k(f)$ is ε_E-concentrated to E in $L^q(\mu_k)$-norm, $q = p/(p-1)$, if and only if

$$\| \, |y|^b\mathcal{F}_k(f) - |y|^b\mathcal{F}_k(S_E f)\|_{L^q(\mu_k)} \le \varepsilon_E \| \, |y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)} \tag{2.6}$$

It is useful to have uncertainty principle for the $L^p(\mu_k)$-norm.

Theorem 2.2. *Let E be measurable subset of $\mathbb{R}^d$; and let $1 < p \le 2$, $q = p/(p-1)$, $f \in L^p(\mu_k)$ and $b > 0$. If $|y|^b\mathcal{F}_k(f)$ is ε_E-concentrated to E in $L^q(\mu_k)$-norm, then for $0 < a < (2\gamma + d)/q$:*

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \le \frac{C(a,b)}{(1-\varepsilon_E)^{\frac{a}{a+b}}} \| \, |x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+b}} \| \, |y|^b\mathcal{F}_k(f)\chi_E\|_{L^q(\mu_k)}^{\frac{a}{a+b}},$$

where $C(a,b)$ is the constant given by (2.2).

Proof. Let $f \in L^p(\mu_k)$, $1 < p \le 2$. Since $|y|^b\mathcal{F}_k(f)$ is ε_E-concentrated to E in $L^q(\mu_k)$-norm, $q = p/(p-1)$, then by (2.6),

$$\| \, |y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)} \le \varepsilon_E \| \, |y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)} + \| \, |y|^b\mathcal{F}_k(f)\chi_E\|_{L^q(\mu_k)}.$$

Thus,

$$\| \, |y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+b}} \le \frac{1}{(1-\varepsilon_E)^{\frac{a}{a+b}}} \| \, |y|^b\mathcal{F}_k(f)\chi_E\|_{L^q(\mu_k)}^{\frac{a}{a+b}}.$$

Multiply this inequality by $C(a,b)\| \, |x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+b}}$ and applying Theorem 2.1 we deduce the desired inequality. □

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