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On the Inner-Product Spaces of Complex Interval Sequences

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Abstract
In recent years, there has been increasing interest in interval analysis. Thanks to interval numbers, many real world problems have been modeled and analyzed. Especially, complex intervals have an important place for interval-valued data and interval-based signal processing. In this paper, firstly we introduce the notion of a complex interval sequence and we present the complex interval sequence spaces $\mathbb{I}(w)$ and $\mathbb{I}(l_p)$, $1 \leq p < \infty$. Secondly, we show that these sequence spaces have an algebraic structure called quasilinear space. Further, we construct an inner-product on $\mathbb{I}(l_2)$ and we show that $\mathbb{I}(l_2)$ is an inner-product quasilinear space.

Keywords: Complex interval, Complex interval sequence, Consolidate space, Inner product quasilinear spaces
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1. Introduction

In many real life situations, it is very difficult to deal with a process with reliable information about the properties of the expected variations. This has naturally led to an increased interest in intervals. Because the most ideal way to represent the loss of information is to use intervals.

An interval x is the compact-convex subset of real numbers and x is denoted by $x = [\underline{x}, \overline{x}]$ where $\underline{x}$ and $\overline{x}$ are the left and right endpoints of x, respectively [1]. Further, if $\underline{x} = \overline{x}$ then we say that x is a degenerate interval and it can be shown by $\{x\}$ or $[x,x]$. The set of all real intervals is denoted by $\mathbb{I}_{\mathbb{R}}$.

The idea of using intervals has been highly preferred by many researchers recently [1]-[4]. The interval sequence spaces have been studied by many authors [5, 6]. Also, we presented the notion of a complex interval which is significant for interval-valued data and interval-based signal processing in [7]. A complex interval is defined by

$$X = \left[\underline{x_r}, \overline{x_r}\right] + i\left[\underline{x_s}, \overline{x_s}\right]$$

where $\left[\underline{x_r}, \overline{x_r}\right]$ and $\left[\underline{x_s}, \overline{x_s}\right]$ are real intervals and $i = \sqrt{-1}$ is the complex unit. $\left[\underline{x_r}, \overline{x_r}\right]$ and $\left[\underline{x_s}, \overline{x_s}\right]$ are called real and imaginary part of X, respectively. Further, $\left[\underline{x_r}, \overline{x_r}\right]$ and $\left[\underline{x_s}, \overline{x_s}\right]$ are called real and imaginary part of X, respectively.

In this work, we introduce the notion of complex interval sequence and we analyze some sequence spaces of the complex intervals, e.g., $\mathbb{I}(w)$ and $\mathbb{I}(l_p)$, $1 \leq p < \infty$. However, each element of these sequence spaces does not have an inverse according to the addition operation. These sequence spaces are not a linear space and the algebraic structure on these spaces is called as "quasilinear space". In 1986, Aseev defined the concept of quasilinear space [8]. Further, he present an approach for analysis of set-valued functions. This work has motivated a lot of authors the introduce new results on set-valued analysis [9]-[12]. Let us give the definition:

A set $\mathscr{X}$ is called a quasilinear space on field $\mathbb{K}$ if a partial order relation "$\preceq$", an algebraic sum operation, and an operation of multiplication by real or complex numbers are defined in it in such a way that the following conditions hold for any elements $x,y,z,v \in X$ and any $\alpha,\beta \in \mathbb{K}$:

$x \preceq x$,

$x \preceq z$ if $x \preceq y$ and $y \preceq z$,

$x = y$ if $x \preceq y$ and $y \preceq x$,

$x + y = y + x$,

$x + (y+z) = (x+y) + z$,

there exists an element (zero) $\theta \in \mathscr{X}$ such that $x + \theta = x$,

$\alpha(\beta x) = (\alpha\beta)x$,

$\alpha(x+y) = \alpha x + \alpha y$,

$1x = x$,

$0x = \theta$,

$(\alpha+\beta)x \preceq \alpha x + \beta x$,

$x + z \preceq y + v$ if $x \preceq y$ and $z \preceq v$,

$\alpha x \preceq \alpha y$ if $x \preceq y$.

The most popular examples are $\Omega(E)$ and $\Omega_C(E)$ which are defined as the sets of all non-empty closed bounded and non-empty convex closed bounded subsets of any normed linear space E, respectively. Both are a quasilinear space with the inclusion relation"$\subseteq$", the algebraic sum operation

$$A + B = \overline{\{a + b : a \in A,\ b \in B\}}$$

where the closure is taken on the norm topology of E and the real-scalar multiplication

$$\lambda A = \{\lambda a : a \in A\}.$$

Actually, $\Omega_C(\mathbb{R})$ is the set $\mathbb{I}_{\mathbb{R}}$ and for $x,y \in \mathbb{I}_{\mathbb{R}}$ and $\lambda \in \mathbb{R}$, the Minkowski sum and scalar multiplication operations are defined by

$$x + y = [\underline{x},\overline{x}] + [\underline{y},\overline{y}] = [\underline{x}+\underline{y},\overline{x}+\overline{y}]$$

and

$$\lambda x = \begin{cases} [\lambda\underline{x},\lambda\overline{x}] & , \quad \lambda \geq 0 \\ [\lambda\overline{x},\lambda\underline{x}] & , \quad \lambda < 0, \end{cases}$$

respectively. Further, the product of two intervals $x = [\underline{x},\overline{x}]$ and $y = [\underline{y},\overline{y}]$ is given by

$$x \cdot y = [\underline{x},\overline{x}]\,[\underline{y},\overline{y}] = [\min S, \max S] \tag{1.1}$$

where $S = \{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}$, [1].

The Minkowski sum and scalar multiplication on $\mathbb{I}_{\mathbb{C}}$ are defined by

$$\begin{aligned} X + Y &= [\underline{x_r},\overline{x_r}] + i\,[\underline{x_s},\overline{x_s}] + [\underline{y_r},\overline{y_r}] + i\,[\underline{y_s},\overline{y_s}] \\ &= [\underline{x_r}+\underline{y_r},\overline{x_r}+\overline{y_r}] + i\,[\underline{x_s}+\underline{y_s},\overline{x_s}+\overline{y_s}] \\ &= \left\{a + ib : a \in [\underline{x_r}+\underline{y_r},\overline{x_r}+\overline{y_r}],\ b \in [\underline{x_s}+\underline{y_s},\overline{x_s}+\overline{y_s}]\right\} \end{aligned}$$

and

$$\begin{aligned} \lambda X &= \lambda\,[\underline{x_r},\overline{x_r}] + i\left(\lambda\,[\underline{x_s},\overline{x_s}]\right) \\ &= \left\{\lambda a + i\lambda b : a \in [\underline{x_r},\overline{x_r}],\ b \in [\underline{x_s},\overline{x_s}]\right\} \end{aligned}$$

on $\mathbb{I}_{\mathbb{C}}$ where $i = \sqrt{-1}$ and $\lambda \in \mathbb{C}$. Further, the relation

$$X \preceq Y \text{ iff } [\underline{x_r},\overline{x_r}] \subseteq [\underline{y_r},\overline{y_r}] \text{ and } [\underline{x_s},\overline{x_s}] \subseteq [\underline{y_s},\overline{y_s}]$$

is a partial order relation on $\mathbb{I}_{\mathbb{C}}$. Thus, $\mathbb{I}_{\mathbb{C}}$ is a quasilinear space [7].

2. Preliminaries

Let us start with some main definitions, notions and theorems.

Suppose that $\mathscr{X}$ is a quasilinear space and $\mathscr{Y} \subseteq \mathscr{X}$. Then $\mathscr{Y}$ is called a *subspace of* $\mathscr{X}$ whenever $\mathscr{Y}$ is a quasilinear space with the same partial order and the restriction to $\mathscr{Y}$ of the operations on $\mathscr{X}$. $\mathscr{Y}$ is subspace of a quasilinear space $\mathscr{X}$ if and only if for every $x, y \in \mathscr{Y}$ and $\alpha, \beta \in \mathbb{K}$, $\alpha x + \beta y \in \mathscr{Y}$. Proof of this theorem is quite similar to its classical linear space analogue. Let $\mathscr{Y}$ be a subspace of a quasilinear space $\mathscr{X}$ and suppose each element x in $\mathscr{Y}$ has an inverse in $\mathscr{Y}$. Then the partial order on $\mathscr{Y}$ is determined by the equality. In this case $\mathscr{Y}$ is a linear subspace of $\mathscr{X}$, [14].

An element x in a quasilinear space $\mathscr{X}$ is said to be *symmetric* if $-x = x$ and $\mathscr{X}_{sym}$ denotes the set of all symmetric elements. Also, $\mathscr{X}_r$ stands for the set of all regular elements of $\mathscr{X}$ while $\mathscr{X}_s$ stands for the sets of all singular elements and zero in $\mathscr{X}$. Further, it can be easily shown that $\mathscr{X}_r$, $\mathscr{X}_{sym}$ and$\mathscr{X}_s$ are subspaces of $\mathscr{X}$. They are called *regular, symmetric* and *singular subspaces* of $\mathscr{X}$, respectively. Furthermore, it isn't hard to prove that summation of a regular element with a singular element is a singular element and the regular subspace of $\mathscr{X}$ is a linear space while the singular one is nonlinear at all. Further, $\mathbb{I}_{\mathbb{C}}$ is a closed subspace of $\Omega(\mathbb{C})$, [13].

A real-valued function $\|.\|$ on the quasilinear space $\mathscr{X}$ is called a *norm* if the following conditions hold:

$$\|x\| > 0 \text{ if } x \neq 0, \tag{2.1}$$

$$\|x+y\| \leq \|x\| + \|y\|, \tag{2.2}$$

$$\|\alpha x\| = |\alpha| \, \|x\|, \tag{2.3}$$

$$\text{if } x \preceq y, \text{ then } \|x\| \leq \|y\|, \tag{2.4}$$

$$\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in \mathscr{X} \text{ such that } x \preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\| \leq \varepsilon \text{ then } x \preceq y, \tag{2.5}$$

here x, y, x_ε are arbitrary element in $\mathscr{X}$ and α is any scalar. A quasilinear space $\mathscr{X}$ with a norm defined on it, is called *normed quasilinear space*, [8].

For a normed linear space E, a norm on $\Omega(E)$ is defined by

$$\|A\|_\Omega = \sup_{a \in E} \|a\|_E \ .$$

Hence $\Omega_C(E)$ and $\Omega(E)$ are normed quasilinear spaces. A norm on $\mathbb{I}_{\mathbb{R}}$ is defined by

$$\|x\| = \|[\underline{x}, \overline{x}]\| = \sup_{t \in [\underline{x}, \overline{x}]} |t| .$$

Moreover, $\mathbb{I}_{\mathbb{C}}$ is a normed quasilinear space with the norm

$$\begin{aligned} \|X\|_{\mathbb{I}_{\mathbb{C}}} &= \sup\{|z| : z \in X\} \\ &= \sup\{|a+ib| : a \in [\underline{x_r}, \overline{x_r}], b \in [\underline{x_s}, \overline{x_s}]\}, \end{aligned}$$

for $X = [\underline{x_r}, \overline{x_r}] + i[\underline{x_s}, \overline{x_s}]$, [12].

Now we will give the notion of consolidate quasilinear space defined in [12]. Thanks to this definition, we were able to give a representation to every element in a quasilinear space and we were able to define an inner-product quasilinear space.

Definition 2.1. *[12] Let $\mathscr{X}$ be a quasilinear space and $y \in \mathscr{X}$. The floor of y is the set of all regular elements y of $\mathscr{X}$ such that $x \preceq y$. It is denoted by $F_y^{\mathscr{X}}$ and $F_y^{\mathscr{X}} \subset \mathscr{X}$. Hence $F_y^{\mathscr{X}} = \{x \in \mathscr{X}_r : x \preceq y\}$.*

Definition 2.2. *[12] A quasilinear space $\mathscr{X}$ is called* consolidate *or* Solid-Floored *whenever*

$$\sup_{\preceq}\{x \in \mathscr{X}_r : x \preceq y\} = \sup_{\preceq} F_y^{\mathscr{X}}$$

exists and

$$y = \sup_{\preceq}\{x \in \mathscr{X}_r : x \preceq y\}$$

for each $y \in \mathscr{X}$. Otherwise, $\mathscr{X}$ is called a non-consolidate quasilinear space.

From above example immediately we can see that $\mathbb{I}_{\mathbb{R}}$ is consolidate while $(\mathbb{I}_{\mathbb{R}})_s$ is not. Analogous results are also true for the spaces $\mathbb{I}_{\mathbb{C}}$ and $(\mathbb{I}_{\mathbb{C}})_s$.

Definition 2.3. *[13] Let $\mathscr{X}$ be a consolidate quasilinear space. A mapping $\langle\ ,\ \rangle : \mathscr{X} \times \mathscr{X} \to \Omega(\mathbb{K})$ is called an inner product on $\mathscr{X}$ if for any $x,y,z \in \mathscr{X}$ and $\alpha \in \mathbb{K}$ the following conditions are satisfied :*

If $x,y \in \mathscr{X}_r$ then $\langle x,y\rangle \in \Omega_C(\mathbb{K})_r \equiv \mathbb{K}$,

$$\langle x+y,z\rangle \subseteq \langle x,z\rangle + \langle y,z\rangle,$$

$$\langle \alpha x,y\rangle = \alpha \langle x,y\rangle,$$

$$\langle x,y\rangle = \langle y,x\rangle,$$

$\langle x,x\rangle \geq 0$ for $x \in \mathscr{X}_r$ and $\langle x,x\rangle = 0 \Leftrightarrow x = 0$,

$$\|\langle x,y\rangle\|_{\Omega} = \sup\left\{\|\langle a,b\rangle\|_{\Omega} : a \in F_x^{\mathscr{X}}, b \in F_y^{\mathscr{X}}\right\},$$

if $x \preceq y$ and $u \preceq v$ then $\langle x,u\rangle \subseteq \langle y,v\rangle$,

if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in \mathscr{X}$ such that
$x \preceq y + x_\varepsilon$ and $\langle x_\varepsilon, x_\varepsilon\rangle \subseteq S_\varepsilon(\theta)$ then $x \preceq y$.

A quasilinear space with an inner product is called as an *inner-product quasilinear space*.
$\mathscr{X}$ is a linear Hilbert space, then the space $\Omega(\mathscr{X})$ is a Hilbert quasilinear space with the inner product defined by

$$\langle A,B\rangle_{\Omega} = \overline{\{\langle a,b\rangle_{\mathscr{X}} : a \in A, b \in B\}}$$

for $A,B \in \Omega(\mathscr{X})$. Especially, the inner product on $\Omega(\mathbb{C})$ given by

$$\langle A,B\rangle_{\Omega} = \{\langle a,b\rangle_{\mathbb{C}} : a \in A, b \in B\}. \tag{2.6}$$

If $A,B \in \mathbb{I}_{\mathbb{C}}$ then the inner-product (2.6) is equivalent to the following:

$$\langle A,B\rangle = [\underline{a_1},\overline{a_1}]\,[\underline{b_1},\overline{b_1}] + [\underline{a_2},\overline{a_2}]\,[\underline{b_2},\overline{b_2}] + i([\underline{a_2},\overline{a_2}]\,[\underline{b_1},\overline{b_1}] - [\underline{a_1},\overline{a_1}]\,[\underline{b_2},\overline{b_2}])$$

where $A = [\underline{a_1},\overline{a_1}] + i\,[\underline{a_2},\overline{a_2}]$, $B = [\underline{b_1},\overline{b_1}] + i\,[\underline{b_2},\overline{b_2}]$ and the product of two intervals is given in (1.1). Namely, the above equality is the reduction of the inner-product on $\Omega(\mathbb{C})$ to the inner-product on $\mathbb{I}_{\mathbb{C}}$.

3. Complex Interval Sequence Spaces

In this section, firstly we present the complex interval sequence spaces $\mathbb{I}(w)$ and $\mathbb{I}(l_p)$, $1 \leq p < \infty$ and we show that these spaces are the normed quasilinear spaces. Later, we construct a set-valued inner-product on $\mathbb{I}(l_2)$.

The sequence $X = (X_i)_{i=1}^{\infty}$ is called as complex interval sequence if $X_i \in \mathbb{I}_{\mathbb{C}}$, $i = 1,2,...$. The set $\mathbb{I}(l_w)$ denotes the set of all complex interval sequences $X = (X_i)_{i=1}^{\infty}$. The addition and multiplication operations $\mathbb{I}(w)$ are defined by

$$\begin{aligned} X+Y &= (X_1, X_2, ...) + (Y_1, Y_2, ...) \\ &= (X_1 + Y_1, X_2 + Y_2, ...) \end{aligned}$$

and

$$\alpha X = \alpha(X_1, X_2, ...) = (\alpha X_1, \alpha X_2, ...),$$

respectively where $X_i + Y_i$ is the sum of two complex intervals and αX_i is the multiplication of a complex interval with the scalar α. Further, the partial order relation on $\mathbb{I}(w)$ is that

$$X \ll Y \text{ iff } X_i \preceq Y_i,\ i = 1,2,...$$

where the relation "$\preceq$" is the partial order relation on $\mathbb{I}_{\mathbb{C}}$. Thus, $\mathbb{I}(l_w)$ is a quasilinear space with the above operations and the partial order relation.

For $1 \leq p < \infty$, $\mathbb{I}(l_p)$ is the set of all complex interval sequences $X = (X_i)_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} \|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p < \infty.$$

The space $\mathbb{I}(l_p)$ is a quasilinear space with the operations and the partial order relation on $\mathbb{I}(l_w)$. Really, for $X, Y \in \mathbb{I}(l_p)$ we write that by the Minkowski inequality

$$\begin{aligned} \sum_{i=1}^{\infty} (\|X_i + Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} &\leq \sum_{i=1}^{\infty} (\|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p + \|Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} \\ &\leq (\sum_{i=1}^{\infty} \|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} + (\sum_{i=1}^{\infty} \|Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} < \infty. \end{aligned}$$

Further,

$$\sum_{i=1}^{\infty} \|\lambda X_i\|_{\mathbb{I}_{\mathbb{C}}}^p = |\lambda|^p (\sum_{i=1}^{\infty} \|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p) < \infty$$

for $X \in \mathbb{I}(l_p)$ and $\lambda \in \mathbb{C}$.

Proposition 3.1. *$\mathbb{I}(l_p)$, $1 \leq p < \infty$ is a normed quasilinear space with the norm defined by*

$$\|X\| = (\sum_{i=1}^{\infty} \|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p}.$$

Proof. It is obvious that $\|X\| \geq 0$ for any $X \in \mathbb{I}(l_p)$. Further, for any $X, Y \in \mathbb{I}(l_p)$ and $\lambda \in \mathbb{C}$ by the triangle inequality and Minkowski inequality we write that

$$\|X+Y\| = \sum_{i=1}^{\infty} (\|X_i + Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} \leq (\sum_{i=1}^{\infty} \|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} + (\sum_{i=1}^{\infty} \|Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} = \|X\| + \|Y\|$$

and

$$\|\lambda X\| = (\sum_{i=1}^{\infty} \|\lambda X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} = |\lambda| (\sum_{i=1}^{\infty} \|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} = |\lambda| \, \|X\|.$$

Let us assume that $X \ll Y$ for any $X, Y \in \mathbb{I}(l_p)$. Then $\|X_i\|_{\mathbb{I}_{\mathbb{C}}} \leq \|Y_i\|_{\mathbb{I}_{\mathbb{C}}}$ for $i = 1,2,...$ since $X_i \preceq Y_i$, $i = 1,2,...$ and $\mathbb{I}_{\mathbb{C}}$ is a normed quasilinear space. This implies that

$$\|X\| = (\sum_{i=1}^{\infty} \|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} \leq (\sum_{i=1}^{\infty} \|Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} = \|Y\|.$$

Now suppose that there exists an element $X^\varepsilon \in \mathbb{I}(l_p)$ such that $X \ll Y + X^\varepsilon$ and $\|X^\varepsilon\| \leq \varepsilon$ for any $\varepsilon > 0$. Then we have that $X_i \subseteq Y_i + X_i^\varepsilon$ and for $i = 1, 2, ...$ and

$$\|X^\varepsilon\| = (\sum_{i=1}^{\infty} \|X_i^\varepsilon\|_{\mathbb{I}_\mathbb{C}}^p)^{1/p} \leq \varepsilon.$$

Hence, we obtain that $\|X_i\|_{\mathbb{I}_\mathbb{C}} < \varepsilon$ for $i = 1, 2, ...$ By the fourth condition of norm on $\mathbb{I}_\mathbb{C}$ we write that $X_i \preceq Y_i$ for $i = 1, 2, ...$ and so $X \ll Y$. □

Example 3.2. *Let us take the complex interval sequence $X = (X_k)_{k=1}^\infty$ given as follows:*

$$(X_k)_{k=1}^\infty = (\frac{1}{2^k} + i[0, \frac{1}{2^k}])_{k=1}^\infty = (\frac{1}{2} + i[0, \frac{1}{2}], \frac{1}{2^2} + i[0, \frac{1}{2^2}], ...).$$

We can say that $X = (X_k)_{k=1}^\infty \in \mathbb{I}(l_2)$ since

$$\begin{aligned}
\|X\|^2 &= \sum_{k=1}^{\infty} \left\| \frac{1}{2^k} + i[0, \frac{1}{2^k}] \right\|_{\mathbb{I}_\mathbb{C}}^2 \\
&= \sum_{k=1}^{\infty} (\sup\{|a+ib| : a = \frac{1}{2^k}, b \in [0, \frac{1}{2^k}]\})^2 \\
&= \sum_{k=1}^{\infty} (\frac{1}{2^{2k}} + \frac{1}{2^{2k}}) = 2\sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{2} \frac{1}{1-1/4} = 2/3.
\end{aligned}$$

Hence, the norm of the sequence $X = (X_k)_{k=1}^\infty$ is that

$$\|X\| = (\sum_{k=1}^{\infty} \|X_k\|_{\mathbb{I}_\mathbb{C}}^2)^{1/2} = \sqrt{\frac{2}{3}}.$$

Among the $\mathbb{I}(l_p)$ spaces, $\mathbb{I}(l_2)$ has an important place. Because $\mathbb{I}(l_2)$ is an inner-product quasilinear space. Before we construct an inner-product on $\mathbb{I}(l_2)$, we must show that it is a consolidate space.

Lemma 3.3. *The space $\mathbb{I}(l_p)$, $1 \leq p < \infty$ is a consolidate quasilinear space.*

Proof. To complete the proof we will show that

$$X = \sup_{"\ll"}\{Y \in (\mathbb{I}(l_p))_r : Y \ll X\}.$$

If $Y \ll X$ for $Y \in (\mathbb{I}(l_p))_r$ then we write that $Y_i \preceq X_i$ for $i = 1, 2, ...$ and $X_i \in \mathbb{I}_\mathbb{C}$. Since $\mathbb{I}_\mathbb{C}$ is a consolidate quasilinear space, we obtain that

$$X_i = \sup F_{X_i} = \sup\{Y_i \in \mathbb{I}_\mathbb{C} : Y_i \preceq X_i\}$$

for each $i = 1, 2, ...$ This means that $\sup F_X = X$ for $X = (X_i)_{i=1}^\infty \in \mathbb{I}(l_p)$. □

Theorem 3.4. *The quasilinear space $\mathbb{I}(l_2)$ with the inner-product*

$$\langle X, Y \rangle = \sum_{i=1}^{\infty} \langle X_i, Y_i \rangle_{\mathbb{I}_\mathbb{C}} \tag{3.1}$$

is an inner-product quasilinear space where

$$\begin{aligned}
\langle X_i, Y_i \rangle_{\mathbb{I}_\mathbb{C}} &= \left\langle \left[\underline{x_i^r}, \overline{x_i^r}\right] + i\left[\underline{x_i^s}, \overline{x_i^s}\right], \left[\underline{y_i^r}, \overline{y_i^r}\right] + i\left[\underline{y_i^s}, \overline{y_i^s}\right] \right\rangle \\
&= \left[\underline{x_i^r}, \overline{x_i^r}\right] + \left[\underline{y_i^s}, \overline{y_i^s}\right] + i(\left[\underline{x_i^s}, \overline{x_i^s}\right]\left[\underline{y_i^r}, \overline{y_i^r}\right] - \left[\underline{x_i^r}, \overline{x_i^r}\right]\left[\underline{y_i^s}, \overline{y_i^s}\right]).
\end{aligned}$$

Proof. Firstly, we will show that the equality (3.1) is well-defined, i.e., $\langle X,Y\rangle \in \Omega(\mathbb{C})$:

By the Hölder and Schwartz inequalities we observe that

$$\begin{aligned}
\|\langle X,Y\rangle\| &= \left\|\sum_{i=1}^{\infty}\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}}\right\| \leq \sum_{i=1}^{\infty}\left\|\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}}\right\|_{\Omega} \\
&\leq \sum_{i=1}^{\infty}(\|X_i\|_{\mathbb{I}_\mathbb{C}}\|Y_i\|_{\mathbb{I}_\mathbb{C}}) \leq (\sum_{i=1}^{\infty}\|X_i\|^2_{\mathbb{I}_\mathbb{C}})^{1/2}(\sum_{i=1}^{\infty}\|Y_i\|^2_{\mathbb{I}_\mathbb{C}})^{1/2} \\
&= \|X\|\,\|Y\|
\end{aligned}$$

for $X,Y\in\mathbb{I}(l_2)$. This means that the set $\langle X,Y\rangle$ is bounded. Now let us take a sequence $(X_n)_{n=1}^{\infty}$ in the set $\langle X,Y\rangle$ such that $X_n\to X_0$. Then $\{X_n\}\to\{X_0\}$ for $n=1,2,...$ in $\Omega(\mathbb{C})$ since $X_n\in\langle X,Y\rangle$ for $n=1,2,....$ Further, we can say that $\langle X_n,Y_n\rangle\to\langle X,Y\rangle$. The Lemma 4-a in [8] implies $\{X_0\}\subseteq\langle X,Y\rangle$. Consequently, we obtain that $X_0\in\langle X,Y\rangle$.

1. If $X,Y\in(\mathbb{I}(l_2))_r$ then

$$\langle X,Y\rangle = \sum_{i=1}^{\infty}\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}} \in (\Omega(\mathbb{C}))_r \equiv \mathbb{C}$$

since $X_i,Y_i\in\mathbb{C}$ for $i=1,2,....$

2. By the second condition of inner-product on $\mathbb{I}_\mathbb{C}$ we write that

$$\begin{aligned}
\langle X+Y,Z\rangle &= \sum_{i=1}^{\infty}\langle X_i+Y_i,Z_i\rangle_{\mathbb{I}_\mathbb{C}} \\
&\subseteq \sum_{i=1}^{\infty}(\langle X_i,Z_i\rangle_{\mathbb{I}_\mathbb{C}}+\langle Y_i,Z_i\rangle_{\mathbb{I}_\mathbb{C}}) \\
&= \sum_{i=1}^{\infty}\langle X_i,Z_i\rangle_{\mathbb{I}_\mathbb{C}}+\sum_{i=1}^{\infty}\langle Y_i,Z_i\rangle_{\mathbb{I}_\mathbb{C}} \\
&= \langle X,Z\rangle+\langle Y,Z\rangle.
\end{aligned}$$

3. By the third condition of inner-product on $\mathbb{I}_\mathbb{C}$ we obtain that

$$\langle\alpha X,Y\rangle = \sum_{i=1}^{\infty}\langle\alpha X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}} = \sum_{i=1}^{\infty}\alpha\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}} = \alpha\sum_{i=1}^{\infty}\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}} = \alpha\langle X,Y\rangle.$$

Further, it can be easily shown that $\langle X,\alpha Y\rangle = \bar{\alpha}\langle X,Y\rangle$.

4. By the fourth condition of inner-product on $\mathbb{I}_\mathbb{C}$,

$$\langle X,Y\rangle = \sum_{i=1}^{\infty}\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}} = \langle X,Y\rangle = \sum_{i=1}^{\infty}\langle Y_i,X_i\rangle_{\mathbb{I}_\mathbb{C}} = \langle Y,X\rangle.$$

5.

$$\langle X,X\rangle = \{0\} \Leftrightarrow \sum_{i=1}^{\infty}\langle X_i,X_i\rangle_{\mathbb{I}_\mathbb{C}} = \{0\} \Leftrightarrow X_i=\theta, i=1,2,... \Leftrightarrow X=\theta.$$

and for any $X\in(\mathbb{I}(l_2))_r$ we write that

$$\langle X,X\rangle = \sum_{i=1}^{\infty}\langle X_i,X_i\rangle_{\mathbb{I}_\mathbb{C}} = \sum_{i=1}^{\infty}|X_i|^2 \geq 0.$$

6.

$$
\begin{aligned}
\|\langle X,Y\rangle\|_{\Omega} &= \left\| \sum_{i=1}^{\infty} \langle X_i, Y_i\rangle_{\mathbb{I}_{\mathbb{C}}} \right\|_{\Omega} \\
&= \sup\{|z| : z \in \sum_{i=1}^{\infty} \langle X_i, Y_i\rangle_{\mathbb{I}_{\mathbb{C}}}\} \\
&= \sup\{|z| : z \in \sum_{i=1}^{\infty} (\left[\underline{x_i^r}, \overline{x_i^r}\right] + \left[\underline{y_i}^s, \overline{y_i^s}\right] + i(\left[\underline{x_i}^s, \overline{x_i^s}\right] \left[\underline{y_i^r}, \overline{y_i^r}\right] - \left[\underline{x_i^r}, \overline{x_i^r}\right] \left[\underline{y_i}^s, \overline{y_i^s}\right]) \\
&= \sup\{|\langle x,y\rangle| : x \in F_X, y \in F_Y\}.
\end{aligned}
$$

7. If $X \ll Y$ and $Z \ll T$ then $X_i \preceq Y_i$ and $Z_i \preceq T_i$ for $i = 1,2,\ldots$. By the seventh condition of inner-product on $\mathbb{I}_{\mathbb{C}}$ we write that $\langle X_i, Z_i\rangle_{\mathbb{I}_{\mathbb{C}}} \subseteq \langle Y_i, T_i\rangle_{\mathbb{I}_{\mathbb{C}}}$ for $i = 1,2,\ldots$ and so $\langle X,Z\rangle \subseteq \langle Y,T\rangle$.

8. Suppose that for any $\varepsilon > 0$ there exists an element $X^{\varepsilon} \in \mathbb{I}(l_2)$ such that $X \ll Y + X^{\varepsilon}$ and $\langle X^{\varepsilon}, X^{\varepsilon}\rangle \subseteq S_{\varepsilon}(\theta)$. Then we say that $X_i \subseteq Y_i + X_i^{\varepsilon}$ for $i = 1,2,\ldots$. By the hypotesis we write that

$$\sum_{i=1}^{\infty} \langle X_i^{\varepsilon}, X_i^{\varepsilon}\rangle_{\mathbb{I}_{\mathbb{C}}} \subseteq S_{\varepsilon}(\theta).$$

Since $\mathbb{I}_{\mathbb{C}}$ is an inner-product quasilinear space, if $X_i \subseteq Y_i + X_i^{\varepsilon}$ for $i = 1,2,\ldots$ and $\|X_i^{\varepsilon}\|_{\Omega} \leq \varepsilon$ then $X_i \subseteq Y_i$ for $i = 1,2,\ldots$ This implies $X \ll Y$.

□

4. Conclusion

In this paper, we have presented the notion of complex interval sequence and some important complex interval sequence spaces. In this way, we brought a new perspective to sequence spaces with the help of interval analysis and quasilinear functional analysis. We also have defined the inner product function on the complex interval sequence space $\mathbb{I}(l_2)$, which is one of the most important sequence spaces. Thus, by using quasilinear functional analysis techniques, we have introduced a new type of space to the literature.

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Author's contributions

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