About one non-local problem for the degenerating parabolic-hyperbolic type equation.

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Abstract.

In the present paper, the existence and uniqueness of solution of the analogue of Frankl's problem for the degenerated equation of the parabolic-hyperbolic type was investigated. Uniqueness of solution of the investigated problem are proved with principle an extremum and existence of solution with method of integral equations.

Key words.

Boundary value problem, existence and uniqueness of solution, degenerating equation, parabolic-hyperbolic type, a principle an extremum, method of integral equations.

1.Introduction.

As we know, in 1959 year in the first by I.M Gelfand [3] was offered to studying of boundary value problems for the equations parabolic-hyperbolic type.

Since A.V.Bitsadze's works, in the theory partial differential equations there was a new direction, in which the problem of the type of Frankl for the first time is formulated and investigated for the modeling equations of the mixed type. We note following works that are connected with studying Frankl problem for the mixed type equations. In the books [1],[2] the Frankl problem was discussed for the special mixed type equation of second order: $u_{xx} + signy u_{yy} = 0$. The Frankl problem for the mixed equation with parabolic degeneracy $signy|y|^m u_{xx} + u_{yy} = 0$ with is a mathematical model of problem of gas dynamic, was discussed in the book of M.M.Smirnov [8]. Existence of solution of Frankl problem for general Lavrent'ev-Bitsadze equations was proved in work of Guo-chun Wen and H.Begehr [4].

The basic review of boundary value problems for the mixed type equations with Frankl condition it is possible will receive in the work J. M. Rassias [9].

2. Initial necessary dates

Definition. Let's, the function $f(x)$ is any function from a class $L(a,b)$ $a < x < \infty$. An operator in the form

$$D_{ax}^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_{a}^{x} \dfrac{f(t)}{(x-t)^{1+c}}, & \alpha < 0, \\[3mm] \dfrac{d^{n+1}}{dx^{n+1}} D_{ax}^{\alpha-(n+1)} f(x), & \alpha > 0, \end{cases}$$

$$D_{xb}^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_{x}^{b} \dfrac{f(t)}{(t-x)^{1+\alpha}}, & \alpha < 0, \\[3mm] (-1)^{n+1} \dfrac{d^{n+1}}{dx^{n+1}} D_{xb}^{\alpha-(n+1)} f(x), & \alpha > 0, \end{cases} \qquad (2.1)$$

where D_{ax}^{α} and D_{xb}^{α} is called as the integral operator of fractionally integration α, at $\alpha < 0$, and the generalized derivatives in understand of Liuvill on the order α, at $\alpha > 0$, $n = [\alpha]$; $[\alpha]$ the whole part of number α.

Some properties integral differential operators of fractionally order.

1^0. If $f(x) \in L(a,b)$, then for all $\alpha > 0$ almost for all $x \in (a,b)$

$$D_{xb}^{\alpha} D_{xb}^{-\alpha} f(x) = f(x), \qquad (2.2)$$

2^0. Lets $f(x) \in L(a,b)$, then:

1) if $\beta \geq \alpha > 0$, then

$$D_{ax}^{\alpha} D_{ax}^{-\beta} f(x) = D_{ax}^{-(\beta-\alpha)} f(x), \ D_{xb}^{\alpha} D_{xb}^{-\beta} f(x) = D_{xb}^{-(\beta-\alpha)} f(x), \ x \in (a,b);$$

2) if $\alpha > \beta \geq 0$ and the function $f(x)$ have a derivative of $D_{ax}^{\alpha-\beta} f(x)$, $D_{xb}^{\alpha-\beta} f(x)$ then

$$D_{ax}^{\alpha} D_{ax}^{-\beta} f(x) = D_{ax}^{(\alpha-\beta)} f(x),$$

$$D_{xb}^{\alpha} D_{xb}^{-\beta} f(x) = D_{xb}^{(\alpha-\beta)} f(x), \ x \in (a,b);$$

3^0. Let $0 < 2\beta < 1$ $(b-x)^{-\beta} f(x) \in L(a,b)$. then almost everywhere on (a,b) it is fair identities:

$$D_{xb}^{\beta}(b-x)^{2\beta-1} D_{xb}^{\beta-1}(b-x)^{-\beta} f(x) = (b-x)^{\beta-1} D_{xb}^{2\beta-1} f(x). \qquad (2.3)$$

4^0. *A principle of an extremum for the fractional derivative operations D_{ax}^{α} and $D_{xb}^{\alpha}(0 < \alpha < 1)$. Let positive not decreasing function $\omega(t)$ and a function $f(t)$ continuously in $[a,b]$. Then, if the function $f(t)$ reaches the positive maximum (a negative minimum) in the segment $[a,b]$ on the point $t = x$, $a < x < b$ and in as much as small vicinity of this point derivative of function $\omega(t)f(t)$ satisfy Gelder condition with an indicator $\gamma > \alpha$, then $D_{ax}^{\alpha}\omega f > 0$, $(D_{xb}^{\alpha}\omega f < 0)$.*

The similar remark takes place for the operator D_{xb}^{α}, if $\omega(t)$ positive not increasing function on the $[a,b]$.

3. The statement of problems F.

The given work is devoted research of non-local problem of the Frankl type for the equation

$$0 = \begin{cases} y^{m_0} u_{xx} - x^{n_0} u_y, & x > 0, y > 0, \\ (-y)^{n} u_{xx} - x^{n} u_{yy}, & x > 0, y < 0, \end{cases} \qquad (3.1)$$

where $m_0, n_0, n = const$, $m_0 > 0, n_0 > 0, n > 0$.

Let's Ω is, domain restricted at $x < 0$, $y > 0$, by the segments AB, BB_0, A_0B_0, A_0A on the lines $y = 0$, $x = 1$, $y = 1$, $x = 0$ and at $x > 0$, $y < 0$, restricted by line $x = 0$, $(-1 \le y \le 0)$ and characteristics

$$BC : x^{\frac{n+2}{2}} + (-y)^{\frac{n+2}{2}} = 1,$$

of equation (3.1), where $A(0,0)$, $B(1,0)$, $A_0(0,1)$, $B_0(1,1)$.

Let's to put designations:

$$J = \{(x,y) : 0 < x < 1, y = 0\}, \ \Omega_1 = \Omega \cap \{(x,y) : x > 0, y > 0\},$$

$$\Omega_2 = \Omega \cap \{(x,y) : x > 0, y < 0\}, \ \Omega_{21} = \Omega_2 \cap \{(x,y) : x + y > 0\},$$

$$\Omega_{22} = \Omega_2 \cap \{(x,y) : x + y < 0\}, \Omega^* = \Omega_1 \cup \Omega_{21} \cup J, \ 2\beta = \frac{n}{n+2}, \ \alpha_0 = \frac{n_0 + 1}{n_0 + 2},$$

and

$$0 < 2\beta < 1, \quad 1 < 2\alpha_0 < 2. \tag{3.2}$$

we will designate, through

$$\theta(x_0) = \left(\frac{1 + x_0}{2}\right)^{\frac{2}{n+2}} - i \left(\frac{1 - x_0}{2}\right)^{\frac{2}{n+2}}, \tag{3.3}$$

affix of the point of crossing characteristic BC_0 by the characteristic leaving on the point $(x_0, 0) \in J$, parallel characteristic AC_0, where $C_0 \left(2^{1/(n+2)}, 2^{-1/(n+2)}\right)$.

The Problem F. To find a function $u(x,y)$ with following conditions:
1) $u(x,y) \in C(\overline{\Omega}) \cap C^{2,1}(\Omega_1) \cap C^2(\Omega_{21} \cup \Omega_{22})$;
2) $u(x,y)$ satisfies equation (3.1) in the domain $\Omega_1 \cup \Omega_{21} \cup \Omega_{22}$;
3) $u_x(x,y) \in C(\Omega_1 \cup AA_0) \cap C(\Omega_{22} \cup AC)$, $u_x(+0,y) \in C(\Omega_{22} \cup AC)$, $y^{-m_0}u_y \in C(\Omega_1 \cup J)$, $u_y \in C(\Omega_2 \cup J)$ and on AB satisfied gluing condition:

$$\lim_{y \to +0} y^{-m_0} u_y(x,y) = \lim_{y \to -0} u_y(x,y), \quad (x,y) \in J, \tag{3.4}$$

4) $u(x,y)$ satisfies boundary conditions :

$$u(x,y)|_{AA_0} = \tau_0(y), \quad u(x,y)|_{BB_0} = \varphi_0(y), \quad 0 \le y \le 1, \tag{3.5}$$

$$D_{x^2 1}^\beta (1 - x^2)^{2\beta-1} u[\theta(x)] = a(x)u\left(x^{\frac{2}{n+2}}, 0\right) +$$

$$+b(x)(1 - x^2)^{\beta-1}u_y\left(x^{\frac{2}{n+2}}, 0\right) + c(x), \quad x \in (0,1), \tag{3.6}$$

$$u_x(0, +y) = u_x(0, -y), \quad 0 < y < 1, \tag{3.7}$$

where $\varphi_0(y), \tau_0(y), a(x), b(x), c(x)$ are given continuous functions, at that

$$\tau_0(y), \varphi_0(y) \in C[0,1] \cap C^1(0,1), \tag{3.8}$$

$$a(x), b(x), c(x) \in C[0,1] \cap C^3(0,1). \tag{3.9}$$

3.1. Reduction of main functional relations.

A solution of the Cauchy problem satisfying the following conditions $\tau^-(x) = u(x, -0)$, $0 \le x \le 1$, $\nu^-(x) = u_y(x, -0)$, $0 < x < 1$, for the equation(3.1) in the domain of Ω_{21}, looks like[7]:

$$u(x,y) = \gamma_1 \int_0^1 \tau^-\left(z_1^{\frac{1}{n+2}}\right) z^{\beta-1}(1-z)^{\beta-1}dz -$$

$$-\gamma_2 xy \int\limits_0^1 z_1^{-\frac{1}{n+2}} \nu^- \left(z_1^{\frac{1}{n+2}} \right) z^{-\beta}(1-z)^{-\beta} dz, \qquad (3.10)$$

where, $\gamma_1 = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)}$, $\gamma_2 = \frac{\Gamma(2-2\beta)}{\Gamma^2(1-\beta)}$, $z_1 = x^{n+2} + (-y)^{n+2} + 2x^{\frac{n+2}{2}}(-y)^{\frac{n+2}{2}}(2z-1)$.

By virtue (3.3), from (3.10), we have

$$u[\theta(x)] = \gamma_1 \int\limits_0^1 \tau^- \left[(x^2 + (1-x^2)z)^{\frac{1}{2}-\beta} \right] (z(1-z))^{\beta-1} dz + \gamma_2 \left(\frac{1-x^2}{4} \right)^{1-2\beta} \times$$

$$\times \int\limits_0^1 (x^2 + (1-x^2)z)^{\beta-\frac{1}{2}} \nu^- \left[(x^2 + (1-x^2)z)^{\frac{1}{2}-\beta} \right] (z(1-z))^{-\beta} dz.$$

From here, owing to replacement $x^2 + (1-x^2)z = s$, we will receive

$$u[\theta(x)] = \gamma_1(1-x^2)^{1-2\beta} \int\limits_{x^2}^1 (s-x^2)^{\beta-1}(1-s)^{\beta-1}\tau^- \left(s^{\frac{1}{n+2}} \right) ds +$$

$$+\gamma_2 4^{2\beta-1} \int\limits_{x^2}^1 (s-x^2)^{-\beta}(1-s)^{-\beta} s^{\beta-\frac{1}{2}} \nu^- \left(s^{\frac{1}{n+2}} \right) ds.$$

Further, taking properties of integro-differential operators into account (2.1)[8], we have

$$u[\theta(x)] = \gamma_1\Gamma(\beta)(1-x^2)^{1-2\beta} D_{x^2 1}^{-\beta} \tau^- \left(x^{\frac{2}{n+2}} \right) (1-x^2)^{\beta-1} +$$

$$+\gamma_2\Gamma(1-\beta)4^{2\beta-1} D_{x^2 1}^{\beta-1}(1-x^2)^{-\beta} x^{2\beta-1}\nu^- \left(x^{\frac{2}{n+2}} \right). \qquad (3.11)$$

Substituting (3.11),(2.2) to the condition (3.6), and replacing x^2 to x, we have

$$\left[\overline{a}(x) - \gamma_1\Gamma(\beta)(1-x)^{\beta-1} \right] \tau^- \left(x^{\frac{1}{n+2}} \right) + \overline{c}(x) = \gamma_2\Gamma(1-\beta)4^{2\beta-1} \times$$

$$\times D_{x1}^{\beta}(1-x)^{2\beta-1} D_{x1}^{\beta-1}(1-x)^{-\beta} x^{\beta-\frac{1}{2}} \nu^- \left(x^{\frac{1}{n+2}} \right) - (1-x)^{\beta-1}\overline{b}(x)\nu^- \left(x^{\frac{1}{n+2}} \right). \qquad (3.12)$$

From (3.12) and (2.3), we have

$$\overline{a}_1(x)\widetilde{\tau}^-(x) = \gamma_2\Gamma(1-\beta)4^{2\beta-1} D_{x1}^{2\beta-1} x^{\beta-\frac{1}{2}}\widetilde{\nu}^-(x) -$$

$$-\overline{b}(x)\widetilde{\nu}^-(x) - \overline{c}(x)(1-x)^{1-\beta}, \quad 0 < x < 1, \qquad (3.13)$$

where, $\widetilde{\tau}^-(x) = \tau^- \left(x^{\frac{1}{n+2}} \right)$, $\widetilde{\nu}^-(x) = \nu^- \left(x^{\frac{1}{n+2}} \right)$,

$$\overline{a}_1(x) = \overline{a}(x)(1-x)^{1-\beta} - \gamma_1\Gamma(\beta), \overline{a}(x) = a(\sqrt{x}), \overline{b}(x) = b(\sqrt{x}), \overline{c}(x) = c(\sqrt{x}). \qquad (3.14)$$

Let's consider three cases:

I. Let's $b(x) = 0$, $a(x) \neq 0$. Then from (3.13), receive

$$\gamma_2\Gamma(1-\beta)4^{2\beta-1} D_{x1}^{2\beta-1} x^{\beta-\frac{1}{2}}\widetilde{\nu}^-(x) = \overline{a}_1(x)\widetilde{\tau}^-(x) + (1-x)^{1-\beta}\overline{c}(x). \qquad (3.15)$$

Applying the operator $D_{x1}^{1-2\beta}[.]$ to both parts of equality (3.15), we will obtain the basic functional relation between $\widetilde{\tau}^-(x)$ and $\widetilde{\nu}^-(x)$:

$$\gamma_2\Gamma(1-\beta)4^{2\beta-1}\widetilde{\nu}^-(x) = x^{\frac{1}{2}-\beta} D_{x1}^{1-2\beta}\overline{a}_1(x)\widetilde{\tau}^-(x) +$$

$$+x^{\frac{1}{2}-\beta}D_{x1}^{1-2\beta}(1-x)^{1-\beta}\overline{c}\,(x)\,. \tag{3.16}$$

Further, from the equation $u_{xx} - x^{n_0}y^{-m_0}u_y = 0$ at the $y \to +0$ we have receive ordinary differential equation

$$\tau^{''+}(x) - x^{n_0}\nu^+(x) = 0, \quad 0 < x < 1, \tag{3.17}$$

where $\tau^+(x) = u(x, +0)$ and $\nu^+(x) = \lim_{y \to +0} y^{-m_0}u_y(x, y)$.

Solving this equation with conditions $\tau^+(0) = \tau_0(0)$ and $\tau^+(1) = \varphi_0(0)$, deduce functional relation between $\tau^+(x)$ and $\nu^+(x)$:

$$\tau^+(x) = \int_0^1 G(x, t)t^{n_0}\nu^+(t)dt + f(x), \qquad 0x1,$$

here

$$G(x, t) = \begin{cases} t(x - t), & 0 \le t \le x, \\ (t - 1)x, & x \le t \le 1. \end{cases} \tag{3.18}$$

$$f(x) = \tau_0(0) + x(\varphi_0(0) - \tau_0(0)). \tag{3.19}$$

Further, by virtue replace $x \sim x^{\frac{1}{n+2}}$ and $t \sim t^{\frac{1}{n+2}}$ receive functional relation between $\widetilde{\tau}^+(x)$ $\widetilde{\nu}^+(x)$:

$$\widetilde{\tau}^+(x) = \int_0^1 \widetilde{G}(x, t)\widetilde{\nu}^+(t)dt + \widetilde{f}(x), \qquad 0x1, \tag{3.20}$$

where, $\widetilde{f}(x) = f\left(x^{\frac{1}{n+2}}\right)$, $\widetilde{\tau}^+(x) = \tau^+\left(x^{\frac{1}{n+2}}\right)$, $\widetilde{\nu}^+(t) = \nu^+\left(t^{\frac{1}{n+2}}\right)$,

$$G(x, t) = \frac{1}{n+2}t^{\frac{n_0+1}{n+2}-1}G\left(x^{\frac{1}{n+2}}, t^{\frac{1}{n+2}}\right). \tag{3.21}$$

3.2. Uniqueness of the solution.

Theorem 1. If satisfying the conditions (3.2),$b(x) = 0$, $a(x) \ne 0$ and

$$\overline{a}_1(x) > 0, \quad x \in (0, 1), \tag{3.22}$$

then a solution $u(x, y)$ of the problem F is unique.

The Proof. According to the extremum principle for the parabolic equations [5], [10], the solution $u(x, y)$ of the equation(3.1) cannot reach the positive maximum and negative minimum in the domain of Ω_1 and on a piece A_0B_0. We will denote, that the solution $u(x, y)$ does not reach the positive maximum and negative minimum on an interval AB.

Let's assume the return, i.e. let in some point $E(x_0, 0)$ function $u(x, y)$ reaches the positive maximum (negative minimum). Then from (3.16), at $\overline{c}(x) \equiv 0$ we have:

$$\gamma_2\Gamma(1 - \beta)4^{2\beta-1}\widetilde{\nu}^-(x_0) = x_0^{\frac{1}{2}-\beta}D_{x_01}^{1-2\beta}\overline{a}_1(x_0)\widetilde{\tau}^-(x_0)\,. \tag{3.23}$$

From here, owing to a principle for the differential operators fractional order [8], on the point of positive maximum (negative minimum) strictly positively (negatively) $D_{x_01}^{1-2\beta}\overline{a}_1(x_0)\widetilde{\tau}^-(x_0) > 0$, $\left(D_{x_01}^{1-2\beta}\overline{a}_1(x_0)\widetilde{\tau}^-(x_0) < 0\right)$. Accordingly, owing to that $x_0 > 0$, $\gamma_2 > 0$ from (3.23), receive, $\widetilde{\nu}^-(x_0) > 0$, $(\widetilde{\nu}^-(x_0) < 0)$. From here, by virtue (3.4) we have $\widetilde{\nu}^+(x_0) > 0$, $(\widetilde{\nu}^+(x_0) < 0)$. This inequality contradicts an inequality $\widetilde{\nu}^+(x_0)0$, $(\widetilde{\nu}^+(x_0) \ge 0)$, which is direct appears from (3.17).

Thus, the solution $u(x, y)$ of equation (3.1) can't reach the positive maximum and negative minimum on an interval AB. Hence, $u(x, y)$ can to reach the positive maximum (a negative minimum) on the piece of $\overline{AA_0}$ and $\overline{BB_0}$.

From here owing to (3.5), considering continuity of the function $u(x, y)$ in $\overline{\Omega}_1$, a solution of the first boundary value problem for the equation (3.1) in the domain of Ω_1 to identically equally zero at $\varphi_0(y) \equiv \tau_0(y) \equiv 0$.

As $u(x, y) \equiv 0$ in domain $\overline{\Omega}_1$ we have $\widetilde{\tau}(x) \equiv 0$, and by virtue (3.23), $\widetilde{\nu}(x) \equiv 0$. Hence, owing to unequivocal solvability of Cauchy problem it is had $u(x, y) \equiv 0$ in the domain $\overline{\Omega}_{21}$, from here $u(x, -x) \equiv 0$.

Further, from solution homogeneous first boundary value problem for the equation (3.1) in domain of Ω_1 taking into account a condition (3.7), we will receive $u_x(0, y) = u_x(0, -y) = 0$, $0 < y < 1$. Hence, the solution of the Cauchy-Gaursat problem for the equation (3.1) with zero given identically equally to zero in the domain of Ω_{22}, i.e. $u(x, y) \equiv 0$ in the domain $\overline{\Omega}_{22}$.

Thus, from the above-stated we will receive, that $u(x, y) \equiv 0$ in the domain $\overline{\Omega}$. Hence, the solution of a problem F in the domain of Ω is unique. **The theorem 1 was proved.**

3.3. Existence of the solution.

Theorem 2. If satisfying the conditions (3.2), (3.8), (3.9) and $b(x) = 0$, $a(x) \neq 0$ then a solution $u(x, y)$ of the problem F is exist in the domain of Ω.

Proof. Considering a continuity of the solution of a problem F, excluding $\widetilde{\tau}(x) = \widetilde{\tau}^-(x) = \widetilde{\tau}^+(x)$ from (3.16) and (3.20), we have

$$\gamma_2 4^{2\beta-1}\Gamma(1-\beta)\widetilde{\nu}(x) = x^{\frac{1}{2}-\beta}D_{x1}^{1-2\beta}\overline{a}_1(x)\left[\int\limits_0^1 \widetilde{G}(x,t)\widetilde{\nu}(t)dt + \widetilde{f}(x)\right] +$$

$$+x^{\frac{1}{2}-\beta}D_{x1}^{1-2\beta}\overline{c}(x)(1-x)^{1-\beta}.$$

Further, taking into account properties of integro-differential operators (2.1) [11], we find

$$\widetilde{\nu}(x) = k_1 x^{\frac{1}{2}-\beta}\frac{d}{dx}\left(\int\limits_x^1 (t-x)^{2\beta-1}\overline{a}_1(t)dt \int\limits_0^1 \widetilde{G}(t,s)\widetilde{\nu}(s)ds\right) +$$

$$+k_1 x^{\frac{1}{2}-\beta}\frac{d}{dx}\left(\int\limits_x^1 (t-x)^{2\beta-1}\overline{a}_1(t)\widetilde{f}(t)dt + \int\limits_x^1 (t-x)^{2\beta-1}(1-t)^{1-\beta}\overline{c}(t)dt\right),$$

$$(3.24)$$

where, $k_1 = 1/\gamma_2 4^{2\beta-1}\Gamma(1-\beta)\Gamma(2\beta)$

Having executed replacement $t = x + (1-x)\sigma$ and changing an order of integration from (3.24), we have

$$\widetilde{\nu}(x) = k_1 x^{\frac{1}{2}-\beta}\frac{d}{dx}\left((1-x^{2\beta})\int\limits_0^1 \sigma^{2\beta-1}\overline{a}_1(x+(1-x)\sigma)d\sigma \int\limits_0^1 \widetilde{G}(x+(1-x)\sigma,s))\widetilde{\nu}(s)ds\right) +$$

$$+k_1 x^{\frac{1}{2}-\beta}\frac{d}{dx}\left((1-x)^{2\beta}\int\limits_0^1 \sigma^{2\beta-1}\overline{a}_1(x+(1-x)\sigma)\widetilde{f}(x+(1-x)\sigma)d\sigma\right) +$$

$$+k_1 x^{\frac{1}{2}-\beta} \frac{d}{dx}\left((1-x)^{1+\beta}\int_0^1 \sigma^{2\beta-1}(1-\sigma)^{1-\beta}\overline{c}(x+(1-x)\sigma)d\sigma\right).$$

From here, after some evaluations, we will obtain the integral equation

$$\widetilde{\nu}(x) = \int_0^1 \widetilde{K}(x,s)\widetilde{\nu}(s)ds + \widetilde{\Phi}(x), \tag{3.25}$$

where

$$\widetilde{K}(x,s) = \widetilde{K}_1(x,s) + \widetilde{K}_2(x,s), \tag{3.26}$$

$$\widetilde{K}_1(x,s) = -2\beta k_1 x^{\frac{1}{2}-\beta}(1-x)^{2\beta-1}\int_0^1 \sigma^{2\beta-1}\overline{a}_1\left(x+(1-x)\sigma\right)\times$$

$$\times \widetilde{G}\left(x+(1-x)\sigma, s\right)d\sigma, \tag{3.27}$$

$$\widetilde{K}_2(x,s) = k_1 x^{\frac{1}{2}-\beta}(1-x)^{2\beta}\frac{d}{dx}\int_0^1 \sigma^{2\beta-1}\overline{a}_1(x+(1-x)\sigma)\times$$

$$\times \widetilde{G}(x+(1-x)\sigma, s)d\sigma, \tag{3.28}$$

$$\widetilde{\Phi}(x) = -2\beta k_1 x^{\frac{1}{2}-\beta}\int_0^1 \sigma^{2\beta-1}\overline{a}_1(x+(1-x)\sigma)\widetilde{f}(x+(1-x)\sigma)d\sigma+$$

$$+k_1 x^{\frac{1}{2}-\beta}(1-x)^{2\beta}\int_0^1 \sigma^{2\beta-1}\frac{d}{dx}\left[\overline{a}_1(x+(1-x)\sigma)\widetilde{f}(x+(1-x)\sigma)\right]d\sigma-$$

$$-(1+\beta)k_1 x^{\frac{1}{2}-\beta}(1-x)^{\beta}\int_0^1 \sigma^{2\beta-1}(1-\sigma)^{1-\beta}\overline{c}(x+(1-x)\sigma)d\sigma+$$

$$+k_1 x^{\frac{1}{2}-\beta}(1-x)^{1+\beta}\int_0^1 \sigma^{2\beta-1}(1-\sigma)^{1-\beta}\frac{d}{dx}\left[\overline{c}(x+(1-x)\sigma)\right]d\sigma. \tag{3.29}$$

From here, owing to continuity the functions $G(x,t) \in C([0,1]\times[0,1])$ and $a(x)$, we have

$$\left|\widetilde{K}_1(x,s)\right| c_1 s^{\frac{n_0+1}{n+2}-1}(1-x)^{2\beta-1}. \tag{3.30}$$

Also, considering (3.9),(3.14), (3.18),(3.21) from (3.28) we will receive

$$\left|\widetilde{K}_2(x,s)\right| c_2 s^{\frac{n_0+2}{n+2}-1}(1-x)^{2\beta-1}. \tag{3.31}$$

Thus, by virtue (3.30) and (3.31) from (3.26), we have

$$\left|\widetilde{K}(x,s)\right| c_3 s^{\frac{n_0+1}{n+2}-1}(1-x)^{2\beta-1}. \tag{3.32}$$

There under (3.2), (3.9), (3.14), (3.19) appear from (3.29) that the function $\widetilde{\Phi}(x)$. Supposes an estimate

$$\left|\widetilde{\Phi}(x)\right| c_4(1-x)^{2\beta-1}. \tag{3.33}$$

where, $c_1, c_2, c_3, c_4 = const.$

Thus, by virtue (3.32), (3.33) integral equation (3.25) constitute Fredholm integral equation of the second kind[6], with the weak feature which unequivocal solvability appears from the uniqueness of the solution of investigated problem, i.e. the equation (3.25) has the unique solution, and $\nu^+(x) \in C^2(0,1)$.

Hence, it is possible to present its solution on the form of[6]:

$$\widetilde{\nu}^-(x) = \widetilde{\Phi}(x) + \int\limits_0^1 R(x,s)\widetilde{\Phi}(s)ds, \tag{3.34}$$

where $R(x,s)$- resolvent the kernel of $K(x,s)$.

From here, according to gluing condition (3.4) taking into account (3.34) and (3.20) we find function $\widetilde{\tau}^+(x)$,

$$\widetilde{\tau}^+(x) = \int\limits_0^1 \widetilde{G}(x,t)\left[\widetilde{\Phi}(t) + \int\limits_0^1 R(t,z)\widetilde{\Phi}(z)dz\right]dt + \widetilde{f}(x),$$

Further, designating, $\overline{\Phi}(t) = \widetilde{\Phi}(t) + \int\limits_0^1 R(t,z)\widetilde{\Phi}(z)dz$, we have

$$\widetilde{\tau}^+(x) = \int\limits_0^1 \widetilde{G}(x,t)\overline{\Phi}(t)dt + \widetilde{f}(x), \qquad 0x1. \tag{3.35}$$

Hence, by virtue (3.2), (3.9) owing to (3.35) and (3.21), (3.19) conclude, that the function $\tau^+(x)$ in $C[0,1] \cap C^2(0,1)$.

II. Let's $b(x) \neq 0$, $a(x) \neq 0$.

From (3.13), we will receive

$$\widetilde{\nu}^-(x) = \frac{\gamma_2\Gamma(1-\beta)4^{2\beta-1}}{\overline{b}(x)\Gamma(1-2\beta)} \int\limits_x^1 (t-x)^{-2\beta}t^{\beta-\frac{1}{2}}\widetilde{\nu}^-(t)dt -$$

$$-\frac{\overline{a}_1(x)}{\overline{b}(x)}\widetilde{\tau}^-(x) - \frac{\overline{c}(x)}{\overline{b}(x)}(1-x)^{1-\beta}. \tag{3.36}$$

Let's notice, that the integral equation (2.36) is integrated Equation Volterra of the second kind

$$\widetilde{\nu}^-(x) = \lambda\int\limits_x^1 N(x,t)\widetilde{\nu}^-(t)dt + F(x), \tag{3.37}$$

where, $\lambda = \frac{\gamma_2\Gamma(1-\beta)4^{2\beta-1}}{\Gamma(1-2\beta)}$,

$$F(x) = -\frac{\overline{a}_1(x)}{\overline{b}(x)}\widetilde{\tau}^-(x) - \frac{\overline{c}(x)}{\overline{b}(x)}(1-x)^{1-\beta}, \tag{3.38}$$

$$N(x,t) = \frac{1}{\overline{b}(x)}(t-x)^{-2\beta}t^{\beta-\frac{1}{2}}. \tag{3.39}$$

By virtue (3.9), from (3.38) and (3.39) accordingly

$$|N(x,t)| \leq M, \qquad 0 \leq x \leq 1, \tag{3.40}$$

and

$$|F(x)| \leq const \tag{3.41}$$

further, owing to the theory of integrated equations Volterra of the second kind [9], taking into account (3.40) and (3.41), we have $|R(x, s; \lambda)| \leq const$, i. the solution of equation (3.37) it is possible will present on the form of

$$\widetilde{\nu}^-(x) = -\lambda \int\limits_x^1 \frac{\overline{a}_1(s)}{\overline{b}(x)} R(x, s; \lambda) \widetilde{\tau}^-(s) ds - \lambda \int\limits_x^1 \frac{R(x, s; \lambda) \overline{c}(s)}{\overline{b}(s)} ds -$$

$$-\frac{\overline{a}_1(x)}{\overline{b}(x)} \tau^-(x) - \frac{\overline{c}(x)}{\overline{b}(x)} (1-x)^{1-\beta}. \tag{3.42}$$

3.4. Uniqueness of the solution.

Theorem 3. If satisfying the conditions (3.2),(3.8), (3.9), $b(x) \neq 0$, $a(x) \neq 0$ and

$$\overline{a}_1(x) > 0, \quad \overline{b}(x) < 0, \quad 0 < x < 1, \tag{3.43}$$

then the solution $u(x, y)$ of the problem F is unique.

Proof. Let's notice, that justice of the theorem 3 the follows at once from the theorem 1, if is proved, than the solution $u(x, y)$ of the equations (3.1) cannot reach the positive maximum and negative minimum in domain of Ω_1 and on a piece $A_0 B_0$. And this statement is similarly proved as the theorem 1, i.e. by virtue principle of an extremum for the parabolic equations [5], the solution $u(x, y)$ the equation (3.1) cannot reach the positive maximum and a negative minimum in domain Ω_1 and on a piece $A_0 B_0$. Let's show, that the solution $u(x, y)$ does not reach the positive maximum and negative minimum on an interval AB. We will assume the return, i.e. let in some point $(x_0, 0)$ function $u(x, y)$ reaches the positive maximum (negative minimum). Then from (3.42), at $\overline{c}(x) \equiv 0$ we have:

$$\widetilde{\nu}^-(x_0) = -\lambda \int\limits_{x_0}^1 \frac{\overline{a}_1(s)}{\overline{b}(x_0)} R(x_0, s; \lambda) \widetilde{\tau}^-(s) ds - \frac{\overline{a}_1(x_0)}{\overline{b}(x_0)} \widetilde{\tau}^-(x_0).$$

From here considering (3.43), owing to, that $R(x_0, s; \lambda) > 0$ in the point of positive maximum (negative minimum) $\widetilde{\tau}^-(x_0) \geq 0$ $(\widetilde{\tau}^-(x_0) \leq 0)$ we will receive $\widetilde{\nu}^-(x_0) \geq 0$ $(\widetilde{\nu}^-(x_0) \leq 0)$, and this inequality contradicts an inequality $\widetilde{\nu}^+(x_0) 0$, $(\widetilde{\nu}^+(x_0) \geq 0)$, which directly follows from (3.17). Hence the solution $u(x, y)$ the equation (3.1) can't reach the positive maximum and negative minimum in domain Ω_1 and on a piece $A_0 B_0$. **The theorem 3 is proved.**

3.5. Existence of the solution.

Theorem 4. If satisfying the conditions (3.2), (3.8), (3.9) and $b(x) \neq 0$, $a(x) \neq 0$ then the solution $u(x, y)$ of the problem F is exist.

Proof. Substituting (3.42) in (3.20), we have

$$\widetilde{\tau}(x) = \int\limits_0^1 \widetilde{K}(x, s) \widetilde{\tau}(s) ds + \widetilde{f}(x), \qquad 0x1, \tag{3.44}$$

where,

$$\widetilde{K}(x, s) = -\frac{\overline{a}_1(s)}{\overline{b}(s)} \left[\widetilde{G}(x, s) + \lambda \int\limits_0^s \widetilde{G}(x, t) R(t, s; \lambda) dt \right].$$

The equation (3.44) is Fredholm integral equation the second kind[6] and it unequivocal resolubility follows from the uniqueness of the solution the problems F.

III. Let's $a(x) \equiv 0, \ b(x) \neq 0$.

3.6. Uniqueness and existence of the solution.

On the case of $a(x) \equiv 0, \ b(x) \neq 0$ takes place the following uniqueness theorem:

Theorem 5. If satisfying the conditions (3.2) and

$$\bar{b}(x) > 0, \quad 0 < x < 1, \tag{3.45}$$

then the solution $u(x, y)$ of the problem F is unique.

Proof. From the integral equation (3.36) at $a(x) \equiv 0$, taking into account (3.14) we will receive:

$$\widetilde{\nu}^-(x) = \frac{\gamma_2 \Gamma(1-\beta) 4^{2\beta-1}}{\bar{b}(x)\Gamma(1-2\beta)} \int\limits_x^1 (t-x)^{-2\beta} t^{\beta-\frac{1}{2}} \widetilde{\nu}^-(s) ds \ + F_1(x)$$

where

$$F_1(x) = \frac{\gamma_1 \Gamma(\beta)}{\bar{b}(x)} \widetilde{\tau}^-(x) - \frac{\bar{c}(x)}{\bar{b}(x)} (1-x)^{1-\beta},$$

and $|F_1(x)| \, const.$

Hence, from (3.42), at $a(x) \equiv 0$ we obtain main functional relation between $\widetilde{\tau}^-(x)$ and $\widetilde{\nu}^-(x)$:

$$\widetilde{\nu}^-(x) = \lambda \gamma_1 \Gamma(\beta) \int\limits_x^1 \frac{1}{\bar{b}(x)} R(x, s; \lambda) \widetilde{\tau}^-(s) ds - \lambda \int\limits_x^1 \frac{R(x, s; \lambda)\bar{c}(s)}{\bar{b}(s)} ds+$$

$$+\frac{\gamma_1 \Gamma(\beta)}{\bar{b}(x)} \tau^-(x) - \frac{\bar{c}(x)}{\bar{b}(x)}(1-x)^{1-\beta}. \tag{3.46}$$

Let's show, that the solution $u(x, y)$ does not reach the positive maximum and negative minimum on an interval AB. We will assume the return, i.e. let in some point $(x_0, 0)$ function $u(x, y)$ reach the positive maximum (negative minimum). Then from (3.46), at $\bar{c}(x) \equiv 0$ we have:

$$\widetilde{\nu}^-(x_0) = \lambda \gamma_1 \Gamma(\beta) \int\limits_{x_0}^1 \frac{1}{\bar{b}(x_0)} R(x_0, s; \lambda) \widetilde{\tau}^-(s) ds + \frac{\gamma_1 \Gamma(\beta)}{\bar{b}(x_0)} \widetilde{\tau}^-(x_0).$$

From here considering (3.45), owing to, that $R(x_0, s; \lambda) > 0$ in the point of positive maximum (negative minimum) $\widetilde{\tau}^-(x_0) \geq 0$ ($\widetilde{\tau}^-(x_0) \leq 0$) we will receive $\widetilde{\nu}^-(x_0) \geq 0$ ($\widetilde{\nu}^-(x_0) \leq 0$), and this inequality contradicts an inequality $\widetilde{\nu}^+(x_0) 0$, ($\widetilde{\nu}^+(x_0) \geq 0$), which directly follows from (3.17). Hence the solution $u(x, y)$ the equation (3.1) can't reach the positive maximum and negative minimum in domain Ω_1 and on a piece $A_0 B_0$. further, let's notice, that justice of the theorem 4 the follows at once from the theorem 1 and theorem 3. **The theorem 5 is proved.**

Theorem 6. If satisfying the conditions (3.2), (3.8), (3.9) and $a(x) \equiv 0$, $b(x) \neq 0$ then the solution $u(x, y)$ of the problem F is exist.

Proof. Substituting (3.46) in (3.20), we have

$$\widetilde{\tau}(x) = \int\limits_0^1 \widetilde{K}_1(x,s)\widetilde{\tau}(s)ds + \widetilde{f}(x), \qquad 0x1, \tag{3.47}$$

where,

$$\widetilde{K}_1(x,s) = \frac{\gamma_1\Gamma(\beta)}{\overline{b}(s)}\left[\widetilde{G}(x,s) + \lambda\int\limits_0^s \widetilde{G}(x,t)R(t,s;\lambda)dt\right].$$

The equation (3.47) is Fredholm integral equation the second kind[6], and it unequivocal solubility follows from the uniqueness of the solution the problems F.

Thus, the solution of the investigated problem in the domain of Ω_1 is restored as the solution of the first boundary problem which has kind of [11]:

$$u(x,y) = \int\limits_0^1 G_1(x,\xi;y,\alpha_0)\tau^+(\xi)\xi^{n_0}d\xi + y^{-m_0}\frac{\partial}{\partial y}\int\limits_0^y G_2(x,y-t,\alpha_0)\tau_0^+(t)t^{m_0}dt+$$

$$+y^{-m_0}\frac{\partial}{\partial y}\int\limits_0^y G_3(x,y-t,\alpha_0)\varphi_0(t)t^{m_0}dt,$$

where

$$G_3(x,y,\alpha_0) = (1-\alpha_0)^{2(1-\alpha_0)}x - \int\limits_0^1 G_1(x,\xi;y,\alpha_0)\left[(1-\alpha_0)^{2(1-\alpha_0)}\right]\xi^{n_0}d\xi,$$

$$G_2(x,y,\alpha_0) = 1 - (1-\alpha_0)^{2(1-\alpha_0)}x - \int\limits_0^1 G_1(x,\xi;y,\alpha_0)\left[1 - (1-\alpha_0)^{2(1-\alpha_0)}\xi\right]\xi^{n_0}d\xi,$$

$$G_1(x,\xi,y;\alpha_0) = \sum_{k=0}^{\infty} e^{-\frac{\lambda_k^2 y^{m_0+1}}{4(m_0+1)}}(1-\alpha_0)\sqrt{x\xi}\times$$

$$\times\frac{J_{1-\alpha_0}(\lambda_k(1-\alpha_0)(\sqrt{x})^{\frac{1}{1-\alpha_0}})J_{1-\alpha_0}(\lambda_k(1-\alpha_0)(\sqrt{\xi})^{\frac{1}{1-\alpha_0}})}{J_{2-\alpha_0}^2(\lambda_k)},$$

$J_\nu(z) = \sum\limits_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$ is the function of Bessel on the first kind , λ_k are positive solutions of equation $J_{1-\alpha_0}(\lambda_k) = 0$, $k = 0,1,2$. $G_1(x,\xi;y,\alpha_0)$ - the Grin function of the first boundary value problem.

Satisfying condition $\nu_0^+(y) = u_x(0,y)$, $(0 < y < 1)$ to solution of the first boundary value problem, we have:

$$\nu_0^+(y) = \lim_{x\to+0}\frac{\partial}{\partial x}\int\limits_0^1 G_1(x,\xi;y,\alpha_0)\tau_1^+(\xi)\xi^{n_0}d\xi+$$

$$+\lim_{x\to+0}\frac{\partial}{\partial x}\left[y^{-m_0}\frac{\partial}{\partial y}\int\limits_0^y G_2(x,y-t,\alpha_0)\tau_0^+(t)t^{m_0}dt\right]+$$

$$+ \lim_{x \to +0} \frac{\partial}{\partial x} \left[y^{-m_0} \frac{\partial}{\partial y} \int\limits_{0}^{y} G_3(x, y - t, \alpha_0) \varphi_0(t) t^{m_0} dt \right].$$

From here, by virtue condition(3.7), the solution of the problem F on domain of Ω_{22}, it is restored as the solution of problem Cauchy-Gaursat, satisfying to conditions $\nu^+(y) = \nu^-(y) = u_x(0, y)$, $-1 < y < 0$ $u(-y, y) = h(y)$, where $h(y)$ is the trace of solution of problem Cauchy in domain of Ω_{21} on the characteristics $y = -x$. Thus, the existence of solution of the problem F is proved.

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