Konuralp Journal of Mathematics
Volume 2 No. 1 pp. 45–62 (2014) ©KJM

INTEGRAL TRANSFORM METHOD FOR SOLVING DIFFERENT F.S.I.ES AND P.F.D.ES

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Abstract. In this work, the authors used Laplace transform to obtain formal solution to some systems of singular integral equations of fractional type. In the last section, the authors considered certain non homogeneous fractional system of heat equations with different orders which is a generalization to the problem of heat transferring from metallic bar through the surrounding media. Illustrative examples are also provided.

1. Introduction and Definitions

Fractional differential equations have been the focus of many studies due to their frequent appearance in various fields such as chemistry and engineering, physics. The main reason for success of applications fractional calculus is that these new fractional order models are more accurate than integer order models, i.e. there are more degrees of freedom in the fractional order models. The Laplace transform technique is one of most useful tools of applied mathematics. Typical applications include heat transfer, diffusion, waves, vibrations and fluid motion problems. However, contrary to expectations, it is surprising to find that the popularity of Laplace transforms, in comparison to numerical or other methods, is gradually diminishing and Laplace transform is less fashionable today than they were a few decades ago. Nevertheless, the applications of Laplace transforms continue to be an important part of the mathematical education received by students in various fields of natural sciences and engineering. The fractional diffusion equation, the fractional wave equation, the fractional advection-dispersion equation, the fractional kinetic equation and other fractional PDEs have been studied and explicit solutions have been achieved by Mainardi, Pagnini and Saxena [18], Langlands [13], Mainardi, Pagnini and Gorenflo [17], Mainardi and Pagnini [15,16], Yu and Zhang [25], Liu, Anh, Turner and Zhang [14], Saichev and Zaslavsky [21], Saxena, Mathai and Haubold [22], Wyss [24] and several other research works can be found in other literatures. In these works, the techniques of using integral transforms were used to obtain the

1991 *Mathematics Subject Classification.* 26A33; 34A08; 34K37; 35R11.

Key words and phrases. Caputo fractional derivative; Time fractional heat equation; Laplace transform; Fractional order singular integral equation system; Kelvin's functions.

The author is supported by university of Guilan.

formal solutions of fractional PDEs. Integral transforms are extensively used in solving boundary value problems and integral equations. The problem related to partial differential equations can be solved by using a special integral transform thus many authors solved the boundary value problems by using single Laplace transform. Laplace transform is very useful in applied mathematics, for instance for solving some differential equations and partial differential equations, and in automatic control, where it defines a transfer function.

The Caputo fractional derivatives of order $\alpha > 0$ $(n-1 < \alpha \leq n, n \in N)$ is defined by

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx. $$

The Laplace transform of a function $f(t)$ denoted by $F(s)$, is defined by the integral equation

$$ L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) dt := F(s). $$

Definition 1.1. The inverse Laplace transform is given by the contour integral

$$ f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, $$

where $F(s)$ is analytic in the region $Re(s) > c$.

Theorem 1.1. *For $n-1 < \alpha \leq n$, we can get*

$$ L\{{}_{0}^{C}D_{t}^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0). $$

Two-parameter Mittag-Leffler function and Wright function is given by

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(\alpha n+\beta)}, $$

$$ W(\alpha,\beta;z) = \sum_{n=0}^{\infty} \frac{z^{n}}{n!\Gamma(\alpha n+\beta)}. $$

when $\alpha, \beta, z \in C$.

Theorem 1.2. ***Schouten-Van der Pol Theorem****: Consider a function $f(t)$ which has the Laplace transform $F(s)$ which is analytic in the half-plane $Re(s) > s_0$. We can use this knowledge to find $g(t)$ whose Laplace transform $G(s)$ equals $F(\phi(s))$, where $\phi(s)$ is also analytic for $Re(s) > s_0$. This means that if*

$$ G(s) = F(\phi(s)) = \int_{0}^{\infty} f(\tau) \exp(-\phi(s)\tau) d\tau, $$

and

$$ g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\phi(s)) \exp(ts) ds, $$

then

$$g(t) = \int_0^\infty f(\tau) \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(-\phi(s)\tau) \exp(ts) ds \right) d\tau.$$

Proof. See [10]

2. Fractional Order Singular Integral Equations

The mathematical formulation of physical phenomena often involves Cauchy type, or more severe, singular integral equations. There are many applications in many important fields, like fracture mechanics, elastic contact problems, the theory of porous filtering contain integral and integro- differential equation with singular kernel. In following section, Laplace transform has been used to solve certain types of singular integral equations of fractional order. We solve a fractional order singular integral equation system. Special examples are mentioned.

Lemma 2.1. *The fractional Fredholm singular integro-differential equation of the form*

$$ {}_0^C D_x^\alpha \varphi(x) = f(x) + \lambda \int_0^\infty (\frac{x}{t})^{\frac{\nu}{2}} J_\nu (2\sqrt{xt}) \varphi(t) dt, \tag{2.1}$$

where $\varphi(0) = 0$, $0 \le \alpha \le 1$ and $\nu > -1$ has the formal solution as

$$\varphi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s^{-\alpha} F(s) + \frac{\lambda}{s^{\nu+1}} F(\frac{1}{s})}{1-\lambda^2} e^{sx} ds. \tag{2.2}$$

Proof. Let $L(\varphi(x)) = \Phi(s)$ and $L(f(x)) = F(s)$, then by using the Laplace transform of (2-1) we have the following relation

$$s^\alpha \Phi(s) = F(s) + \lambda \frac{1}{s^{\nu+1}} \Phi(\frac{1}{s}). \tag{2.3}$$

In relation (2-3) we replace s by $\frac{1}{s}$, to obtain

$$s^{-\alpha} \Phi(\frac{1}{s}) = F(\frac{1}{s}) + \lambda s^{\nu+1}\, \Phi(s). \tag{2.4}$$

Combination of (2-3) and (2-4), $\Phi(s)$ can be obtained as

$$\Phi(s) = \frac{s^{-\alpha} F(s) + \frac{\lambda}{s^{\nu+1}} F(\frac{1}{s})}{1-\lambda^2}. \tag{2.5}$$

By using the complex inversion formula, relation (2-5) leads to the following,

$$\varphi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s^{-\alpha} F(s) + \frac{\lambda}{s^{\nu+1}} F(\frac{1}{s})}{1-\lambda^2} e^{sx} ds.$$

Example 2.1. Solve the following fractional singular integral equation

$$ {}_0^C D_x^{\frac{2}{3}} \varphi(x) = \frac{1}{\sqrt{\pi x}} + \lambda \int_0^\infty (\frac{x}{t})^{\frac{1}{4}} J_{\frac{1}{2}} (2\sqrt{xt}) \varphi(t) dt,$$

Solution. By using the formula (2-2), we get

$$\begin{aligned}\varphi(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{s^{-\frac{2}{3}}F(s)+\frac{\lambda}{s^{\frac{3}{2}}}F(\frac{1}{s})}{1-\lambda^2}e^{sx}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\frac{1}{s^{\frac{7}{6}}}+\frac{\lambda}{s}}{1-\lambda^2}e^{sx}ds\\ &= \frac{1}{1-\lambda^2}(\frac{x^{\frac{1}{6}}}{\Gamma(\frac{7}{6})}+\lambda).\end{aligned}$$

Lemma 2.2. *The system of fractional Fredholm singular integro-differential equation of the form*

$$ {}_0^C D_x^{\alpha}\varphi_1(x) = f(x)+\lambda\int_0^\infty(\frac{x}{t})^{\frac{\nu}{2}}J_\nu(2\sqrt{xt})\varphi_2(t)dt,$$
$$ {}_0^C D_x^{\alpha}\varphi_2(x) = g(x)+\lambda\int_0^\infty(\frac{x}{t})^{\frac{\mu}{2}}J_\mu(2\sqrt{xt})\varphi_2(t)dt,$$

where $\varphi_1(0)=\varphi_2(0)=0$ *and* $0<\alpha,\beta\le 1$ *has the formal solutions*

(2.6)
$$\varphi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(s^{-\alpha}\left(F(s)+\frac{\lambda^2}{1-\lambda^2}\frac{G(s)}{s^{\nu-\mu}}\right)+\frac{\lambda}{1-\lambda^2}\frac{1}{s^{\nu+1}}G(\frac{1}{s})\right)e^{xs}ds,$$

$$\varphi_2(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{s^{-\alpha}G(s)+\frac{\lambda}{s^{\mu+1}}G(\frac{1}{s})}{1-\lambda^2}e^{xs}ds. \tag{2.7}$$

Proof. Applying the Laplace transform term wise to both equations and using the initial conditions yields

$$s^\alpha\Phi_1(s) = F(s)+\frac{\lambda}{s^{\nu+1}}\Phi_2(\frac{1}{s}), \tag{2.8}$$

$$s^\alpha\Phi_2(s) = G(s)+\frac{\lambda}{s^{\mu+1}}\Phi_2(\frac{1}{s}). \tag{2.9}$$

Following the same procedure as in lemma 2.1, we get $\Phi_2(s)$ as

$$\Phi_2(s) = \frac{s^{-\alpha}G(s)+\frac{\lambda}{s^{\mu+1}}G(\frac{1}{s})}{1-\lambda^2},$$

then, changing s to $\frac{1}{s}$ leads to

$$\Phi_2(\frac{1}{s}) = \frac{s^{\alpha}G(\frac{1}{s})+\lambda s^{\mu+1}G(s)}{1-\lambda^2}.$$

By replacing $\Phi_2(\frac{1}{s})$ in (2-8), we will have

$$\Phi_1(s) = s^{-\alpha}\left(F(s)+\frac{\lambda^2}{1-\lambda^2}\frac{G(s)}{s^{\nu-\mu}}\right)+\frac{\lambda}{1-\lambda^2}\frac{1}{s^{\nu+1}}G(\frac{1}{s}).$$

At this point, using the complex inversion formula, the final solutions are as follows

$$\varphi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(s^{-\alpha}\left(F(s)+\frac{\lambda^2}{1-\lambda^2}\frac{G(s)}{s^{\nu-\mu}}\right)+\frac{\lambda}{1-\lambda^2}\frac{1}{s^{\nu+1}}G(\frac{1}{s})\right)e^{x\,s}ds,$$

$$\varphi_2(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{s^{-\alpha}G(s) + \frac{\lambda}{s^{\mu+1}}G(\frac{1}{s})}{1-\lambda^2} e^{xs} ds.$$

Example 2.2. Let us solve the system

$$ {}_0^C D_x^{\frac{1}{2}} \varphi_1(x) = \frac{e^{-\frac{1}{4x}}}{2\sqrt{\pi x^3}} + \lambda \int_0^\infty (\frac{x}{t})^{\frac{3}{4}} J_{\frac{3}{2}}(2\sqrt{xt})\varphi_2(t)dt,$$

$$ {}_0^C D_x^{\frac{1}{2}} \varphi_2(x) = 1 + \lambda \int_0^\infty (\frac{x}{t})^{\frac{1}{4}} J_{\frac{1}{2}}(2\sqrt{xt})\varphi_2(t)dt,$$

where $\varphi_1(0) = \varphi_2(0) = 0$ and $0 < \alpha, \beta \leq 1$. Direct use of relations (2-6) and (2-7), leads to

$$\begin{aligned} \varphi_1(x) &= L^{-1}\left\{\frac{e^{-\sqrt{s}}}{\sqrt{s}} + \frac{\lambda^2}{1-\lambda^2}\frac{1}{s^{\frac{5}{2}}} + \frac{\lambda}{1-\lambda^2}\frac{1}{s^{\frac{3}{2}}}\right\} \\ &= \frac{e^{-\frac{1}{4x}}}{\sqrt{\pi x}} + \frac{4\lambda^2 x^{\frac{3}{2}}}{3\sqrt{\pi}(1-\lambda^2)} + \frac{2\lambda x^{\frac{1}{2}}}{\sqrt{\pi}(1-\lambda^2)}, \end{aligned}$$

$$\varphi_2(x) = L^{-1}\left\{\frac{s^{-\frac{3}{2}} + \lambda s^{-\frac{1}{2}}}{1-\lambda^2}\right\} = \frac{\frac{2}{\sqrt{\pi}}x^{\frac{1}{2}} + \frac{\lambda}{\sqrt{\pi x}}}{1-\lambda^2}.$$

2.1. **Evaluation of the Integrals.** In applied mathematics, the **Kelvin functions** $Ber_\nu(x)$ and $Bei_\nu(x)$ are the real and imaginary parts, respectively, of $J_\nu(xe^{3\pi i/4})$,where x is real, and $J_\nu(z)$ is the ν-th order Bessel function of the first kind. Similarly, the functions $Ker_\nu(x)$ and $Kei_\nu(x)$ are the real and imaginary parts, respectively, of $K_\nu(xe^{\pi i/4})$, where $K_\nu(z)$is the ν-th order modified Bessel function of the second kind. These functions are named after William Thomson, 1st Baron Kelvin. The Kelvin functions were investigated because they are involved in solutions of various engineering problems occurring in the theory of electrical currents, elasticity and in fluid mechanics. One of the main applications of Laplace transform is evaluating the integrals as discussed in the following.

Lemma 2.3. *The following integral relationship holds true*

$$\int_1^\infty \frac{bei(\sqrt{2\lambda})d\lambda}{\sqrt{\lambda^2-1}} = \frac{\pi}{2}J_0(1)I_0(1).$$

Proof. Let us define the following function

$$I(x) = \int_1^\infty \frac{bei(\sqrt{2x\lambda})d\lambda}{\sqrt{\lambda^2-1}}.$$

Laplace transform of $I(x)$ leads to

$$L\{I(x)\} = \int_0^\infty e^{-sx}\left(\int_1^\infty \frac{bei(\sqrt{2x\lambda})d\lambda}{\sqrt{\lambda^2-1}}\right)dx.$$

By changing the order of integration, which is permissible, we obtain

$$L\{I(x)\} = \int_1^\infty \frac{1}{\sqrt{\lambda^2-1}} \left(\int_0^\infty e^{-sx} bei(\sqrt{2x\lambda}) dx \right) d\lambda,$$

or

$$L\{I(x)\} = \int_1^\infty \frac{1}{\sqrt{\lambda^2-1}} (\frac{1}{s} \sin \frac{\lambda}{2s}) d\lambda.$$

At this point, let us introduce the new variable $\lambda = \cosh \xi$, we get the following

$$L\{I(x)\} = \frac{1}{s} \int_0^\infty \sin((2s)^{-1} \cosh \xi) d\xi,$$

using the following well-known integral representation for $J_0(\varphi)$

$$J_0(\varphi) = \frac{2}{\pi} \int_0^\infty \sin(\varphi \cosh \vartheta) d\vartheta.$$

One gets finally

$$L\{I(x)\} = \frac{\pi}{2s} J_0(\frac{1}{2s}),$$

now, taking inverse Laplace transform of the above relationship leads to

$$I(x) = L^{-1}\{\frac{\pi}{2s} J_0(\frac{1}{2s})\} = \frac{\pi}{2} J_0(\sqrt{x}) I_0(\sqrt{x}).$$

Letting $x = 1$ we get

$$\int_1^\infty \frac{bei(\sqrt{2\lambda}) d\lambda}{\sqrt{\lambda^2-1}} = \frac{\pi}{2} J_0(1) I_0(1).$$

Lemma 2.4. *The following integral relations hold true*

$$\int_0^1 x^{\mu-1} ber(2\sqrt{\ln x}) dx = \frac{1}{\mu} \cos \frac{1}{\mu},$$

$$\int_0^1 \frac{ber(2\sqrt{\ln x})}{\sqrt{x}} dx = 2\cos 2.$$

Proof. Let us define the following function

$$I(\xi) = \int_0^1 x^{\mu-1} ber(2\sqrt{(\ln x)\xi}) dx.$$

Laplace transform of $I(\xi)$ leads to

$$L\{I(\xi)\} = \int_0^\infty e^{-s\xi} \left(\int_0^1 x^{\mu-1} ber(2\sqrt{(\ln x)\xi}) dx \right) d\xi.$$

By changing the order of integration, which is permissible, we will have

$$L\{I(\xi)\} = \int_0^1 x^{\mu-1} \int_0^\infty e^{-s\xi} ber(2\sqrt{(\ln x)\xi}) d\xi dx.$$

But the value of inner integral is as following

$$\int_0^\infty e^{-s\xi} ber(2\sqrt{(\ln x)\xi}) d\xi = \frac{1}{s} \cos \frac{(\ln x)}{s}.$$

To prove the second relationship, by setting this value in the integral, one gets

$$L\{I(\xi)\} = \int_0^1 x^{\mu-1}\frac{1}{s}\cos\frac{(\ln x)}{s}dx = \frac{1}{s}\int_0^1 x^{\mu-1}\cos\frac{(\ln x)}{s}dx.$$

At this point, we introduce the new variable $\ln x = -w$. One gets after easy calculation

$$L\{I(\xi)\} = \frac{1}{s}\int_0^\infty e^{-\mu w}\cos(\frac{w}{s})dw = \frac{1}{\mu}\{\frac{s}{s^2+(\mu^{-1})^2}\}.$$

Taking inverse Laplace transform to obtain

$$I(\xi) = \int_0^1 x^{\mu-1}ber(2\sqrt{(\ln x)\xi})dx = \frac{1}{\mu}\cos\frac{\xi}{\mu},$$

from the above relationship, we get

$$I(1) = I_0(\mu) = \int_0^1 x^{\mu-1}ber(2\sqrt{\ln x})dx = \frac{1}{\mu}\cos\frac{1}{\mu}.$$

In the above integral, by setting 0.5 for the parameter,we obtain the second assertion

$$I_0(0.5) = \int_0^1 \frac{ber(2\sqrt{\ln x})dx}{\sqrt{x}} = 2\cos 2.$$

3. Bobylev-Cercignani Theorem and Their Applications

Bobylev and Cercignani developed a theorem [8] concerning the inversion of multivalued transforms that are analytic everywhere in the $s-$ plane except along the negative real axis. The theorem is as follows:

Theorem 3.1. ***Bobylev-Cercignani Theorem****: Let $f(t)$ denote a real-valued function, where its Laplace transform $F(s)$ exists. Let $F(s)$ satisfy the following hypothesis:*

1) $F(s)$ is a multi-valued function which has no singularities in the cut $s-$ plane. The branch cut lies along the negative real axis $(-\infty, 0]$.

2) $F^*(s) = F(s^*)$, where the star denotes the complex conjugate.

3) $F^{\pm}(\eta) = \lim_{\phi\to\pi^-} F(\eta e^{\pm\phi i})$ and $F^+(\eta) = (F_-(\eta))^*$.

4) $F(s\ ) = o(1)$ as $|s| \to \infty$ and $F(s) = o(\frac{1}{|s|})$ as $|s| \to 0$, uniformly in any sector $|\arg(s)| < \pi - \eta$, $0 < \eta < \pi$.

5) There exists $\varepsilon > 0$, such that for every $\pi - \varepsilon < \phi \le \pi$, $\frac{F(re^{\pm\phi i})}{1+r} \in L_1(R^+)$ and $|F(re^{\pm\phi i})| < a(r)$, where $a(r)$ does not depend on ϕ and $a(r)e^{-\delta r} \in L_1(R^+)$ for any $\delta > 0$.Then

$$f(t) = \frac{1}{\pi}\int_0^\infty Im(\,F^-(\eta))e^{-t\eta}d\eta.$$

In following lemma, we apply this theorem.

Lemma 3.1. *The following relationship holds true*

$$L^{-1}\left\{\frac{1}{s+1}\exp\left(-x\sqrt{\frac{\mu+s^{\alpha}}{\lambda+s^{\alpha}}}\right)\right\}=\frac{1}{\pi}\int_{0}^{\infty}Im(F^{-}(\eta))e^{-t\eta}d\eta.$$

where $0<\alpha<1$, $\lambda,\mu>0$ *and*

$$Im(F^{-}(\eta))=\frac{e^{-x\sqrt{\frac{\rho_1}{\rho_2}}\cos(\frac{\theta_1-\theta_2}{2})}}{\eta-1}\sin\left(x\sqrt{\frac{\rho_1}{\rho_2}}\sin(\frac{\theta_1-\theta_2}{2})\right).$$

Proof. $F(s)$ satisfies all of the conditions listed in the theorem 3.1. Then

$$\begin{aligned}F^{-}(\eta)&=\lim_{\phi\to\pi}F(\eta e^{-\phi i})=\frac{1}{\eta e^{-\pi i}+1}\exp\left(-x\sqrt{\frac{\eta^{\alpha}e^{-\pi\alpha i}+\mu}{\eta^{\alpha}e^{-\pi\alpha i}+\lambda}}\right)\\&=\frac{1}{1-\eta}\exp\left(-x\sqrt{\frac{\rho_1}{\rho_2}}e^{\frac{i(\theta_1-\theta_2)}{2}}\right)\\&=\frac{1}{1-\eta}\exp\left(-x\sqrt{\frac{\rho_1}{\rho_2}}(\cos(\frac{\theta_1-\theta_2}{2})+i\sin(\frac{\theta_1-\theta_2}{2}))\right),\end{aligned}$$

where

$$\rho_1=\sqrt{\eta^{2\alpha}+2\mu\eta^{\alpha}\cos\pi\alpha+\mu^2},\rho_2=\sqrt{\eta^{2\alpha}+2\lambda\eta^{\alpha}\cos\pi\alpha+\lambda^2},$$

$$\theta_1=-\tan^{-1}\left(\frac{\eta^{\alpha}\sin\alpha\pi}{\eta^{\alpha}\cos\alpha\pi+\mu}\right),\theta_2=-\tan^{-1}\left(\frac{\eta^{\alpha}\sin\alpha\pi}{\eta^{\alpha}\cos\alpha\pi+\lambda}\right)\ (0<\theta<\pi).$$

Image part of $F^{-}(\eta)$ is founded as

$$Im(F^{-}(\eta))=\frac{e^{-x\sqrt{\frac{\rho_1}{\rho_2}}\cos(\frac{\theta_1-\theta_2}{2})}}{\eta-1}\sin\left(x\sqrt{\frac{\rho_1}{\rho_2}}\sin(\frac{\theta_1-\theta_2}{2})\right).$$

Finally, the inverse Laplace transform is as

$$f(t)=\frac{1}{\pi}\int_{0}^{\infty}Im(F^{-}(\eta))e^{-t\eta}d\eta.$$

Problem 1. *Let us consider the following* ***four terms*** *partial fractional differential equation*

$$\frac{\partial}{\partial x}\{\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}\}+a\frac{\partial^{\beta}u(x,t)}{\partial t^{\beta}}=\lambda u(x,t)-b\frac{\partial u(x,t)}{\partial x},$$

where $0<\alpha<1$, $0<\beta\leq1$, $0<x<\infty$, $t,a,b>0$ *with the boundary conditions*

$$u(0,t)=\frac{t^{\gamma-1}}{\Gamma(\gamma)}\ (\gamma>0),\lim_{x\to\infty}|u(x,t)|<\infty,$$

and the initial conditions $u(x,0)=u_x(x,0)=0$.

Solution. Applying the Laplace transform of the equation and using the boundary and initial conditions leads to differential equation with respect to x as

$$U_x(x,t) + \frac{as^\beta - \lambda}{s^\alpha + b} U(x,t) = 0,$$

when $L\{u(x,t)\} = U(x,s)$. Solution of the above equation yields

$$U(x,s) = \frac{1}{s^\gamma} \exp\left(-x\frac{as^\beta - \lambda}{s^\alpha + b}\right).$$

$U(x,s)$ satisfies all of the conditions explained in the theorem 3.1. Hence

$$\begin{aligned} U^-(x,\eta) &= \lim_{\phi\to\pi} U(x,\eta e^{-\phi i}) = \frac{1}{\eta^\gamma e^{-\pi\gamma i}} \exp\left(-x\frac{a\eta^\beta e^{-\pi\beta i} - \lambda}{\eta^\alpha e^{-\pi\alpha i} + b}\right) \\ &= \frac{e^{\pi\gamma i}}{\eta^\gamma} \exp\left(-x\frac{(a\eta^\beta e^{-\pi\beta i} - \lambda)(\eta^\alpha e^{\pi\alpha i} + b)}{\rho}\right), \end{aligned}$$

where $\rho = \eta^{2\alpha} + 2b\eta^\alpha \cos \pi\alpha + b^2$. Therefore

$$Im(U^-(x,\eta)) =$$

$$\frac{1}{\eta^\gamma} \exp\left\{-x\frac{ab\eta^\beta \cos\beta\pi + a\eta^{\alpha+\beta}\cos(\alpha-\beta)\pi - \lambda\eta^\alpha \cos\alpha\pi - \lambda b}{\rho}\right\}$$
$$\times \sin\left\{\pi\gamma - x\frac{a\eta^{\alpha+\beta}\sin(\alpha-\beta)\pi - ab\eta^\beta \sin\beta\pi - \lambda\eta^\alpha \sin\alpha\pi}{\rho}\right\}.$$

Finally, $u(x,\, t)$ is found to be

$$u(x,t) = \frac{1}{\pi}\int_0^\infty Im(U^-(x,\eta))e^{-t\eta}d\eta.$$

4. Partial Fractional Differential Equation (PFDE) with Moving Boundary

In PFDE problems, Laplace transforms are particularly useful when the boundary conditions are time dependent. We consider now the case when one of the boundaries is moving. This type of problem arises in combustion problems where the boundary moves due to the burning of the fuel [10]. Such fractional partial differential equations have not been studied in the literature.

Problem 2. *Let us solve the following three terms time-fractional heat equation with moving boundaries*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a^2\frac{\partial^2 u(x,t)}{\partial x^2} + \lambda\frac{\partial u(x,t)}{\partial x} \quad (0 < \alpha \leq 1), \tag{4.1}$$

where $\lambda \in R$, $\beta t < x < \infty$, $t > 0$ *and subject to the boundary conditions*

$$u(x,t)\bigg|_{x=\beta t} = \frac{1}{\sqrt{\pi t}}\exp(-\frac{1}{4t}),\ \lim_{x\to\infty}|u(x,t)| < \infty,$$

and the initial condition $u(x,0) = 0$ $(0 < x < \infty)$.

Solution: We introduce the change of variable $\eta = x - \beta t$. The above equation can be reformulated as

$$\frac{\partial^\alpha w(\eta,t)}{\partial t^\alpha} - \beta \frac{\partial}{\partial \eta}\left({}_0I_t^{1-\alpha} w(\eta,t)\right) = a^2 \frac{\partial^2 w(\eta,t)}{\partial \eta^2} + \lambda \frac{\partial w(\eta,t)}{\partial \eta}, \tag{4.2}$$

where $0 < \eta < \infty$, $t > 0$ subject to the boundary conditions

$$w(0,t) = \frac{1}{\sqrt{\pi t}} \exp(-\frac{1}{4t}), \lim_{\eta \to \infty} |w(\eta,t)| < \infty,$$

and the initial condition $w(\eta,0) = 0$ $(0 < \eta < \infty)$. By applying the Laplace transform of the equation (4-2), we obtain

$$\frac{\partial^2 W(\eta,s)}{\partial \eta^2} + \frac{1}{a^2}(\frac{\beta}{s^{1-\alpha}} + \lambda)\frac{\partial W(\eta,s)}{\partial \eta} - \frac{s^\alpha}{a^2} W(\eta,s) = 0, \tag{4.3}$$

with conditions

$$W(0,s) = \frac{e^{-\sqrt{s}}}{\sqrt{s}}, \lim_{\eta \to \infty} |W(\eta,t)| < \infty.$$

Differential equation (4-3) has the solution as

$$W(\eta,s) = \frac{e^{-\sqrt{s}}}{\sqrt{s}} \exp\left(-\frac{\lambda \eta}{2a^2} - \frac{\beta \eta}{2a^2 s^{1-\alpha}} - \frac{\eta}{2}\sqrt{\frac{1}{a^4}(\frac{\beta}{s^{1-\alpha}} + \lambda)^2 + \frac{4s^\alpha}{a^2}}\right).$$

Case 1: If $\alpha = 1$, then

$$W(\eta,s) = e^{-(\lambda+\beta)\frac{\eta}{2a^2}} \frac{e^{-\sqrt{s}}}{\sqrt{s}} \exp\left(-\frac{\eta}{a}\sqrt{\frac{1}{4\,a^2}(\beta+\lambda)^2 + s}\right).$$

Using the fact that

$$L^{-1}\left\{\exp\left(-\frac{\eta}{a}\sqrt{\frac{1}{4a^2}(\beta+\lambda)^2 + s}\right)\right\} = e^{-\frac{1}{4a^2}(\beta+\lambda)^2 t} \frac{\eta}{2a\sqrt{\pi t^3}} e^{-\frac{\eta^2}{4\,a^2 t}},$$

and using the Laplace transform inversion and then applying the convolution theorem in this transform, we get $w(\eta,t)$ as

$$\begin{aligned} w(\eta,t) &= L^{-1}\{W(\eta,s)\} \\ &= \frac{\eta}{2a\pi} e^{-(\lambda+\beta)\frac{\eta}{2a^2}} \int_0^t \frac{e^{-\frac{1}{4(t-\tau)}}}{\sqrt{\tau^3(t-\tau)}} e^{-\frac{1}{4a^2}\beta+\lambda)^2 \tau} e^{-\frac{\eta^2}{4\,a^2 \tau}} d\tau. \end{aligned}$$

Therefore we obtain $u(x,t)$ as following

$$u(x,t) = \frac{x - \beta t}{2a\pi} e^{-(\lambda+\beta)\frac{x-\beta t}{2a^2}} \int_0^t \frac{e^{-\frac{1}{4(t-\tau)}}}{\sqrt{\tau^3(t-\tau)}} e^{-\frac{1}{4a^2}(\beta+\lambda)^2 \tau} e^{-\frac{(x-\beta t)^2}{4a^2 \tau}} d\tau.$$

Case 2: If $\alpha \neq 1$, then

$$W(\eta,s)=e^{-\frac{\lambda\eta}{2a^2}}\frac{1}{\sqrt{s}}\exp\left(-\sqrt{s}-\frac{\beta\eta}{2a^2s^{1-\alpha}}-\frac{\eta}{2}\sqrt{\frac{\beta^2}{a^4s^{2-2\alpha}}+\frac{2\beta\lambda}{a^4s^{1-\alpha}}+\frac{4s^\alpha}{a^2}+\lambda^2}\right),$$

and we can use the theorem 3.1, hence

$$W^-(\eta,\xi)=\lim_{\phi\to\pi}W(\eta,\xi e^{-\phi i})=\frac{e^{-\frac{\lambda\eta}{2a^2}}e^{\sqrt{\xi}e^{-\frac{\pi i}{2}}}}{\sqrt{\xi}e^{-\frac{\pi i}{2}}}\times$$

$$\exp\left(-\frac{\beta\eta}{2a^2\xi^{1-\alpha}e^{(\alpha-1)\pi i}}-\frac{\eta}{2}\sqrt{\frac{\beta^2}{a^4\xi^{2-2\alpha}e^{(2\alpha-2)\pi i}}+\frac{2\beta\lambda}{a^4\xi^{1-\alpha}e^{(\alpha-1)\pi i}}+\frac{4\xi^\alpha e^{-\alpha\pi i}}{a^2}+\lambda^2}\right)$$

$$=\frac{e^{-\frac{\lambda\eta}{2a^2}}e^{i(\frac{\pi}{2}-\sqrt{\xi})}}{\sqrt{\xi}}\exp\left(\frac{\beta\eta e^{-\alpha\pi i}}{2a^2\xi^{1-\alpha}}-\frac{\eta}{2}\sqrt{\frac{\beta^2e^{-2\alpha\pi i}}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})e^{-\alpha\pi i}+\lambda^2}\right)$$

$$=\frac{e^{-\frac{\lambda\eta}{2a^2}}e^{i(\frac{\pi}{2}-\sqrt{\xi})}}{\sqrt{\xi}}\times$$

$$e^{\frac{\beta\eta(\cos\alpha\pi-i\sin\alpha\pi)}{2a^2\xi^{1-\alpha}}-\frac{\eta}{2}\sqrt{\left[\frac{\beta^2\cos2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\cos\alpha\pi+\lambda^2\right]-i\left[\frac{\beta^2\sin2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\sin\alpha\pi\right]}}$$

$$=\frac{e^{-\frac{\lambda\eta}{2a^2}}e^{i(\frac{\pi}{2}-\sqrt{\xi})}}{\sqrt{\xi}}e^{\frac{\beta\eta(\cos\alpha\pi-i\sin\alpha\pi)}{2a^2\xi^{1-\alpha}}-\frac{\eta}{2}\sqrt{\rho}e^{\frac{\theta i}{2}}},$$

where

$$\rho=\sqrt{\left\{\frac{\beta^2\cos2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\cos\alpha\pi+\lambda^2\right\}^2+\left\{\frac{\beta^2\sin2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\sin\alpha\pi\right\}^2}$$

$$\theta=-\tan^{-1}\left(\frac{\frac{\beta^2\sin2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\sin\alpha\pi}{\frac{\beta^2\cos2\alpha\pi}{a^4\xi^{2-2\alpha}}+(\frac{4\xi^\alpha}{a^2}-\frac{2\beta\lambda}{a^4\xi^{1-\alpha}})\cos\alpha\pi+\lambda^2}\right)\quad(0<\theta<\pi).$$

Then imaginary part of $W^-(\eta,\xi)$ is

$$Im(W^-(\eta,\xi))=\frac{e^{-\frac{\lambda\eta}{2a^2}}}{\sqrt{\xi}}e^{\frac{\beta\eta\cos\alpha\pi}{2a^2\xi^{1-\alpha}}-\frac{\eta}{2}\sqrt{\rho}\cos\frac{\theta}{2}}\cos(\sqrt{\xi}+\frac{\beta\eta}{2a^2\xi^{1-\alpha}}\sin\alpha\pi+\frac{\eta}{2}\sqrt{\rho}\sin\frac{\theta}{2}).$$

The formal solution will be as follows,

$$u(x,t)=\frac{1}{\pi}\int_0^\infty Im(W^-(x-\beta\xi,\xi))e^{-t\xi}d\xi.$$

5. A Non-Homogenous System of Fractional Heat equations with different orders

In this section, we consider certain non-homogeneous fractional system of heat equations (different orders) which is a generalization to the problem of heat transferring from metallic bar through the surrounding media studied by V.A. Ditkin, P.A. Prudnikov [9]. The basic goal of this work has been to implement the Laplace transform method for studying the above mentioned problem. The goal has been achieved by formally deriving exact analytical solution.

Problem 3. *We consider the following system of fractional PDE with different orders in Caputo sense*

$$ {}^cD_t^{\alpha}u+\gamma u=1+\frac{\partial^2 u}{\partial x^2}+\lambda a\frac{\partial v}{\partial r}\bigg|_{r=a}, \tag{5.1}$$

$$ {}^cD_t^{\delta}v-\beta v=\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}, \tag{5.2}$$

where $0<\alpha,\delta<1$*,* $t>0$*,* $-l\le x\le l$*,* $r\ge a$ *and* $\beta,\gamma\in R$ *with the boundary conditions*

$$u(x,0)=v(x,r,0)=0,u(-l,t)=u(l,t)=0,$$

and

$$v(x,a,t)=u(x,t),\ \lim_{r\to\infty}v(x,r,t)=0.$$

Solution: By taking the Laplace transform of relation (5-2), we get

$$r^2V_{rr}+rV_r+(i\sqrt{s^{\delta}-\beta})^2r^2V=0.$$

Let us assume that $L\{v(x,r,t)\}=V(x,r,s)$, then one has

$$V(x,r,s)=c_1J_0(i\sqrt{s^{\delta}-\beta}r)+c_2Y_0(i\sqrt{s^{\delta}-\beta}r),$$

where J_0 and Y_0 are Bessel functions of the first and second kind of order zero, respectively. Using this fact that $\lim_{r\to\infty}v(x,r,t)=0$, we get

$$V(x,r,s)=c_1J_0(i\sqrt{s^{\delta}-\beta}r).$$

But $v(x,a,t)=u(x,t)$, therefore

$$V(x,r,s)=\frac{J_0(i\sqrt{s^{\delta}-\beta}r)}{J_0(i\sqrt{s^{\delta}-\beta}a)}U(x,s),$$

where $L\{u(x,t)\}=U(x,s)$. On the other hand, we have

$$\frac{\partial V}{\partial r}\bigg|_{r=a}=U(x,s)\left(-i\sqrt{s^{\delta}-\beta}\frac{J_1(i\sqrt{s^{\delta}-\beta}a)}{J_0(i\sqrt{s^{\delta}-\beta}a)}\right).$$

Applying the Laplace transform term wise to relation (5-1), we obtain

$$s^{\alpha}U=U_{xx}-i\lambda a\sqrt{s^{\delta}-\beta}\frac{J_1(i\sqrt{s^{\delta}-\beta}a)}{J_0(i\sqrt{s^{\delta}-\beta}a)}U+\frac{1}{s}-\gamma U,$$

or

$$U_{xx} - h(s)U = -\frac{1}{s}, \tag{5.3}$$

where

$$h(s) = s^{\alpha} + \gamma + i\lambda a\sqrt{s^{\delta} - \beta}\frac{J_1(i\sqrt{s^{\delta} - \beta}a)}{J_0(i\sqrt{s^{\delta} - \beta}a)}.$$

Differential equation (5-3) has the following solution

$$U(x,s) = c_1\cosh(\sqrt{h(s)}x) + c_2\sinh(\sqrt{h(s)}x) + \frac{1}{sh(s)}.$$

Using the boundary conditions $u(-l,t) = u(l,t) = 0$ leads to

$$U(x,s) = \frac{1}{sh(s)}\left(1 - \frac{\cosh(\sqrt{h(s)}x)}{\cosh(\sqrt{h(s)}l)}\right).$$

Let us assume that

$$F(x,h(s)) = 1 - \frac{\cosh(\sqrt{h(s)}x)}{\cosh(\sqrt{h(s)}l)},$$

then we get

$$U(x,s) = \frac{F(x,h(s))}{sh(s)}.$$

Now, if

$$L_t\{\phi(x,t)\} = \frac{F(x,s)}{s}, L_t\{\psi(\xi,t)\} = \frac{e^{-\xi h(s)}}{s},$$

then

$$u(x,t) = L_t^{-1}\{U(x,s)\} = L^{-1}\{\frac{F(x,h(s))}{sh(s)}\} = \int_0^{\infty}\psi(\xi,t)\phi(x,\xi)d\xi.$$

Finally, we will have

$$\begin{aligned}
\phi(x,t) &= L_t^{-1}\{\frac{F(x,s)}{s}\} = L_t^{-1}\left\{\frac{1}{s}(1 - \frac{\cosh(\sqrt{s}x)}{\cosh(\sqrt{s}l)})\right\} \\
&= 1 - L_t^{-1}\left\{\frac{\cosh(\sqrt{s}x)}{s\cosh(\sqrt{s}l)}\right\} = 1 - L_t^{-1}\left\{e^{\sqrt{s}(x-l)}\frac{1+e^{-2\sqrt{s}x}}{s(1+e^{-2\sqrt{s}l})}\right\} \\
&= 1 - \sum_{n=0}^{\infty} L_t^{-1}\left\{\frac{\exp(-((2n+1)l-x)\sqrt{s})}{s} - \frac{\exp(-((2n+1)l+x)\sqrt{s})}{s}\right\} \\
&= 1 - \sum_{n=0}^{\infty}\left(erfc(\frac{(2n+1)l-x}{2\sqrt{t}}) - erfc(\frac{(2n+1)l+x}{2\sqrt{t}})\right).
\end{aligned}$$

Also,

Figure 1

$$h(s) = s^{\alpha} + \gamma + i\lambda a\sqrt{s^{\delta}-\beta}\frac{J_1(i\sqrt{s^{\delta}-\beta}a)}{J_0(i\sqrt{s^{\delta}-\beta}a)},$$

hence

$$\begin{aligned}
\psi(\xi,t) &= L_t^{-1}\{\frac{e^{-\xi h(s)}}{s}\} \\
&= L_t^{-1}\left\{e^{-\xi\gamma}\frac{e^{-\xi s^{\alpha}}}{s}\exp\left(-i\xi\lambda a\sqrt{s^{\delta}-\beta}\frac{J_1(i\sqrt{s^{\delta}-\beta}a)}{J_0(i\sqrt{s^{\delta}-\beta}a)}\right)\right\} \\
&= e^{-\xi\gamma}L_t^{-1}\left\{\frac{\sqrt{s^{\delta}-\beta}}{s}\frac{e^{-\xi s^{\alpha}}}{\sqrt{s^{\delta}-\beta}}\exp\left(-i\xi\lambda a\sqrt{s^{\delta}-\beta}\frac{J_1(i\sqrt{s^{\delta}-\beta}a)}{J_0(i\sqrt{s^{\delta}-\beta}a)}\right)\right\}.
\end{aligned}$$

Case 1: Assume that $\delta = 1$, therefore

$$\begin{aligned}
f_1(\xi,t) &= L_t^{-1}\left\{\frac{1}{\sqrt{s-\beta}}\exp\left(-i\xi\lambda a\sqrt{s-\beta}\frac{J_1(i\sqrt{s-\beta}a)}{J_0(i\sqrt{s-\beta}a)}\right)\right\} \\
&= e^{\beta t}L_t^{-1}\left\{\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)\right\}.
\end{aligned}$$

The inverse Laplace transform is given by

$$L_t^{-1}\left\{\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds.$$

The integrand has a branch point at the origin and it is thus necessary to choose a contour which does not contain the origin. We deform the Bromwich contour so that the circular arc BDE is terminated just short of the horizontal axis and the arc LNA starts just below the horizontal axis. In between the contour follows an inclined path EH followed by a circular arcHJK enclosing the origin and a return section KL meeting the arc LNA (see figure). As there are no singularities inside this contour C, we have

$$\int_C \frac{1}{\sqrt{s}}\exp\left(-i\xi\lambda a\sqrt{s}\frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)}\right)e^{ts}ds = 0.$$

Now on BDE and LNA, we get

$$\left| \frac{1}{\sqrt{s}} \exp\left(-i\xi\lambda a\sqrt{s} \frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)} \right) \right| \leq \frac{1}{\sqrt{s}},$$

so that the integrals over these arcs tend to zero as $R \to \infty$. Over the circular arc HJK as its radius $\varepsilon \to 0$, we have $s = \varepsilon e^{i\theta}$, $\phi \leq \theta \leq -\phi$. Thus

$$\lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \int_{HJK} \frac{1}{\sqrt{s}} \exp\left(-i\xi\lambda a\sqrt{s} \frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)} \right) e^{ts} ds = 0.$$

Along EH, $s = ue^{i\phi}$, $\sqrt{s} = \sqrt{u} e^{\frac{i\phi}{2}}$, hence

$$\lim_{\substack{R \to \infty \\ \varepsilon \to 0, \phi \to \pi}} \int_{EH} \frac{1}{\sqrt{s}} \exp\left(-i\xi\lambda a\sqrt{s} \frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)} \right) e^{ts} ds =$$
$$\int_0^\infty \frac{1}{i\sqrt{u}} \exp\left(-\xi\lambda a\sqrt{u} \frac{J_1(\sqrt{u}a)}{J_0(\sqrt{u}a)} \right) e^{-tu} du.$$

Similarly, along KL, $s = ue^{-i\phi}$, $\sqrt{s} = \sqrt{u} e^{-\frac{i\phi}{2}}$, then

$$\lim_{\substack{R \to \infty \\ \varepsilon \to 0, \phi \to \pi}} \int_{KL} \frac{1}{\sqrt{s}} \exp\left(-i\xi\lambda a\sqrt{s} \frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)} \right) e^{ts} ds =$$
$$\int_0^\infty \frac{1}{i\sqrt{u}} \exp\left(-\xi\lambda a\sqrt{u} \frac{J_1(\sqrt{u}a)}{J_0(\sqrt{u}a)} \right) e^{-tu} du.$$

Consequently, we have

$$\frac{1}{2\pi i} \int_C \frac{1}{\sqrt{s}} \exp\left(-i\xi\lambda a\sqrt{s} \frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)} \right) e^{ts} ds = \frac{1}{2\pi i} \int_{AB} ds + \frac{1}{2\pi i} \int_{BDE} ds$$
$$+ \frac{1}{2\pi i} \int_{EH} ds + \frac{1}{2\pi i} \int_{HJK} ds + \frac{1}{2\pi i} \int_{KL} ds + \frac{1}{2\pi i} \int_{LNA} ds = 0.$$

The final result is as

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\sqrt{s}} \exp\left(-i\xi\lambda a\sqrt{s} \frac{J_1(i\sqrt{s}a)}{J_0(i\sqrt{s}a)} \right) e^{ts} ds =$$
$$\frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{u}} \exp\left(-\xi\lambda a\sqrt{u} \frac{J_1(\sqrt{u}a)}{J_0(\sqrt{u}a)} \right) e^{-tu} du.$$

Thus we obtain

$$\begin{aligned} f_1(\xi, t) &= L_t^{-1} \left\{ \frac{1}{\sqrt{s-\beta}} \exp\left(-i\xi\lambda a\sqrt{s-\beta} \frac{J_1(i\sqrt{s-\beta}a)}{J_0(i\sqrt{s-\beta}a)} \right) \right\} \\ &= \frac{1}{\pi} e^{\beta t} \int_0^\infty \frac{1}{\sqrt{u}} \exp\left(-\xi\lambda a\sqrt{u} \frac{J_1(\sqrt{u}a)}{J_0(\sqrt{u}a)} \right) e^{-tu} du. \end{aligned}$$

In case of $0 < \delta < 1$, we get

$$\begin{aligned} f_2(\xi,t) &= L_t^{-1}\left\{\frac{1}{\sqrt{s^\delta-\beta}}\exp\left(-i\xi\lambda a\sqrt{s^\delta-\beta}\frac{J_1(i\sqrt{s^\delta-\beta}a)}{J_0(i\sqrt{s^\delta-\beta}a)}\right)\right\} \\ &= \frac{1}{t}\int_0^\infty f_1(\xi,\tau)W(-\delta,0;-\tau t^{-\delta})d\tau. \end{aligned}$$

Also, for $0<\delta<1$,

$$\begin{aligned} f_3(t) &= L_t^{-1}\left\{\frac{\sqrt{s^\delta-\beta}e^{-\xi s^\alpha}}{s}\right\} = L_t^{-1}\left\{s^{\frac{\delta}{2}-1}(1-\beta s^{-\delta})^{\frac{1}{2}}e^{-\xi s^\alpha}\right\} \\ &= \sum_{n=0}^\infty(-\beta)^n\begin{pmatrix}\frac{1}{2}\\ n\end{pmatrix}L_t^{-1}\{s^{-\delta n+\frac{\delta}{2}-1}e^{-\xi s^\alpha}\} \\ &= \sum_{n=0}^\infty(-\beta)^n\begin{pmatrix}\frac{1}{2}\\ n\end{pmatrix}L_t^{-1}\left\{s^{-\delta n+\frac{\delta}{2}-1}\sum_{k=0}^\infty\frac{(-\xi)^k s^{\alpha k}}{k!}\right\} \\ &= \sum_{n=0}^\infty(-\beta)^n\begin{pmatrix}\frac{1}{2}\\ n\end{pmatrix}\left\{\sum_{k=0}^\infty\frac{(-\xi)^k}{k!}\frac{t^{\delta n-\alpha k-\frac{\delta}{2}}}{\Gamma(\delta n-\alpha k-\frac{\delta}{2}+1)}\right\}. \end{aligned}$$

Consequently

$$\psi(\xi,t)=L_t^{-1}\{\frac{1}{s}\exp(-\xi h(s))\}=e^{-\xi\gamma}\int_0^t f_2(\xi,\eta)f_3(t-\eta)d\eta \quad : 0<\alpha,\delta<1$$

Finally, we obtain $u(x,t)$ as follows

$$\begin{aligned} u(x,t) &= \int_0^\infty\psi(\xi,t)\phi(x,\xi)d\xi \\ &= \int_0^\infty e^{-\xi\gamma}\left(\int_0^t f_2(\xi,\eta)f_3(t-\eta)d\eta\right) \end{aligned}$$

$$\times\left(1-\sum_{n=0}^\infty\left(erfc(\frac{(2n+1)l-x}{2\sqrt{\xi}})-erfc(\frac{(2n+1)l+x}{2\sqrt{\xi}})\right)\right)d\xi.$$

Now, we should determine the inverse Laplace transform of the following term

$$V(x,r,s)=U(x,s)\frac{J_0(i\sqrt{s^\delta-\beta}r)}{J_0(i\sqrt{s^\delta-\beta}a)}.$$

If $\delta=1$, we obtain

$$g_1(r,t)=L_t^{-1}\left\{\frac{J_0(i\sqrt{s-\beta}r)}{J_0(i\sqrt{s-\beta}a)}\right\}=\frac{2}{a^2}\sum_{k=0}^\infty\frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}\exp(-(\frac{\lambda_k^2}{a^2}-\beta)t),$$

where $\lambda_1,\lambda_2,\lambda_3,\ldots$ are roots of $J_0(i\sqrt{s-\beta}a)$. For $0<\delta<1$, we conclude that

$$
\begin{aligned}
g_2(r,t) &= L_t^{-1}\left\{\frac{J_0(i\sqrt{s^\delta-\beta}r)}{J_0(i\sqrt{s^\delta-\beta}a)}\right\}\\
&= \frac{2}{a^2t}\sum_{k=0}^{\infty}\frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}\int_0^\infty \exp(-(\frac{\lambda_k^2}{a^2}-\beta)\tau)W(-\delta,0;-\tau t^{-\delta})d\tau\\
&= \frac{2}{a^2t}\sum_{k=0}^{\infty}\frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}L\left\{W(-\delta,0;-\tau t^{-\delta});\tau\to s\right\}\Bigg|_{\,s=\frac{\lambda_k^2}{a^2}-\beta}\\
&= \frac{2}{t}\sum_{k=0}^{\infty}\frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)}\left(\sum_{n=0}^{\infty}\frac{(-a^2)^n t^{-\delta n}}{\Gamma(-\delta n)(\lambda_k^2-a^2\beta)^{n+1}}\right)\\
&= \frac{2}{t}\sum_{k=0}^{\infty}\frac{\lambda_k J_0(\frac{\lambda_k}{a}r)}{J_1(\lambda_k)(\lambda_k^2-a^2\beta)}E_{-\delta,0}(-\frac{a^2t^{-\delta}}{\lambda_k^2-a^2\beta}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
v(x,r,t) &= L_t^{-1}\{V(x,r,s)\} = L_t^{-1}\left(U(x,s)\frac{J_0(i\sqrt{s^\delta-\beta}r)}{J_0(i\sqrt{s^\delta-\beta}a)}\right)\\
&= \int_0^t u(x,\eta)g_2(r,t-\eta)d\eta \quad : 0<\alpha,\delta<1.
\end{aligned}
$$

6. Acknowledgment

The authors would like to thank the anonymous referee for helpful comments and suggestions.

7. Conclusion

The paper is devoted to study and applications of Laplace transform. The main purpose of this work is to develop a method for finding formal solution of certain systems of Fredholm fractional singular integral equations of second kind, analytic solution of the time fractional heat equation and system of partial fractional differential equations with different orders, which is a generalization to certain types of problems in the literature. Numerous non trivial examples and exercises provided throughout the paper. We hope that it will also benefit many researchers in the disciplines of applied mathematics, mathematical physics and engineering.

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