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ON SPECTRAL PROPERTIES FOR A REGULAR STURM-LIOUVILLE PROBLEM

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Abstract. In this work we study a discontinuous boundary-value problem with retarded argument which contains a spectral parameter in the transmission conditions. We firstly prove the existence theorem and then obtain asymptotic representation of eigenvalues and eigenfunctions.

1. Introduction

The theory of differential equations with retarded arguments is one of the actual branch of the theory of ordinary differential equations. Particularly, there has been increasing interest in spectral analysis of boundary value problems. There is quite substantial literature concerning such problems. Here we mention the results of [1–19].

In this paper we study the eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument and spectral parameters in the transmission conditions. Namely we consider the boundary value problem for the differential equation

$$y''(x) + q(x)y(x - \Delta(x)) + \lambda y(x) = 0 \tag{1.1}$$

on $[0, h_1) \cup (h_1, h_2) \cup (h_2, \pi]$, with boundary conditions

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \tag{1.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \tag{1.3}$$

and transmission conditions

$$y(h_1 - 0) - \sqrt[3]{\lambda}\delta y(h_1 + 0) = 0, \tag{1.4}$$

$$y'(h_1 - 0) - \sqrt[3]{\lambda}\delta y'(h_1 + 0) = 0, \tag{1.5}$$

$$y(h_2 - 0) - \gamma y(h_2 + 0) = 0, \tag{1.6}$$

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$$y'(h_2-0)-\gamma y'(h_2+0)=0, \tag{1.7}$$

where the real-valued function $q(x)$ is continuous in $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$ and has finite limits $q(h_1\pm 0)=\lim_{x\to h_1\pm 0}q(x)$, $q(h_2\pm 0)=\lim_{x\to h_2\pm 0}q(x)$ the real valued function $\Delta(x)\geq 0$ continuous in $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$ and has finite limits $\Delta(h_1\pm 0)=\lim_{x\to h_1\pm 0}\Delta(x)$, $\Delta(h_2\pm 0)=\lim_{x\to h_2\pm 0}\Delta(x)$, $x-\Delta(x)\geq 0$, if $x\in[0,h_1)\,;x-\Delta(x)\geq h_1$, if $x\in(h_1,h_2)\,,\,x-\Delta(x)\geq h_2$, if $x\in(h_2,\pi]\,;\,\lambda$ is a real spectral parameter; δ,γ are arbitrary real numbers and $\sin\alpha\sin\beta\neq 0$.

Let $w_1(x,\lambda)$ be a solution of Equation (1.1) on $[0,h_1]$, satisfying the initial conditions

$$w_1(0,\lambda)=\sin\alpha,w_1'(0,\lambda)=-\cos\alpha. \tag{1.8}$$

The conditions (1.8) define a unique solution of Equation (1.1) on $[0,h_1]$ ([2], p. 12).

After defining the above solution we shall define the solution $w_2(x,\lambda)$ of Equation (1.1) on $[h_1,h_2]$ by means of the solution $w_1(x,\lambda)$ using the initial conditions

$$w_2(h_1,\lambda)=\lambda^{-1/3}\delta^{-1}w_1(h_1,\lambda)\,,\quad w_2'(h_1,\lambda)=\lambda^{-1/3}\delta^{-1}w_1'(h_1,\lambda). \tag{1.9}$$

The conditions (1.9) are defined as a unique solution of Equation (1.1) on $[h_1,h_2]$.

After defining the above solution we shall define the solution $w_3(x,\lambda)$ of Equation (1.1) on $[h_2,\pi]$ by means of the solution $w_2(x,\lambda)$ using the initial conditions

$$w_3(h_2,\lambda)=\gamma^{-1}w_2(h_2,\lambda)\,,\quad w_3'(h_2,\lambda)=\gamma^{-1}w_2'(h_2,\lambda). \tag{1.10}$$

The conditions (1.10) are defined as a unique solution of Equation (1.1) on $[h_2,\pi]$.

Consequently, the function $w(x,\lambda)$ is defined on $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$ by the equality

$$w(x,\lambda)=\begin{cases} w_1(x,\lambda), & x\in[0,h_1)\,,\\ w_2(x,\lambda), & x\in(h_1,h_2)\,,\\ w_3(x,\lambda), & x\in(h_2,\pi]\end{cases}$$

is a solution of the Equation (1.1) on $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$;which satisfies one of the.boundary conditions and transmission conditions (1.4)-(1.5).

Lemma 1.1. *Let $w(x,\lambda)$ be a solution of Equation* (1.1) *and $\lambda>0$. Then the following integral equations hold:*

$$\begin{aligned} w_1(x,\lambda)&=\sin\alpha\cos sx-\frac{\cos\alpha}{s}\sin sx\\ &-\frac{1}{s}\int\limits_0^x q(\tau)\sin s(x-\tau)\,w_1(\tau-\Delta(\tau),\lambda)\,d\tau\quad\left(s=\sqrt{\lambda},\lambda>0\right), \end{aligned}\tag{1.11}$$

$$\begin{aligned} w_2(x,\lambda)&=\frac{1}{s^{2/3}\delta}w_1(h_1,\lambda)\cos s(x-h_1)+\frac{w_1'(h_1,\lambda)}{s^{5/3}\delta}\sin s(x-h_1)\\ &-\frac{1}{s}\int\limits_{h_1}^x q(\tau)\sin s(x-\tau)\,w_2(\tau-\Delta(\tau),\lambda)\,d\tau\quad\left(s=\sqrt{\lambda},\lambda>0\right), \end{aligned}\tag{1.12}$$

$$w_3(x,\lambda) = \frac{1}{\gamma} w_2(h_2,\lambda)\cos s(x-h_2) + \frac{w_2'(h_2,\lambda)}{s\gamma}\sin s(x-h_2)$$

$$- \frac{1}{s}\int_{h_2}^{x} q(\tau)\sin s(x-\tau)\, w_3(\tau-\Delta(\tau),\lambda)\, d\tau \quad \left(s=\sqrt{\lambda}, \lambda>0\right). \tag{1.13}$$

Proof. To prove this, it is enough to substitute $-s^2 w_1(\tau,\lambda) - w_1''(\tau,\lambda)$, $-s^2 w_2(\tau,\lambda) - w_2''(\tau,\lambda)$ and $-s^2 w_3(\tau,\lambda) - w_3''(\tau,\lambda)$ instead of $-q(\tau)w_1(\tau-\Delta(\tau),\lambda)$, $-q(\tau)w_2(\tau-\Delta(\tau),\lambda)$ and $-q(\tau)w_3(\tau-\Delta(\tau),\lambda)$ in the integrals in (1.11), (1.12) and (1.13) respectively and integrate by parts twice. □

Theorem 1.1. *The problem* (1.1) – (1.7) *can have only simple eigenvalues.*

Proof. Let $\widetilde{\lambda}$ be an eigenvalue of the problem (1.1) – (1.7) and

$$\widetilde{y}(x,\widetilde{\lambda}) = \begin{cases} \widetilde{y}_1(x,\widetilde{\lambda}), & x\in[0,h_1), \\ \widetilde{y}_2(x,\widetilde{\lambda}), & x\in(h_1,h_2), \\ \widetilde{y}_3(x,\widetilde{\lambda}), & x\in(h_2,\pi] \end{cases}$$

be a corresponding eigenfunction. Then from (1.2) and (1.8) it follows that the determinant

$$W\left[\widetilde{y}_1(0,\widetilde{\lambda}), w_1(0,\widetilde{\lambda})\right] = \begin{vmatrix} \widetilde{y}_1(0,\widetilde{\lambda}) & \sin\alpha \\ \widetilde{y}_1'(0,\widetilde{\lambda}) & -\cos\alpha \end{vmatrix} = 0,$$

and by Theorem 2.2 in [2] the functions $\widetilde{y}_1(x,\widetilde{\lambda})$ and $w_1(x,\widetilde{\lambda})$ are linearly dependent on $[0,h_1]$. We can also prove that the functions $\widetilde{y}_2(x,\widetilde{\lambda})$, $w_2(x,\widetilde{\lambda})$ are linearly dependent on $[h_1,h_2]$ and $\widetilde{y}_3(x,\widetilde{\lambda})$, $w_3(x,\widetilde{\lambda})$ are linearly dependent on $[h_2,\pi]$. Hence

$$\widetilde{y}_i(x,\widetilde{\lambda}) = K_i w_i(x,\widetilde{\lambda}) \quad \left(i=\overline{1,3}\right) \tag{1.14}$$

for some $K_1\neq 0$, $K_2\neq 0$ and $K_3\neq 0$. We must show that $K_1=K_2=K_3$. Suppose that $K_1\neq K_2$. From the equalities (1.4) and (1.14), we have

$$\begin{aligned} \widetilde{y}(h_1-0,\widetilde{\lambda}) - \sqrt[3]{\widetilde{\lambda}}\delta\widetilde{y}(h_1+0,\widetilde{\lambda}) &= \widetilde{y}_1(h_1,\widetilde{\lambda}) - \sqrt[3]{\widetilde{\lambda}}\delta\widetilde{y}_2(h_1,\widetilde{\lambda}) \\ &= K_1 w_1(h_1,\widetilde{\lambda}) - \sqrt[3]{\widetilde{\lambda}}\delta K_2 w_2(h_1,\widetilde{\lambda}) \\ &= \sqrt[3]{\widetilde{\lambda}}\delta K_1 w_2(h_1,\widetilde{\lambda}) - \sqrt[3]{\widetilde{\lambda}}\delta K_2 w_2(h_1,\widetilde{\lambda}) \\ &= \sqrt[3]{\widetilde{\lambda}}\delta\left(K_1-K_2\right) w_2(h_1,\widetilde{\lambda}) = 0. \end{aligned}$$

Since $\delta(K_1-K_2)\neq 0$ it follows that

$$w_2\left(h_1,\widetilde{\lambda}\right) = 0. \tag{1.15}$$

By the same procedure from equality (1.5), we can derive that

$$w_2'\left(h_1,\widetilde{\lambda}\right) = 0. \tag{1.16}$$

From the fact that $w_2(x,\widetilde{\lambda})$ is a solution of the differential Equation (1.1) on $[h_1,h_2]$ and satisfies the initial conditions (1.15) and (1.16) it follows that $w_2(x,\widetilde{\lambda})=0$ identically on $[h_1,h_2]$ (cf. [2, p. 12, Theorem 1.2.1]).

By using this, we may also find

$$w_1\left(h_1,\widetilde{\lambda}\right) = w_1'\left(h_1,\widetilde{\lambda}\right) = 0$$

From the latter discussions of $w_2(x,\widetilde{\lambda})$ it follows that $w_1(x,\widetilde{\lambda})=0$ identically on $[0,h_1]$. But this contradicts (1.8), thus completing the proof. Analogically we can show that $K_2=K_3$. □

2. An existence theorem

The function $w(x,\lambda)$ defined in section 1 is a nontrivial solution of Equation (1.1) satisfying conditions (1.2) and (1.4)-(1.7). Putting $w(x,\lambda)$ into (1.3), we get the characteristic equation

$$F(\lambda)\equiv w(\pi,\lambda)\cos\beta+w'(\pi,\lambda)\sin\beta=0. \tag{2.1}$$

By Theorem 1.1, the set of eigenvalues of boundary-value problem (1.1)-(1.7) coincides with the set of real roots of Eq. (2.1). Let $q_1=\int\limits_0^{h_1}|q(\tau)|d\tau$, $q_2=\int\limits_{h_1}^{h_2}|q(\tau)|\,d\tau$ and $q_3=\int\limits_{h_2}^{\pi}|q(\tau)|\,d\tau$.

Lemma 2.1. (1) *Let $\lambda\geq 4q_1^2$. Then for the solution $w_1(x,\lambda)$ of Equation* (1.11)*, the following inequality holds*:

$$|w_1(x,\lambda)|\leq\frac{1}{|q_1|}\sqrt{4q_1^2\sin^2\alpha+\cos^2\alpha},\quad x\in[0,h_1]. \tag{2.2}$$

(2) *Let $\lambda\geq\max\left\{4q_1^2,4q_2^2\right\}$. Then for the solution $w_2(x,\lambda)$ of Equation* (1.12)*, the following inequality holds*:

$$|w_2(x,\lambda)|\leq\frac{2.5198421}{\sqrt[3]{q_1^5}\delta}\sqrt{4q_1^2\sin^2\alpha+\cos^2\alpha},\quad x\in[h_1,h_2]. \tag{2.3}$$

(3) *Let $\lambda\geq\max\left\{4q_1^2,4q_2^2,4q_3^2\right\}$. Then for the solution $w_2(x,\lambda)$ of Equation* (1.13)*, the following inequality holds*:

$$|w_3(x,\lambda)|\leq\frac{10.0793684}{\sqrt[3]{q_1^5}\delta\gamma}\sqrt{4q_1^2\sin^2\alpha+\cos^2\alpha},\quad x\in[h_2,\pi]. \tag{2.4}$$

Proof. Let $B_{1\lambda}=\max_{[0,h_1]}|w_1(x,\lambda)|$. Then from (1.11), it follows that, for every $\lambda>0$, the following inequality holds:

$$B_{1\lambda}\leq\sqrt{\sin^2\alpha+\frac{\cos^2\alpha}{s^2}}+\frac{1}{s}B_{1\lambda}q_1.$$

If $s\geq 2q_1$ we get (2.2). Differentiating (1.11) with respect to x, we have

$$w_1'(x,\lambda)=-s\sin\alpha\sin sx-\cos\alpha\cos sx-\int\limits_0^x q(\tau)\cos s(x-\tau)\,w_1(\tau-\Delta(\tau),\lambda)d\tau. \tag{2.5}$$

From (2.5) and (2.2), it follows that, for $s\geq 2q_1$, the following inequality holds:.

$$\frac{|w_1'(x,\lambda)|}{s^{5/3}}\leq\frac{1}{\sqrt[3]{4q_1^5}}\sqrt{4q_1^2\sin^2\alpha+\cos^2\alpha}. \tag{2.6}$$

Let $B_{2\lambda} = \max_{[h_1,h_2]} |w_2(x,\lambda)|$. Then from (1.12), (2.2) and (2.6) it follows that, for $s \geq 2q_1$ and $s \geq 2q_2$, the following inequalities hold:

$$B_{2\lambda} \leq \frac{2}{\sqrt[3]{4q_1^5}\delta}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha} + \frac{1}{2q_2}B_{2\lambda}q_2,$$

$$B_{2\lambda} \leq \frac{2\sqrt[3]{2}}{\sqrt[3]{q_1^5}\delta}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}.$$

Hence if $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$ we get (2.3).

Differentiating (1.12) with respect to x, we have

$$w_2'(x,\lambda) = -\frac{\sqrt[3]{s}}{\delta}w_1(h_1,\lambda)\sin s(x-h_1) + \frac{w_1'(h_1,\lambda)}{\sqrt[3]{s^2}\delta}\cos s(x-h_1)$$

$$-\int_{h_1}^{x} q(\tau)\cos s(x-\tau)\, w_2(\tau-\Delta(\tau),\lambda)d\tau. \tag{2.7}$$

From (2.7) and (2.3), it follows that, for $s \geq 2q_1$, the following inequality holds:.

$$\frac{|w_2'(x,\lambda)|}{s} \leq \frac{\sqrt[3]{16}}{\delta\sqrt[3]{q_1^5}}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}. \tag{2.8}$$

Let $B_{3\lambda} = \max_{[h_2,\pi]} |w_3(x,\lambda)|$. Then from (1.13), (2.2), (2.3) and (2.8) it follows that, for $s \geq 2q_1$, $s \geq 2q_2$ and $s \geq 2q_3$, the following inequalities hold:

$$B_{3\lambda} \leq \frac{\sqrt[3]{2^4}}{\sqrt[3]{q_1^5}\delta\gamma}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha} + \frac{\sqrt[3]{2^4}}{\sqrt[3]{q_1^5}\delta\gamma}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha} + \frac{1}{2q_3}B_{3\lambda}q_3,$$

$$B_{3\lambda} \leq \frac{\sqrt[3]{2^{10}}}{\sqrt[3]{q_1^5}\delta\gamma}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}.$$

Hence if $\lambda \geq \max\left\{4q_1^2, 4q_2^2, 4q_3^2\right\}$ we get (2.4). □

Theorem 2.1. *The problem* (1.1)-(1.7) *has an infinite set of positive eigenvalues.*

Proof. Differentiating (1.9) with respect to$x$, we get

$$w_3'(x,\lambda) = -\frac{s}{\gamma}w_2(h_2,\lambda)\sin s(x-h_2) + \frac{w_2'(h_2,\lambda)}{\gamma}\cos s(x-h_2)$$

$$-\int_{h_2}^{x} q(\tau)\cos s(x-\tau)\, w_3(\tau-\Delta(\tau),\lambda)d\tau. \tag{2.9}$$

From (1.11), (1.12), (1.13), (2.1), (2.5), (2.7) and (2.9), we get

$$\left[\frac{1}{\gamma}\left\{\frac{1}{s^{2/3}\delta}\left(\sin\alpha\cos sh_1 - \frac{\cos\alpha}{s}\sin sh_1 - \frac{1}{s}\int_0^{h_1} q(\tau)\sin s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\right.\right.$$

$$\times\cos s(h_2-h_1)$$

$$-\frac{1}{s^{5/3}\delta}\left(s\sin\alpha\sin sh_1 + \cos\alpha\cos sh_1 + \int_0^{h_1} q(\tau)\cos s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$

$$\times \sin s\,(h_2-h_1) - \frac{1}{s}\int_{h_1}^{h_2} q(\tau)\sin s(h_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau \Bigg\} \cos s\,(\pi-h_2) +$$

$$\frac{1}{s\gamma}\Bigg\{-\frac{s^{1/3}}{\delta}\left(\sin\alpha\cos sh_1 - \frac{\cos\alpha}{s}\sin sh_1 - \frac{1}{s}\int_0^{h_1} q(\tau)\sin s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$

$$\times \sin s\,(h_2-h_1)$$

$$-\frac{1}{s^{2/3}\delta}\left(s\sin\alpha\sin sh_1 + \cos\alpha\cos sh_1 + \int_0^{h_1} q(\tau)\cos s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$

$$\times \cos s\,(h_2-h_1) - \int_{h_1}^{h_2} q(\tau)\cos s(h_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau \Bigg\} \sin s\,(\pi-h_2)$$

$$\left.-\frac{1}{s}\int_{h_2}^{\pi} q(\tau)\sin s\,(\pi-\tau)\,w_3(\tau-\Delta(\tau),\lambda)d\tau\right] \cos\beta +$$

$$\Bigg[-\frac{s}{\gamma}\Bigg\{\frac{1}{s^{2/3}\delta}\left(\sin\alpha\cos sh_1 - \frac{\cos\alpha}{s}\sin sh_1 - \frac{1}{s}\int_0^{h_1} q(\tau)\sin s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$

$$\times \cos s\,(h_2-h_1)$$

$$-\frac{1}{s^{5/3}\delta}\left(s\sin\alpha\sin sh_1 + \cos\alpha\cos sh_1 + \int_0^{h_1} q(\tau)\cos s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$

$$\times \sin s\,(h_2-h_1) - \frac{1}{s}\int_{h_1}^{h_2} q(\tau)\sin s(h_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau \Bigg\} \sin s\,(\pi-h_2) +$$

$$\frac{1}{\gamma}\Bigg\{-\frac{s^{1/3}}{\delta}\left(\sin\alpha\cos sh_1 - \frac{\cos\alpha}{s}\sin sh_1 - \frac{1}{s}\int_0^{h_1} q(\tau)\sin s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$

$$\times \sin s\,(h_2-h_1)$$

$$-\frac{1}{s^{2/3}\delta}\left(s\sin\alpha\sin sh_1 + \cos\alpha\cos sh_1 + \int_0^{h_1} q(\tau)\cos s(h_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$

$$\times \cos s\,(h_2-h_1) - \int_{h_1}^{h_2} q(\tau)\cos s(h_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau \Bigg\} \cos s\,(\pi-h_2)$$

$$\left.-\int_{h_2}^{\pi} q(\tau)\cos s\,(\pi-\tau)\,w_3(\tau-\Delta(\tau),\lambda)d\tau\right] \sin\beta = 0. \tag{2.10}$$

Let λ be sufficiently large. Then, by (2.2)-(2.4), Equation (2.10) may be rewritten in the form

$$\sqrt[3]{s}\sin s\pi + O(1) = 0. \tag{2.11}$$

Obviously, for large s Equation (2.11) has an infinite set of roots. Thus the theorem is proved. □

3. Asymptotic Formulas for Eigenvalues and Eigenfunctions

Now we begin to study asymptotic properties of eigenvalues and eigenfunctions. In the following we shall assume that$s$is sufficiently large. From (1.11) and (2.2), we get

$$w_1(x,\lambda) = O(1) \quad \text{on} \quad [0, h_1]. \tag{3.1}$$

From (1.12) and (2.3), we get

$$w_2(x,\lambda) = O(1) \quad \text{on} \quad [h_1, h_2]. \tag{3.2}$$

From (1.13) and (2.4), we get

$$w_3(x,\lambda) = O(1) \quad \text{on} \quad [h_2, \pi]. \tag{3.3}$$

The existence and continuity of the derivatives $w'_{1s}(x,\lambda)$ for $0 \le x \le h_1, |\lambda| < \infty$, $w'_{2s}(x,\lambda)$ for $h_1 \le x \le h_2, |\lambda| < \infty$ and $w'_{3s}(x,\lambda)$ for $h_2 \le x \le \pi, |\lambda| < \infty$ follows from Theorem 1.4.1 in [2].

$$w'_{1s}(x,\lambda) = O(1), \quad x \in [0, h_1], \tag{3.4}$$

$$w'_{2s}(x,\lambda) = O(1), \quad x \in [h_1, h_2], \tag{3.5}$$

$$w'_{3s}(x,\lambda) = O(1), \quad x \in [h_2, \pi] \tag{3.6}$$

hold.

Proof. By differentiating (1.13) with respect to s, we get, by (3.3)
(3.7)

$$w'_{3s}(x,\lambda) = -\frac{1}{s}\int_0^x q(\tau)\cos s(x-\tau) w'_{3s}\left(\tau - \Delta\left(\tau\right), \lambda\right) + \theta(x,\lambda), \quad (|\theta(x,\lambda)| \le \theta_0).$$

Let $D_\lambda = \max_{[h_2,\pi]} \left|w'_{3s}(x,\lambda)\right|$. Then the existance of D_λ follows from continuity of derivation for $x \in [h_2, \pi]$. From (3.7)

$$D_\lambda \le \frac{1}{s} q_3 D_\lambda + \theta_0.$$

Now let $s \ge 2q_3$. Then $D_\lambda \le 2\theta_0$ and the validity of the asymptotic formula (3.6) follows. Formulas (3.4) and (3.5) may be proved analogically. □

Theorem 3.1. *Let n be a natural number. For each sufficiently large n there is exactly one eigenvalue of the problem* (1.1)-(1.7) *near n^2.*

Proof. We consider the expression which is denoted by $O(1)$ in Equation (2.11):

$$\frac{\delta\gamma}{\sin\alpha\sin\beta}\left\{-\frac{\sin(\alpha-\beta)}{s^{2/3}\delta\gamma}\cos s\pi + \frac{\cos\alpha\cos\beta}{s^{5/3}\delta\gamma}\sin s\pi\right.$$
$$-\frac{1}{\delta\gamma}\int_0^{h_1}\left[\frac{\cos\beta}{s^{5/3}}\sin s(\pi-\tau) + \frac{\sin\beta}{s^{2/3}}\cos s\left(\pi-\tau\right)\right] q\left(\tau\right) w_1\left(\tau - \Delta\left(\tau\right), \lambda\right) d\tau$$
$$+\frac{1}{\gamma}\int_{h_1}^{h_2}\left[\frac{\cos\beta}{s^{5/3}}\sin s(\pi-\tau) + \frac{\sin\beta}{s^{2/3}}\cos s\left(\pi-\tau\right)\right] q\left(\tau\right) w_2\left(\tau - \Delta\left(\tau\right), \lambda\right) d\tau$$
$$\left.+\int_{h_2}^{\pi}\left[\frac{\cos\beta}{s}\sin s(\pi-\tau) + \sin\beta\cos s\left(\pi-\tau\right)\right] q\left(\tau\right) w_3\left(\tau - \Delta\left(\tau\right), \lambda\right) d\tau\right\}.$$

If formulas (3.1)-(3.6) are taken into consideration, it can be shown by differentiation with respect to s that for large s this expression has bounded derivative. It is obvious that for large s the roots of Equation (2.11) are situated close to entire numbers. We shall show that, for large n, only one root (2.11) lies near to each n. We consider the function $\phi(s) = \sqrt[3]{s}\sin s\pi + O(1)$. Its derivative, which has the form $\phi'(s) = \frac{1}{3\sqrt[3]{s^2}}\sin s\pi + \sqrt[3]{s}\pi\cos\pi + O(1)$, does not vanish for s close to n for sufficiently large n. Thus our assertion follows by Rolle's Theorem. □

Let n be sufficiently large. In what follows we shall denote by$\lambda_n = s_n^2$ the eigenvalue of the problem (1.1)-(1.7) situated near n^2. We set $s_n = n + \delta_n$. From (2.11) it follows that $\delta_n = O\left(\frac{1}{n^{1/3}}\right)$.

Consequently

$$s_n = n + O\left(\frac{1}{n^{1/3}}\right). \tag{3.8}$$

Formula (3.8) make it possible to obtain asymptotic expressions for eigenfunction of the problem (1.1)-(1.7). From (1.11), (2.5) and (3.1), we get

$$w_1(x,\lambda) = \sin\alpha\cos sx + O\left(\frac{1}{s}\right), \tag{3.9}$$

$$w_1'(x,\lambda) = -s\sin\alpha\sin sx + O(1). \tag{3.10}$$

From (1.12), 2.6), (3.2), (3.9) and (3.10), we get

$$w_2(x,\lambda) = \frac{\sin\alpha}{s^{2/3}\delta}\cos sx + O\left(\frac{1}{s}\right), \tag{3.11}$$

$$w_2'(x,\lambda) = -\frac{s^{1/3}\sin\alpha}{\delta}\sin sx + O(1). \tag{3.12}$$

From (1.13), (2.7), (3.3), (3.11) and (3.12), we get

$$w_3(x,\lambda) = \frac{\sin\alpha}{s^{2/3}\delta\gamma}\cos sh_2\cos s(x-h_2) - \frac{\sin\alpha}{s^{2/3}\delta\gamma}\sin sh_2\sin s(x-h_2) + O\left(\frac{1}{s}\right),$$

(3.13)

$$w_3(x,\lambda) = \frac{\sin\alpha}{s^{2/3}\delta\gamma}\cos sx + O\left(\frac{1}{s}\right).$$

By substituting (3.8) into (3.9), (3.11) and (3.12), we find that

$$u_{1n} = w_1(x,\lambda_n) = \sin\alpha\cos nx + O\left(\frac{1}{n^{1/3}}\right),$$
$$u_{2n} = w_2(x,\lambda_n) = \frac{\sin\alpha}{\delta n^{2/3}}\cos nx + O\left(\frac{1}{n}\right),$$
$$u_{3n} = w_3(x,\lambda_n) = \frac{\sin\alpha}{\delta\gamma n^{2/3}}\cos nx + O\left(\frac{1}{n}\right).$$

Hence the eigenfunctions $u_n(x)$ have the following asymptotic representation:

$$u_n(x) = \begin{cases} \sin\alpha\cos nx + O\left(\frac{1}{n^{1/3}}\right), & x\in[0,h_1), \\ \frac{\sin\alpha}{\delta n^{2/3}}\cos nx + O\left(\frac{1}{n}\right), & x\in(h_1,h_2), \\ \frac{\sin\alpha}{\delta\gamma n^{2/3}}\cos nx + O\left(\frac{1}{n}\right), & x\in(h_2,\pi]. \end{cases}$$

Under some additional conditions the more exact asymptotic formulas which depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled:

a) The derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded in $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$ and have finite limits $q'(h_i\pm 0)=\lim\limits_{x\to h_i\pm 0}q'(x)$ and $\Delta''(h_i\pm 0)=\lim\limits_{x\to h_i\pm 0}\Delta''(x)$ $(i=1,2)$, respectively.

b) $\Delta'(x)\le 1$ in $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$, $\Delta(0)=0$ and $\lim\limits_{x\to h_i+0}\Delta(x)=0$ $(i=1,2)$.

By using b), we have

$$x-\Delta(x)\ge 0, x\in[0,h_1)\,, \tag{3.14}$$

$$x-\Delta(x)\ge h_1, x\in(h_1,h_2)\,, \tag{3.15}$$

$$x-\Delta(x)\ge h_1, x\in(h_2,\pi]\,. \tag{3.16}$$

From Equations (3.9), (3.11) and (3.13)-(3.16) we have

$$w_1(\tau-\Delta(\tau),\lambda)=\sin\alpha\cos s(\tau-\Delta(\tau))+O\left(\frac{1}{s}\right), \tag{3.17}$$

$$w_2(\tau-\Delta(\tau),\lambda)=\frac{\sin\alpha}{s^{2/3}\delta}\cos s(\tau-\Delta(\tau))+O\left(\frac{1}{s}\right), \tag{3.18}$$

$$w_3(\tau-\Delta(\tau),\lambda)=\frac{\sin\alpha}{s^{2/3}\delta\gamma}\cos s(\tau-\Delta(\tau))+O\left(\frac{1}{s}\right). \tag{3.19}$$

Putting these expressions into Equation (2.10), we have

$$\begin{aligned}&-\frac{s^{1/3}}{\delta\gamma}\sin\alpha\sin\beta\sin s\pi+\frac{\sin(\alpha-\beta)}{s^{2/3}\delta\gamma}\cos s\pi-\frac{\sin\alpha\sin\beta}{s^{2/3}\delta\gamma}\\ &\times\left\{\cos s\pi_0^\pi\frac{q(\tau)}{2}\left[\cos s\Delta(\tau)+\cos s(2\tau-\Delta(\tau))\right]d\tau\right.\\ &\left.+\sin s\pi_0^\pi\frac{q(\tau)}{2}\left[\sin s\Delta(\tau)+\sin s(2\tau-\Delta(\tau))\right]d\tau\right\}+O\left(\frac{1}{s^{5/3}}\right)=0.\end{aligned} \tag{3.20}$$

Let

$$\begin{cases}A(x,s,\Delta(\tau))=\frac{1}{2}\int\limits_0^x q(\tau)\sin(s\Delta(\tau))\,d\tau,\\ B(x,s,\Delta(\tau))=\frac{1}{2}\int\limits_0^x q(\tau)\cos(s\Delta(\tau))\,d\tau.\end{cases} \tag{3.21}$$

It is obvious that these functions are bounded for $0\le x\le\pi, 0<s<+\infty$.

Under the conditions a) and b) the following formulas

$$\int\limits_0^x q(\tau)\cos s(2\tau-\Delta(\tau))d\tau=O\left(\frac{1}{s}\right),\quad\int\limits_0^x q(\tau)\sin s(2\tau-\Delta(\tau))d\tau=O\left(\frac{1}{s}\right) \tag{3.22}$$

can be proved by the same technique in Lemma 3.3 in [2]. From Equations (3.20), (3.21) and(3.22),we have

$$\begin{aligned}&\sin s\pi\left[s\sin\alpha\sin\beta+A(\pi,s,\Delta(\tau))\sin\alpha\sin\beta\right]-\\ &\cos s\pi\left[\sin\alpha\cos\beta-\cos\alpha\sin\beta-B(\pi,s,\Delta(\tau))\sin\alpha\sin\beta\right]+O\left(\frac{1}{s}\right)=0.\end{aligned}$$

Hence

$$\tan s\pi = \frac{1}{s}\left[\cot\beta - \cot\alpha - B\left(\pi, s, \Delta\left(\tau\right)\right)\right] + O\left(\frac{1}{s^2}\right).$$

Again if we take $s_n = n + \delta_n$, then

$$\tan\left(n + \delta_n\right)\pi = \tan\delta_n\pi = \frac{1}{n}\left[\cot\beta - \cot\alpha - B\left(\pi, n, \Delta\left(\tau\right)\right)\right] + O\left(\frac{1}{n^2}\right).$$

Hence for large n,

$$\delta_n = \frac{1}{n\pi}\left[\cot\beta - \cot\alpha - B\left(\pi, n, \Delta\left(\tau\right)\right)\right] + O\left(\frac{1}{n^2}\right)$$

and finally

$$s_n = n + \frac{1}{n\pi}\left[\cot\beta - \cot\alpha - B\left(\pi, n, \Delta\left(\tau\right)\right)\right] + O\left(\frac{1}{n^2}\right). \tag{3.23}$$

Thus, we have proven the following theorem.

Theorem 3.2. *If conditions a) and b) are satisfied then, the positive eigenvalues* $\lambda_n = s_n^2$ *of the problem (1.1)-(1.7) have the asymptotic representation of (3.23) forn* $\to \infty$.

We now may obtain a more exact asymptotic formula for the eigenfunctions. From Equations (1.11), (3.17), (3.21) and (3.22)

$$w_1(x,\lambda) = \sin\alpha\cos sx\left[1 + A\left(x, \rangle\, s, \rangle\, \Delta(\tau)\right)\right.$$

$\bar{s}$

$$-\frac{\sin sx}{s}\left[\cos\alpha + \sin\alpha B\left(x, \rangle\, s, \rangle\, \Delta(\tau) + O\left(\frac{1}{s^2}\right).\right.\right. \tag{3.24}$$

Replacing s by s_n and using Equation (3.23), we have

$$u_{1n}(x) = w_1(x,\lambda_n) = \sin\alpha\left\{\cos nx\left[1 + \frac{A\left(x, n, \Delta\left(\tau\right)\right)}{n}\right]\right.$$

(3.25)

$$\left.-\frac{\sin nx}{n\pi}\left[\left(\cot\beta - \cot\alpha - B\left(\pi, n, \Delta\left(\tau\right)\right)\right)x + \left(\cot\alpha + B\left(x, n, \Delta\left(\tau\right)\right)\right)\pi\right]\right\} + O\left(\frac{1}{n^2}\right).$$

From (2.5), (3.18), (3.21), (3.22) and (3.24), we have

$$w_2\left(x,\lambda\right) = \frac{\sin\alpha}{s^{2/3}\delta}\cos sx\left[1 + \frac{A\left(x, s, \Delta(\tau)\right)}{s}\right]$$

$$-\frac{\sin sx}{s^{5/3}\delta}\left(\cos\alpha + \sin\alpha B\left(x, s, \Delta(\tau)\right)\right) + O\left(\frac{1}{s^2}\right), \tag{3.26}$$

Now, replacing s by s_n and using Equation (3.23), we have

$$u_{2n}(x) = \frac{\sin\alpha}{n^{2/3}\delta}\left\{\cos nx\left[1 + \frac{A\left(x, n, \Delta\left(\tau\right)\right)}{n}\right] - \frac{\sin nx}{n^{5/3}\pi}\right.$$

$$\left.\times\left[\left(\cot\beta - \cot\alpha - B\left(\pi, n, \Delta\left(\tau\right)\right)\right)x + \left(\cot\alpha + B\left(x, n, \Delta\left(\tau\right)\right)\right)\pi\right]\right\} + O\left(\frac{1}{n^2}\right).$$

From (2.9), (3.19), (3.21), (3.22) and (3.26), we have

$$w_3(x,\lambda) = \frac{\sin\alpha}{s^{2/3}\delta\gamma}\cos sx\left[1+\frac{A(x,s,\Delta(\tau))}{s}\right]$$
$$-\frac{\sin sx}{s^{5/3}\delta\gamma}\left(\cos\alpha+\sin\alpha B(x,s,\Delta(\tau))\right)+O\left(\frac{1}{s^2}\right),$$

Now, replacing s by s_n and using Equation (3.23)

$$u_{3n}(x) = \frac{\sin\alpha}{n^{2/3}\delta\gamma}\left\{\cos nx\left[1+\frac{A(x,n,\Delta(\tau))}{n}\right]-\frac{\sin nx}{n^{5/3}\pi}\right.$$
$$\left.\times\left[\left(\cot\beta-\cot\alpha-B(\pi,n,\Delta(\tau))\right)x+\left(\cot\alpha+B(x,n,\Delta(\tau))\right)\pi\right]\right\}+O\left(\frac{1}{n^2}\right).$$

Thus, we have proven the following theorem.

Theorem 3.3. *If conditions a) and b) are satisfied then, the eigenfunctions $u_n(x)$ of the problem (1.1)-(1.7) have the following asymptotic representation for $n\to\infty$:*

$$u_n(x) = \begin{cases} u_{1n}(x), & x\in[0,h_1), \\ u_{2n}(x), & x\in(h_1,h_2), \\ u_{3n}(x), & x\in(h_2,\pi], \end{cases}$$

where $u_{1n}(x)$, $u_{2n}(x)$ and $u_{3n}(x)$ are defined as in (3.12) and (3.14) respectively.

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