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ON SOME SINGULAR VALUE INEQUALITIES FOR MATRICES

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Abstract. Some singular value inequalities for matrices are given. Among other inequalities it is proved that if f and g be nonnegative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$, for all $t \in [0,\infty)$, then

$$s_j(A_1^*XB_1 + A_2^*XB_2) \\ \leq s_j((A_1^*f^2(\mid X^* \mid)A_1 + A_2^*f^2(\mid X^* \mid)A_2) \oplus (B_1^*g^2(\mid X \mid)B_1 + B_2^*g^2(\mid X \mid)B_2)),$$

for $j = 1,2,...,n$, where A_1, A_2, B_1, B_2, X are square matrices. Our results in this article generalize some existing singular value inequalities of matrices.

1. Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. Let $\|\cdot\|$ stand for any unitarily invariant norm on M_n, i.e., a norm with the property that $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. Any matrix $A \in M_n$ is called positive semidefinite, denoted as $A \geq 0$ if for all $x \in C^n$, $x^*Ax \geq 0$ and it is called positive definite if for all nonzero $x \in C^n$, $x^*Ax > 0$ and it is denoted as $A > 0$. The singular values of matrix A are the eigenvalues of positive semidefinite matrix $\mid A \mid = (AA^*)^{\frac{1}{2}}$, enumerated as $s_1(A) \geq s_2(A) \geq ... \geq s_n(A)$ and repeated according to multiplicity. The direct sum $A \oplus B$ represent the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

The well-known classical arithmetic-geometric mean inequality for $a, b \geq 0$ defined as

$$a^{\frac{1}{2}}b^{\frac{1}{2}} \leq \frac{a+b}{2}. \tag{1.1}$$

Arithmetic-geometric mean inequality is important in matrix theory, functional analysis, electrical networks, etc. For $A, B, X \in M_n$, such that $A, B \geq 0$, R. Bhatia and F. Kittaneh formulated some matrix versions of this inequality in [3,4] one of

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which is the following

$$\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leq \frac{1}{2}\|AX+XB\|. \tag{1.2}$$

From (1.2), for $X = I$ we have the following inequality for positive semidefinite matrices.

$$\|A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{1}{2}\|A+B\|, \tag{1.3}$$

R. Bhatia and F. Kittaneh also have proved in [5] that if $A, B \in M_n$ such that $A, B \geq 0$, then

$$\|A^{\frac{3}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{3}{2}}\| \leq \frac{1}{2}\|(A+B)^2\|. \tag{1.4}$$

From (1.3), (1.4) and also by triangle inequality, we obtain the following inequality

$$\|A^{\frac{3}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{3}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{1}{2}\|(A+B)^2\| + \frac{1}{2}\|A+B\|. \tag{1.5}$$

In [2] L. Zou and Y. Jiang proved that for positive semidefinite matrices $A, B \in M_n$ and $1 \leq j \leq n$, the following inequality also holds

$$2s_j(A^{\frac{3}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{3}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}}) \leq s_j((A+B)^2 + (A+B)), \tag{1.6}$$

and consequently,

$$\|A^{\frac{3}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{3}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{1}{2}\|(A+B)^2 + (A+B)\|. \tag{1.7}$$

The inequality (1.7) is an improvement of the inequality (1.5).

One another interesting inequality for sum and direct sum of matrices proved by R. Bhatia and F. Kittaneh [6] is

$$s_j(A^*B + B^*A) \leq s_j((A^*A + B^*B) \oplus (A^*A + B^*B)), \tag{1.8}$$

where $A, B \in M_n$ and $1 \leq j \leq n$.

In Section 2, we give generalized form of the inequality (1.6) and also, we obtain the X-version of the inequality (1.8).

2. Singular values inequalities for matrices

In this section, we generalize the inequalities (1.6) and also, we obtain X-version of the inequality (1.8). Our results based on Several lemmas. First two lemmas have been given by F. Kittaneh in [1] and Lemma 2.3 can be found in [8, Theorem 1].

Lemma 2.1. Let $T \in M_n$, then the block matrix $\begin{pmatrix} |T| & T^* \\ T & |T^*| \end{pmatrix} \geq 0$.

Lemma 2.2. Let $A, B, C \in M_n$, such that A and B are positive semidefinite, $BC = CA$ and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$, for all $t \in [0, \infty)$. If the block matrix $\begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \geq 0$, then so $\begin{pmatrix} f^2(A) & C^* \\ C & g^2(B) \end{pmatrix} \geq 0$.

Lemma 2.3. Let $A, B, C \in M_n$ such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$, then

$$2s_j(B) \leq s_j\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}, \; j = 1, 2, ..., n. \tag{2.1}$$

The following Lemma was proved in [7].

Lemma 2.4. Let $A, B, C \in M_n$, such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$, then

$$s_j(B) \leq s_j(A \oplus C),\ j = 1, 2, ..., n. \tag{2.2}$$

To give the general form of (1.6), first we prove the following result.

Theorem 2.5. Let $A, B \in M_n$ be any two matrices and r be a positive integer, then

$$2s_j(A(|A|^2 + |B|^2)^{r-1}B^* + AB^*) \leq s_j((|A|^2 + |B|^2)^r + (|A|^2 + |B|^2)),$$

for $j = 1, 2, ..., n$.

Proof. Let $X = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$. Then,

$$X^*X = \begin{pmatrix} A^*A + B^*B & 0 \\ 0 & 0 \end{pmatrix},\ XX^* = \begin{pmatrix} AA^* & AB^* \\ BA^* & BB^* \end{pmatrix}.$$

So, we have

$$(X^*X)^r = \begin{pmatrix} (A^*A + B^*B)^r & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned} (XX^*)^r &= X(X^*X)^{(r-1)}X^* \\ &= \begin{pmatrix} A(A^*A + B^*B)^{(r-1)}A^* & A(A^*A + B^*B)^{(r-1)}B^* \\ B(A^*A + B^*B)^{(r-1)}A^* & B(A^*A + B^*B)^{(r-1)}B^* \end{pmatrix}. \end{aligned}$$

Therefore, we obtain

$$(X^*X)^r + X^*X = \begin{pmatrix} (A^*A + B^*B)^r + A^*A + B^*B & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned} & (XX^*)^r + XX^* \\ = & \begin{pmatrix} A(A^*A + B^*B)^{(r-1)}A^* + AA^* & A(A^*A + B^*B)^{(r-1)}B^* + AB^* \\ B(A^*A + B^*B)^{(r-1)}A^* + BA^* & B(A^*A + B^*B)^{(r-1)}B^* + BB^* \end{pmatrix}. \end{aligned}$$

So, by Lemma 2.3, from the positive semidefinite block matrix $(XX^*)^r + XX^*$, we have

$$\begin{aligned} 2s_j(A(A^*A + B^*B)^{(r-1)}B^* + AB^*) &\leq s_j((XX^*)^r + XX^*) \\ &= s_j((X^*X)^r + X^*X) \\ &= s_j((A^*A + B^*B)^r + (A^*A + B^*B)), \end{aligned}$$

for $j = 1, 2, ..., n$.

The proof is completed. □

When $A, B \in M_n$ be positive semidefinite in Theorem 2.5 and A is replaced by $A^{\frac{1}{2}}$ and B is replaced by $B^{\frac{1}{2}}$, then we obtain the following promised generalization of the inequality (1.6).

Corollary 2.6. Let $A, B \in M_n$ be positive semidefinite and r be a positive integer. Then,

$$2s_j(A^{\frac{1}{2}}(A+B)^{(r-1)}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}}) \le s_j((A+B)^r + (A+B)),$$

for $j = 1, 2, ..., n$.
Remark 2.7. When we take $r = 2$ in Corollary 2.6 , then we obtain the inequality (1.6).

To give the X-version of the inequality (1.8), first we obtain the following result.

Theorem 2.8. Let $A_1, A_2, B_1, B_2, X \in M_n$. If f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$, for all $t \in [0, \infty)$, then

$$\begin{aligned} & s_j(A_1^* X B_1 + A_2^* X B_2) \\ \le \; & s_j((A_1^* f^2(| X^* |)A_1 + A_2^* f^2(| X^* |)A_2) \oplus (B_1^* g^2(| X |)B_1 + B_2^* g^2(| X |)B_2)), \end{aligned}$$

for $j = 1, 2, ..., n$.
Proof. Let $T_1 = \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}$, $T_2 = \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}$.
Since the block matrix $\begin{pmatrix} | X^* | & X \\ X^* & | X | \end{pmatrix}$ is positive semidefinite (by Lemma 2.1) and the block matrix $Y = \begin{pmatrix} f^2(| X^* |) & X \\ X^* & g^2(| X |) \end{pmatrix}$ is positive semidefinite (by Lemma 2.2), so, $T_1^* Y T_1 = \begin{pmatrix} A_1^* f^2(| X^* |)A_1 & A_1^* X B_1 \\ B_1^* X^* A_1 & B_1^* g^2(| X |)B_1 \end{pmatrix} \ge 0$ and also, $T_2^* Y T_2 = \begin{pmatrix} A_2^* f^2(| X^* |)A_2 & A_2^* X B_2 \\ B_2^* X^* A_2 & B_2^* g^2(| X |)B_2 \end{pmatrix} \ge 0$. That is, we have

$$\begin{aligned} & T_1^* Y T_1 + T_2^* Y T_2 \\ = & \begin{pmatrix} A_1^* f^2(| X^* |)A_1 + A_2^* f^2(| X^* |)A_2 & A_1^* X B_1 + A_2^* X B_2 \\ B_1^* X^* A_1 + B_2^* X^* A_2 & B_1^* g^2(| X |)B_1 + B_2^* g^2(| X |)B_2 \end{pmatrix} \ge 0 \end{aligned}$$

So, our desired result now follows by invoking inequality (2.2).
The proof is completed. □

Following is our desired X-version of the inequality (1.8).

Corollary 2.9. Let $A, B, X \in M_n$, then

$$\begin{aligned} & s_j(A^* X B + B^* X A) \\ \le \; & s_j((A^* \mid X^* \mid A + B^* \mid X^* \mid B) \oplus (A^* \mid X \mid A + B^* \mid X \mid B)), \end{aligned}$$

for $j = 1, 2, ..., n$.
Proof. By taking $f(t) = g(t) = t^{\frac{1}{2}}$, $A_1 = B_2 = A$ and $A_2 = B_1 = B$ in Theorem 2.8, we get the desired result.
The proof is completed. □

One another important case follows from Corollary 2.9 for normal matrices.

Corollary 2.10. Let $A, B, X \in M_n$ such that X is normal matrix,then

$$\begin{aligned} & s_j(A^*XB + B^*XA) \\ \leq \quad & s_j((A^* \mid X \mid A + B^* \mid X \mid B) \oplus (A^* \mid X \mid A + B^* \mid X \mid B)), \end{aligned}$$

for $j = 1, 2, ..., n$.
In particular, when X is positive semidefinite matrix , then

$$\begin{aligned} & s_j(A^*XB + B^*XA) \\ \leq \quad & s_j((A^*XA + B^*XB) \oplus (A^*XA + B^*XB)), \end{aligned}$$

for $j = 1, 2, ..., n$.

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