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SOME INEQUALITIES OF OSTROWSKI TYPE IN THREE INDEPENDENT VARIABLES

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Abstract. Some new inequalities of Ostrowski type involving functions of three independent variables are established.

1. INTRODUCTION

In 2001, Cheng in [3] proved the following integral inequality:

Theorem 1.1. *Let $f : [a, b] \to \mathbf{R}$ be an absolutely continuous function such that there exist constants $\gamma, \Gamma \in \mathbf{R}$ with $\gamma \leq f'(t) \leq \Gamma$, $t \in [a, b]$. Then for all $x \in [a, b]$, we have*

$$\begin{aligned} &|\tfrac{1}{2} f(x) - \tfrac{1}{b-a} \int_a^b f(t)\, dt - \tfrac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}| \\ &\leq \tfrac{1}{8(b-a)}((x-a)^2 + (x-b)^2)(\Gamma - \gamma). \end{aligned} \tag{1.1}$$

The constant $\frac{1}{8}$ is sharp (see [4]).

Remark 1.1. If we take $x = a$ or $x = b$ in (1), then we get a sharp trapezoid inequality

$$|\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt| \leq \frac{1}{8}(b-a)(\Gamma - \gamma).$$

In 2010, Sarikaya in [5] established the following inequality of Ostrowski type involving functions of two independent variables.

Theorem 1.2. *Let $f : [a, b] \times [c, d] \to \mathbf{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and suppose that there exist constants $\gamma, \Gamma \in \mathbf{R}$ with $\gamma \leq \frac{\partial^2 f(t,s)}{\partial t \partial s} \leq \Gamma$ for all $(t, s) \in [a, b] \times [c, d]$. Then we have*

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$$
\begin{aligned}
&|\tfrac{1}{4}f(x,y)+\tfrac{1}{4}H(x,y)-\tfrac{1}{2(b-a)}\int_a^b f(t,y)\,dt-\tfrac{1}{2(d-c)}\int_c^d f(x,s)\,ds\\
&-\tfrac{1}{2(b-a)(d-c)}\int_a^b[(y-c)f(t,c)+(d-y)f(t,d)]\,dt\\
&-\tfrac{1}{2(b-a)(d-c)}\int_c^d[(x-a)f(a,s)+(b-x)f(b,s)]\,ds\\
&+\tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,ds\,dt|\\
&\le \tfrac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{32(b-a)(d-c)}(\Gamma-\gamma)
\end{aligned}
\tag{1.2}
$$

for all $(x,y)\in[a,b]\times[c,d]$, *where*

$$
\begin{aligned}
H(x,y)=&\tfrac{(x-a)[(y-c)f(a,c)+(d-y)f(a,d)]+(b-x)[(y-c)f(b,c)+(d-y)f(b,d)]}{(b-a)(d-c)}\\
&+\tfrac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a}+\tfrac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}.
\end{aligned}
$$

Here we have given a revised version for (2) since the expression in [5] and [6] contained a misprint.

Remark 1.2. If we take any one of the four cases $x=a, y=c$; $x=a, y=d$; $x=b, y=c$ and $x=b, y=d$ in (2), then we get a trapezoid type inequality for double integrals.

$$
\begin{aligned}
&|\tfrac{f(a,c)+f(b,d)+f(b,c)+f(a,d)}{4}-\tfrac{1}{2(b-a)}\int_a^b[f(t,c)+f(t,d)]\,dt\\
&-\tfrac{1}{2(d-c)}\int_c^d[f(a,s)+f(b,s)]\,ds+\tfrac{1}{2(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,ds\,dt|\\
&\le\tfrac{(b-a)(d-c)}{32}(\Gamma-\gamma).
\end{aligned}
\tag{1.3}
$$

It is interesting to compare this inequality (3) with the result in [2].

In the literature, we find that Pachpatte was the first author who has established an inequality of Ostrowski type in three independent variables as follows:

Theorem 1.3. *Let* $f:[a,k]\times[b,l]\times[c,m]\to\mathbf{R}$ *be an absolutely continuous function such that the partial derivative of order* 3 *exists and continuous for all* $(t,s,u)\in[a,k]\times[b,l]\times[c,m]$. *Then we have*

$$
\begin{aligned}
&|\tfrac{(k-a)(l-b)(m-c)}{8}[f(a,b,c)+f(a,b,m)+f(a,l,c)+f(a,l,m)\\
&+f(k,b,c)+f(k,b,m)+f(k,l,c)+f(k,l,m)]\\
&-\tfrac{(l-b)(m-c)}{4}\int_a^k[f(t,b,c)+f(t,l,c)+f(t,b,m)+f(t,l,m)]\,dt\\
&-\tfrac{(k-a)(m-c)}{4}\int_b^l[f(a,s,c)+f(k,s,c)+f(a,s,m)+f(k,s,m)]\,ds\\
&-\tfrac{(k-a)(l-b)}{4}\int_c^m[f(a,b,u)+f(a,l,u)+f(k,b,u)+f(k,l,u)]\,du\\
&+\tfrac{(m-c)}{2}\int_a^k\int_b^l[f(t,s,m)+f(t,s,c)]\,ds\,dt\\
&+\tfrac{(k-a)}{2}\int_b^l\int_c^m[f(k,s,u)+f(a,s,u)]\,du\,ds\\
&+\tfrac{(l-b)}{2}\int_a^k\int_c^m[f(t,l,u)+f(t,b,u)]\,du\,dt\\
&-\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt|\\
&\le\tfrac{(k-a)(l-b)(m-c)}{8}\int_a^k\int_b^l\int_c^m|\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}|\,du\,ds\,dt.
\end{aligned}
\tag{1.4}
$$

Here we also have given a revised version for (4) since the expression in [1] contained misprints.

In this paper, we will extend the above result to establish some new Ostrowski type inequalities involving functions of three independent variables.

2. MAIN RESULTS

Theorem 2.1. *Let $f : [a,k] \times [b,l] \times [c,m] \to \mathbf{R}$ be an absolutely continuous function such that the partial derivative of order 3 exists and suppose that there exist constants $\gamma, \Gamma \in \mathbf{R}$ with $\gamma \leq \frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u} \leq \Gamma$ for all $(t,s,u) \in [a,k]\times[b,l]\times[c,m]$. Then we have*

(2.1)

$$\begin{aligned}
&|\tfrac{1}{8}f(x,y,z) + \tfrac{1}{8}H(x,y,z) - \tfrac{1}{4}\int_a^k G_1(t,y,z)\,dt - \tfrac{1}{4}\int_b^l G_2(x,s,z)\,ds - \tfrac{1}{4}\int_c^m G_3(x,y,u)\,du \\
&+\tfrac{1}{2(k-a)(l-b)(m-c)}\{\int_a^k \int_b^l [(z-c)f(t,s,c) + (m-z)f(t,s,m) + (m-c)f(t,s,z)]\,ds\,dt \\
&+\int_b^l \int_c^m [(x-a)f(a,s,u) + (k-x)f(k,s,u) + (k-a)f(x,s,u)]\,du\,ds \\
&+\int_a^k \int_c^m [(y-b)f(t,b,u) + (l-y)f(t,l,u) + (l-b)f(t,y,u)]\,du\,dt\} \\
&-\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k \int_b^l \int_c^m f(t,s,u)\,du\,ds\,dt| \\
&\leq \tfrac{[(x-a)^2+(k-x)^2][(y-b)^2+(l-y)^2][(z-c)^2+(m-z)^2]}{128(k-a)(l-b)(m-c)}(\Gamma - \gamma)
\end{aligned}$$

for all $(x,y,z) \in [a,k] \times [b,l] \times [c,m]$, where

$$\begin{aligned}
&H(x,y,z) = \tfrac{(z-c)f(x,y,c)+(m-z)f(x,y,m)}{m-c} + \tfrac{(x-a)f(a,y,z)+(k-x)f(k,y,z)}{k-a} \\
&+\tfrac{(y-b)f(x,b,z)+(l-y)f(x,l,z)}{l-b} \\
&+\tfrac{(x-a)(y-b)f(a,b,z)+(k-x)(y-b)f(k,b,z)+(x-a)(l-y)f(a,l,z)+(k-x)(l-y)f(k,l,z)}{(k-a)(l-b)} \\
&+\tfrac{(y-b)(z-c)f(x,b,c)+(l-y)(z-c)f(x,l,c)+(y-b)(m-z)f(x,b,m)+(l-y)(m-z)f(x,l,m)}{(l-b)(m-c)} \\
&+\tfrac{(x-a)(z-c)f(a,y,c)+(k-x)(z-c)f(k,y,c)+(x-a)(m-z)f(a,y,m)+(k-x)(m-z)f(k,y,m)}{(k-a)(m-c)} \\
&+\tfrac{1}{(k-a)(l-b)(m-c)}\{(y-b)(z-c)[(x-a)f(a,b,c) + (k-x)f(k,b,c)] \\
&+(l-y)(z-c)[(x-a)f(a,l,c) + (k-x)f(k,l,c)] \\
&+(y-b)(m-z)[(x-a)f(a,b,m) + (k-x)f(k,b,m)] \\
&+(l-y)(m-z)[(x-a)f(a,l,m) + (k-x)f(k,l,m)]\},
\end{aligned}$$

$$\begin{aligned}
G_1(t,y,z) = &\tfrac{f(t,y,z)}{k-a} + \tfrac{(y-b)f(t,b,z)+(l-y)f(t,l,z)}{(k-a)(l-b)} + \tfrac{(z-c)f(t,y,c)+(m-z)f(t,y,m)}{(k-a)(m-c)} \\
&+\tfrac{(z-c)[(y-b)f(t,b,c)+(l-y)f(t,l,c)]+(m-z)[(y-b)f(t,b,m)+(l-y)f(t,l,m)]}{(k-a)(l-b)(m-c)},
\end{aligned}$$

$$\begin{aligned}
G_2(x,s,z) = &\tfrac{f(x,s,z)}{l-b} + \tfrac{(x-a)f(a,s,z)+(k-x)f(k,s,z)}{(k-a)(l-b)} + \tfrac{(z-c)f(x,s,c)+(m-z)f(x,s,m)}{(l-b)(m-c)} \\
&+\tfrac{(z-c)[(x-a)f(a,s,c)+(k-x)f(k,s,c)]+(m-z)[(x-a)f(a,s,m)+(k-x)f(k,s,m)]}{(k-a)(l-b)(m-c)},
\end{aligned}$$

$$\begin{aligned}
G_3(x,y,u) = &\tfrac{f(x,y,u)}{m-c} + \tfrac{(y-b)f(x,b,u)+(l-y)f(x,l,u)}{(l-b)(m-c)} + \tfrac{(x-a)f(a,y,u)+(k-x)f(k,y,u)}{(k-a)(m-c)} \\
&+\tfrac{(x-a)[(y-b)f(a,b,u)+(l-y)f(a,l,u)]+(k-x)[(y-b)f(k,b,u)+(l-y)f(k,l,u)]}{(k-a)(l-b)(m-c)}.
\end{aligned}$$

Proof. Put

$$p(x,t) := \begin{cases} t - \frac{a+x}{2}, & t \in [a,x], \\ t - \frac{k+x}{2}, & t \in (x,k], \end{cases}$$

$$q(y,s) := \begin{cases} s - \frac{b+y}{2}, & s \in [b,y], \\ s - \frac{l+y}{2}, & s \in (y,l], \end{cases}$$

and

$$r(z,u) := \begin{cases} u - \frac{c+z}{2}, & u \in [c,z], \\ u - \frac{m+z}{2}, & u \in (z,m]. \end{cases}$$

We have

$$\begin{aligned}
&\int_a^k \int_b^l \int_c^m p(x,t)q(y,s)r(z,u)\frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&= \int_a^x \int_b^y \int_c^z (t - \tfrac{a+x}{2})(s - \tfrac{b+y}{2})(u - \tfrac{c+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&+ \int_a^x \int_b^y \int_z^m (t - \tfrac{a+x}{2})(s - \tfrac{b+y}{2})(u - \tfrac{m+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&+ \int_a^x \int_y^l \int_c^z (t - \tfrac{a+x}{2})(s - \tfrac{l+y}{2})(u - \tfrac{c+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&+ \int_a^x \int_y^l \int_z^m (t - \tfrac{a+x}{2})(s - \tfrac{l+y}{2})(u - \tfrac{m+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&+ \int_x^k \int_b^y \int_c^z (t - \tfrac{k+x}{2})(s - \tfrac{b+y}{2})(u - \tfrac{c+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&+ \int_x^k \int_b^y \int_z^m (t - \tfrac{k+x}{2})(s - \tfrac{b+y}{2})(u - \tfrac{m+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&+ \int_x^k \int_y^l \int_c^z (t - \tfrac{k+x}{2})(s - \tfrac{l+y}{2})(u - \tfrac{c+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&+ \int_x^k \int_y^l \int_z^m (t - \tfrac{k+x}{2})(s - \tfrac{l+y}{2})(u - \tfrac{m+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt.
\end{aligned} \tag{2.2}$$

Integrating by parts three times, we can state:

$$\begin{aligned}
&\int_a^x \int_b^y \int_c^z (t - \tfrac{a+x}{2})(s - \tfrac{b+y}{2})(u - \tfrac{c+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&= \tfrac{(x-a)(y-b)(z-c)}{8}[f(x,y,z) + f(x,y,c) + f(x,b,z) + f(x,b,c) \\
&+f(a,y,z) + f(a,y,c) + f(a,b,z) + f(a,b,c)] \\
&-\tfrac{(y-b)(z-c)}{4}\int_a^x [f(t,y,z) + f(t,y,c) + f(t,b,z) + f(t,b,c)]\,dt \\
&-\tfrac{(x-a)(z-c)}{4}\int_b^y [f(x,s,z) + f(x,s,c) + f(a,s,z) + f(a,s,c)]\,ds \\
&-\tfrac{(x-a)(y-b)}{4}\int_c^z [f(x,y,u) + f(x,b,u) + f(a,y,u) + f(a,b,u)]\,du \\
&+\tfrac{x-a}{2}\int_b^y \int_c^z [f(x,s,u) + f(a,s,u)]\,du\,ds \\
&+\tfrac{y-b}{2}\int_a^x \int_c^z [f(t,y,u) + f(t,b,u)]\,du\,dt \\
&+\tfrac{z-c}{2}\int_a^x \int_b^y [f(t,s,z) + f(t,s,c)]\,ds\,dt \\
&-\int_a^x \int_b^y \int_c^z f(t,s,u)\,du\,ds\,dt.
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\int_a^x \int_b^y \int_z^m (t - \tfrac{a+x}{2})(s - \tfrac{b+y}{2})(u - \tfrac{m+z}{2})\frac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\,du\,ds\,dt \\
&= \tfrac{(x-a)(y-b)(m-z)}{8}[f(x,y,m) + f(x,y,z) + f(x,b,m) + f(x,b,z) \\
&+f(a,y,m) + f(a,y,z) + f(a,b,m) + f(a,b,z)] \\
&-\tfrac{(y-b)(m-z)}{4}\int_a^x [f(t,y,m) + f(t,y,z) + f(t,b,m) + f(t,b,z)]\,dt \\
&-\tfrac{(x-a)(m-z)}{4}\int_b^y [f(x,s,m) + f(x,s,z) + f(a,s,m) + f(a,s,z)]\,ds \\
&-\tfrac{(x-a)(y-b)}{4}\int_z^m [f(x,y,u) + f(x,b,u) + f(a,y,u) + f(a,b,u)]\,du \\
&+\tfrac{x-a}{2}\int_b^y \int_z^m [f(x,s,u) + f(a,s,u)]\,du\,ds \\
&+\tfrac{y-b}{2}\int_a^x \int_z^m [f(t,y,u) + f(t,b,u)]\,du\,dt \\
&+\tfrac{m-z}{2}\int_a^x \int_b^y [f(t,s,m) + f(t,s,z)]\,ds\,dt \\
&-\int_a^x \int_b^y \int_z^m f(t,s,u)\,du\,ds\,dt.
\end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&\int_a^x \int_y^l \int_c^z (t - \tfrac{a+x}{2})(s - \tfrac{l+y}{2})(u - \tfrac{c+z}{2}) \tfrac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\, du\, ds\, dt \\
&= \tfrac{(x-a)(l-y)(z-c)}{8}[f(x,l,z) + f(x,l,c) + f(x,y,z) + f(x,y,c) \\
&+ f(a,l,z) + f(a,l,c) + f(a,y,z) + f(a,y,c)] \\
&- \tfrac{(l-y)(z-c)}{4} \int_a^x [f(t,l,z) + f(t,l,c) + f(t,y,z) + f(t,y,c)]\, dt \\
&- \tfrac{(x-a)(z-c)}{4} \int_y^l [f(x,s,z) + f(x,s,c) + f(a,s,z) + f(a,s,c)]\, ds \\
&- \tfrac{(x-a)(l-y)}{4} \int_c^z [f(x,l,u) + f(x,y,u) + f(a,l,u) + f(a,y,u)]\, du \\
&+ \tfrac{x-a}{2} \int_y^l \int_c^z [f(x,s,u) + f(a,s,u)]\, du\, ds \\
&+ \tfrac{l-y}{2} \int_a^x \int_c^z [f(t,l,u) + f(t,y,u)]\, du\, dt \\
&+ \tfrac{z-c}{2} \int_a^x \int_y^l [f(t,s,z) + f(t,s,c)]\, ds\, dt \\
&- \int_a^x \int_y^l \int_c^z f(t,s,u)\, du\, ds\, dt.
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
&\int_a^x \int_y^l \int_z^m (t - \tfrac{a+x}{2})(s - \tfrac{l+y}{2})(u - \tfrac{m+z}{2}) \tfrac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\, du\, ds\, dt \\
&= \tfrac{(x-a)(l-y)(m-z)}{8}[f(x,l,m) + f(x,l,z) + f(x,y,m) + f(x,y,z) \\
&+ f(a,l,m) + f(a,l,z) + f(a,y,m) + f(a,y,z)] \\
&- \tfrac{(l-y)(m-z)}{4} \int_a^x [f(t,l,m) + f(t,l,z) + f(t,y,m) + f(t,y,z)]\, dt \\
&- \tfrac{(x-a)(m-z)}{4} \int_y^l [f(x,s,m) + f(x,s,z) + f(a,s,m) + f(a,s,z)]\, ds \\
&- \tfrac{(x-a)(l-y)}{4} \int_z^m [f(x,l,u) + f(x,y,u) + f(a,l,u) + f(a,y,u)]\, du \\
&+ \tfrac{x-a}{2} \int_y^l \int_z^m [f(x,s,u) + f(a,s,u)]\, du\, ds \\
&+ \tfrac{l-y}{2} \int_a^x \int_z^m [f(t,l,u) + f(t,y,u)]\, du\, dt \\
&+ \tfrac{m-z}{2} \int_a^x \int_y^l [f(t,s,m) + f(t,s,z)]\, ds\, dt \\
&- \int_a^x \int_y^l \int_z^m f(t,s,u)\, du\, ds\, dt.
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
&\int_x^k \int_b^y \int_c^z (t - \tfrac{k+x}{2})(s - \tfrac{b+y}{2})(u - \tfrac{c+z}{2}) \tfrac{\partial^3 f(t,s,u)}{\partial t \partial s \partial u}\, du\, ds\, dt \\
&= \tfrac{(k-x)(y-b)(z-c)}{8}[f(k,y,z) + f(k,y,c) + f(k,b,z) + f(k,b,c) \\
&+ f(x,y,z) + f(x,y,c) + f(x,b,z) + f(x,b,c)] \\
&- \tfrac{(y-b)(z-c)}{4} \int_x^k [f(t,y,z) + f(t,y,c) + f(t,b,z) + f(t,b,c)]\, dt \\
&- \tfrac{(k-x)(z-c)}{4} \int_b^y [f(k,s,z) + f(k,s,c) + f(x,s,z) + f(x,s,c)]\, ds \\
&- \tfrac{(k-x)(y-b)}{4} \int_c^z [f(k,y,u) + f(k,b,u) + f(x,y,u) + f(x,b,u)]\, du \\
&+ \tfrac{k-x}{2} \int_b^y \int_c^z [f(k,s,u) + f(x,s,u)]\, du\, ds \\
&+ \tfrac{y-b}{2} \int_x^k \int_c^z [f(t,y,u) + f(t,b,u)]\, du\, dt \\
&+ \tfrac{z-c}{2} \int_x^k \int_b^y [f(t,s,z) + f(t,s,c)]\, ds\, dt \\
&- \int_x^k \int_b^y \int_c^z f(t,s,u)\, du\, ds\, dt.
\end{aligned} \tag{2.7}
$$

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\begin{aligned}
&\int_x^k \int_b^y \int_z^m (t-\tfrac{k+x}{2})(s-\tfrac{b+y}{2})(u-\tfrac{m+z}{2})\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt\\
&= \tfrac{(k-x)(y-b)(m-z)}{8}[f(k,y,m)+f(k,y,z)+f(k,b,m)+f(k,b,z)\\
&+f(x,y,m)+f(x,y,z)+f(x,b,m)+f(x,b,z)]\\
&-\tfrac{(y-b)(m-z)}{4}\int_x^k[f(t,y,m)+f(t,y,z)+f(t,b,m)+f(t,b,z)]\,dt\\
&-\tfrac{(k-x)(m-z)}{4}\int_b^y[f(k,s,m)+f(k,s,z)+f(x,s,m)+f(x,s,z)]\,ds\\
&-\tfrac{(k-x)(y-b)}{4}\int_z^m[f(k,y,u)+f(k,b,u)+f(x,y,u)+f(x,b,u)]\,du\\
&+\tfrac{k-x}{2}\int_b^y\int_z^m[f(k,s,u)+f(x,s,u)]\,du\,ds\\
&+\tfrac{y-b}{2}\int_x^k\int_z^m[f(t,y,u)+f(t,b,u)]\,du\,dt\\
&+\tfrac{m-z}{2}\int_x^k\int_b^y[f(t,s,m)+f(t,s,z)]\,ds\,dt\\
&-\int_x^k\int_b^y\int_z^m f(t,s,u)\,du\,ds\,dt.
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&\int_x^k \int_y^l \int_c^z (t-\tfrac{k+x}{2})(s-\tfrac{l+y}{2})(u-\tfrac{c+z}{2})\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt\\
&= \tfrac{(k-x)(l-y)(z-c)}{8}[f(k,l,z)+f(k,l,c)+f(k,y,z)+f(k,y,c)\\
&+f(x,l,z)+f(x,l,c)+f(x,y,z)+f(x,y,c)]\\
&-\tfrac{(l-y)(z-c)}{4}\int_x^k[f(t,l,z)+f(t,l,c)+f(t,y,z)+f(t,y,c)]\,dt\\
&-\tfrac{(k-x)(z-c)}{4}\int_b^y[f(k,s,z)+f(k,s,c)+f(x,s,z)+f(x,s,c)]\,ds\\
&-\tfrac{(k-x)(l-y)}{4}\int_c^z[f(k,l,u)+f(k,y,u)+f(x,l,u)+f(x,y,u)]\,du\\
&+\tfrac{k-x}{2}\int_y^l\int_c^z[f(k,s,u)+f(x,s,u)]\,du\,ds\\
&+\tfrac{l-y}{2}\int_x^k\int_c^z[f(t,l,u)+f(t,y,u)]\,du\,dt\\
&+\tfrac{z-c}{2}\int_x^k\int_y^l[f(t,s,z)+f(t,s,c)]\,ds\,dt\\
&-\int_x^k\int_y^l\int_c^z f(t,s,u)\,du\,ds\,dt.
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&\int_x^k \int_y^l \int_z^m (t-\tfrac{k+x}{2})(s-\tfrac{l+y}{2})(u-\tfrac{m+z}{2})\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt\\
&= \tfrac{(k-x)(l-y)(m-z)}{8}[f(k,l,m)+f(k,l,z)+f(k,y,m)+f(k,y,z)\\
&+f(x,l,m)+f(x,l,z)+f(x,y,m)+f(x,y,z)]\\
&-\tfrac{(l-y)(m-z)}{4}\int_x^k[f(t,l,m)+f(t,l,z)+f(t,y,m)+f(t,y,z)]\,dt\\
&-\tfrac{(k-x)(m-z)}{4}\int_y^l[f(k,s,m)+f(k,s,z)+f(x,s,m)+f(x,s,z)]\,ds\\
&-\tfrac{(k-x)(l-y)}{4}\int_z^m[f(k,l,u)+f(k,y,u)+f(x,l,u)+f(x,y,u)]\,du\\
&+\tfrac{k-x}{2}\int_y^l\int_z^m[f(k,s,u)+f(x,s,u)]\,du\,ds\\
&+\tfrac{l-y}{2}\int_x^k\int_z^m[f(t,l,u)+f(t,y,u)]\,du\,dt\\
&+\tfrac{m-z}{2}\int_x^k\int_y^l[f(t,s,m)+f(t,s,z)]\,ds\,dt\\
&-\int_x^k\int_y^l\int_z^m f(t,s,u)\,du\,ds\,dt.
\end{aligned}
\tag{2.10}
$$

From (6)-(14), we can easily deduce that

$$
\begin{aligned}
&\int_a^k \int_b^l \int_c^m p(x,t)q(y,s)r(z,u)\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&= \tfrac{1}{8}\{(k-a)(l-b)(m-c)f(x,y,z)+(k-a)(l-b)[(z-c)f(x,y,c)+(m-z)f(x,y,m)] \\
&+(l-b)(m-c)[(x-a)f(a,y,z)+(k-x)f(k,y,z)] \\
&+(k-a)(m-c)[(y-b)f(x,b,z)+(l-y)f(x,l,z)] \\
&+(m-c)[(x-a)(y-b)f(a,b,z)+(k-x)(y-b)f(k,b,z) \\
&+(x-a)(l-y)f(a,l,z)+(k-x)(l-y)f(k,l,z)] \\
&+(k-a)[(y-b)(z-c)f(x,b,c)+(l-y)(z-c)f(x,l,c) \\
&+(y-b)(m-z)f(x,b,m)+(l-y)(m-z)f(x,l,m)] \\
&+(l-b)[(x-a)(z-c)f(a,y,c)+(k-x)(z-c)f(k,y,c) \\
&+(x-a)(m-z)f(a,y,m)+(k-x)(m-z)f(k,y,m)] \\
&+(y-b)(z-c)[(x-a)f(a,b,c)+(k-x)f(k,b,c)] \\
&+(l-y)(z-c)[(x-a)f(a,l,c)+(k-x)f(k,l,c)] \\
&+(y-b)(m-z)[(x-a)f(a,b,m)+(k-x)f(k,b,m)] \\
&+(l-y)(m-z)[(x-a)f(a,l,m)+(k-x)f(k,l,m)]\} \\
&-\tfrac{1}{4}\int_a^k\{(l-b)(m-c)f(t,y,z)+(m-c)[(y-b)f(t,b,z)+(l-y)f(t,l,z)] \\
&+(l-b)[(z-c)f(t,y,c)+(m-z)f(t,y,m)]+(z-c)[(y-b)f(t,y,c)+(l-y)f(t,l,c)] \\
&+(m-z)[(y-b)f(t,b,m)+(l-y)f(t,l,m)]\}\,dt \\
&-\tfrac{1}{4}\int_b^l\{(k-a)(m-c)f(x,s,z)+(m-c)[(x-a)f(a,s,z)+(k-x)f(k,s,z)] \\
&+(k-a)[(z-c)f(x,s,c)+(m-z)f(x,s,m)]+(z-c)[(x-a)f(a,s,c)+(k-x)f(k,s,c)] \\
&+(m-z)[(x-a)f(a,s,m)+(k-x)f(k,s,m)]\}\,ds \\
&-\tfrac{1}{4}\int_c^m\{(k-a)(l-b)f(x,y,u)+(k-a)[(y-b)f(x,b,u)+(l-y)f(x,l,u)] \\
&+(l-b)[(x-a)f(a,y,u)+(k-x)f(k,y,u)]+(y-b)[(x-a)f(a,b,u)+(k-x)f(k,b,u)] \\
&+(l-y)[(x-a)f(a,l,u)+(k-x)f(k,l,u)]\}\,du \\
&+\tfrac{1}{2}\int_a^k\int_b^l[(z-c)f(t,s,c)+(m-z)f(t,s,m)+(m-c)f(t,s,z)]\,ds\,dt \\
&+\tfrac{1}{2}\int_b^l\int_c^m[(x-a)f(a,s,u)+(k-x)f(k,s,u)+(k-a)f(x,s,u)]\,du\,ds \\
&+\tfrac{1}{2}\int_a^k\int_c^m[(y-b)f(t,b,u)+(l-y)f(t,l,u)+(l-b)f(t,y,u)]\,du\,dt\} \\
&-\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt,
\end{aligned}
$$

and it follows that

(2.11)

$$
\begin{aligned}
&\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k \int_b^l \int_c^m p(x,t)q(y,s)r(z,u)\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt \\
&= \tfrac{1}{8}f(x,y,z)+\tfrac{1}{8}H(x,y,z)-\tfrac{1}{4}\int_a^k G_1(t,y,z)\,dt-\tfrac{1}{4}\int_b^l G_2(x,s,z)\,ds-\tfrac{1}{4}\int_c^m G_3(x,y,u)\,du \\
&+\tfrac{1}{2(k-a)(l-b)(m-c)}\{\int_a^k\int_b^l[(z-c)f(t,s,c)+(m-z)f(t,s,m)+(m-c)f(t,s,z)]\,ds\,dt \\
&+\int_b^l\int_c^m[(x-a)f(a,s,u)+(k-x)f(k,s,u)+(k-a)f(x,s,u)]\,du\,ds \\
&+\int_a^k\int_c^m[(y-b)f(t,b,u)+(l-y)f(t,l,u)+(l-b)f(t,y,u)]\,du\,dt\} \\
&-\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt.
\end{aligned}
$$

We also have

$$
\int_a^k\int_b^l\int_c^m p(x,t)q(y,s)r(z,u)\,du\,ds\,dt=0. \tag{2.12}
$$

Let $M=\frac{\Gamma+\gamma}{2}$. From (16), it follows that

$$\begin{aligned}(2.13)\qquad &\int_a^k\int_b^l\int_c^m p(x,t)q(y,s)r(z,u)[\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}-M]\,du\,ds\,dt\\ &=\int_a^k\int_b^l\int_c^m p(x,t)q(y,s)r(z,u)\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}\,du\,ds\,dt.\end{aligned}$$

On the other hand, we have

(2.14)

$$\begin{aligned}&|\int_a^k\int_b^l\int_c^m p(x,t)q(y,s)r(z,u)[\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}-M]\,du\,ds\,dt|\\ \le &\max_{(t,s,u)\in[a,k]\times\in[b,l]\times[c,m]}|\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}-M|\int_a^k\int_b^l\int_c^m|p(x,t)q(y,s)r(z,u)|\,du\,ds\,dt.\end{aligned}$$

Moreover,

$$(2.15)\qquad \max_{(t,s,u)\in[a,k]\times\in[b,l]\times[c,m]}|\frac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}-M|\le\frac{\Gamma-\gamma}{2}$$

and

(2.16)

$$\int_a^k\int_b^l\int_c^m|p(x,t)q(y,s)r(z,u)|\,du\,ds\,dt=\frac{[(x-a)^2+(k-x)^2][(y-b)^2+(l-y)^2][(z-c)^2+(m-z)^2]}{64}.$$

From (18)-(20), we get

$$\begin{aligned}(2.17)\qquad &|\int_a^k\int_b^l\int_c^m p(x,t)q(y,s)r(z,u)[\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}-M]\,du\,ds\,dt|\\ &\le\tfrac{[(x-a)^2+(k-x)^2][(y-b)^2+(l-y)^2][(z-c)^2+(m-z)^2]}{128}(\Gamma-\gamma).\end{aligned}$$

Finally, from (15), (17) and (21), we see that the inequality (5) holds.

The proof of Theorem 4 is complete.

Remark 2.1. If we take any one of the eight cases $x=a,y=b,z=c$; $x=a,y=b,z=m$; $x=a,y=l,z=c$; $x=a,y=l,z=m$; $x=k,y=b,z=c$; $x=k,y=b,z=m$; $x=k,y=l,z=c$ and $x=k,y=l,z=m$ in (5), then we get the following inequality for triple integrals.

$$\begin{aligned}&|\tfrac{f(a,b,c)+f(a,b,m)+f(a,l,c)+f(a,l,m)+f(k,b,c)+f(k,b,m)+f(k,l,c)+f(k,l,m)}{8}\\ &-\tfrac{1}{4(k-a)}\int_a^k[f(t,b,c)+f(t,l,c)+f(t,b,m)+f(t,l,m)]\,dt\\ &-\tfrac{1}{4(l-b)}\int_b^l[f(a,s,c)+f(k,s,c)+f(a,s,m)+f(k,s,m)]\,ds\\ &-\tfrac{1}{4(m-c)}\int_c^m[f(a,b,u)+f(a,l,u)+f(k,b,u)+f(k,l,u)]\,du\\ &+\tfrac{1}{2(k-a)(l-b)}\int_a^k\int_b^l[f(t,s,m)+f(t,s,c)]\,ds\,dt\\ &+\tfrac{1}{2(l-b)(m-c)}\int_b^l\int_c^m[f(k,s,u)+f(a,s,u)]\,du\,ds\\ &+\tfrac{1}{2(k-a)(m-c)}\int_a^k\int_c^m[f(t,l,u)+f(t,b,u)]\,du\,dt\\ &-\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt|\\ &\le\tfrac{(k-a)(l-b)(m-c)}{128}(\Gamma-\gamma).\end{aligned}$$

Theorem 2.2. *Let $f:[a,k]\times[b,l]\times[c,m]\to\mathbf{R}$ be an absolutely continuous function such that the partial derivative of order 3 exists and continuous for all $(t,s,u)\in[a,k]\times[b,l]\times[c,m]$. Then we have*

(2.18)

$$\begin{aligned}&|\tfrac{1}{8}f(x,y,z)+\tfrac{1}{8}H(x,y,z)-\tfrac{1}{4}\int_a^k G_1(t,y,z)\,dt-\tfrac{1}{4}\int_b^l G_2(x,s,z)\,ds-\tfrac{1}{4}\int_c^m G_3(x,y,u)\,du\\&+\tfrac{1}{2(k-a)(l-b)(m-c)}\{\int_a^k\int_b^l[(z-c)f(t,s,c)+(m-z)f(t,s,m)+(m-c)f(t,s,z)]\,ds\,dt\\&+\int_b^l\int_c^m[(x-a)f(a,s,u)+(k-x)f(k,s,u)+(k-a)f(x,s,u)]\,du\,ds\\&+\int_a^k\int_c^m[(y-b)f(t,b,u)+(l-y)f(t,l,u)+(l-b)f(t,y,u)]\,du\,dt\}\\&-\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt|\\&\leq \tfrac{[\frac{k-a}{2}+|x-\frac{a+k}{2}|][\frac{l-b}{2}+|y-\frac{b+l}{2}|][\frac{m-c}{2}+|x-\frac{c+m}{2}|]}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m|\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}|\,du\,ds\,dt\end{aligned}$$

for all $(x,y,z)\in[a,b]\times[b,l]\times[c,m]$, *where* $H(x,y,z)$, $G_1(t,y,z)$, $G_2(x,s,z)$ *and* $G_3(x,y,u)$ *are as defined in Theorem 4.*

Proof. From (15) we get

$$\begin{aligned}&|\tfrac{1}{8}f(x,y,z)+\tfrac{1}{8}H(x,y,z)-\tfrac{1}{4}\int_a^k G_1(t,y,z)\,dt-\tfrac{1}{4}\int_b^l G_2(x,s,z)\,ds-\tfrac{1}{4}\int_c^m G_3(x,y,u)\,du\\&+\tfrac{1}{2(k-a)(l-b)(m-c)}\{\int_a^k\int_b^l[(z-c)f(t,s,c)+(m-z)f(t,s,m)+(m-c)f(t,s,z)]\,ds\,dt\\&+\int_b^l\int_c^m[(x-a)f(a,s,u)+(k-x)f(k,s,u)+(k-a)f(x,s,u)]\,du\,ds\\&+\int_a^k\int_c^m[(y-b)f(t,b,u)+(l-y)f(t,l,u)+(l-b)f(t,y,u)]\,du\,dt\}\\&-\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt|\\&\leq \tfrac{\max_{(t,s,u)\in[a,k]\times\in[b,l]\times[c,m]}|p(x,t)q(y,s)r(z,u)|}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m|\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}|\,du\,ds\,dt\end{aligned}$$

and observe that

(2.19)

$$\max_{(t,s,u)\in[a,k]\times\in[b,l]\times[c,m]}|p(x,t)q(y,s)r(z,u)|=[\frac{k-a}{2}+|x-\frac{a+k}{2}|][\frac{l-b}{2}+|y-\frac{b+l}{2}|][\frac{m-c}{2}+|x-\frac{c+m}{2}|],$$

we can easily obtain the inequality (22).

Remark 2.2. If we take any one of the eight cases $x=a,y=b,z=c$; $x=a,y=b,z=m$; $x=a,y=l,z=c$; $x=a,y=l,z=m$; $x=k,y=b,z=c$; $x=k,y=b,z=m$; $x=k,y=l,z=c$ and $x=k,y=l,z=m$ in (22), then we get the following inequality for triple integrals.

$$\begin{aligned}&|\tfrac{f(a,b,c)+f(a,b,m)+f(a,l,c)+f(a,l,m)+f(k,b,c)+f(k,b,m)+f(k,l,c)+f(k,l,m)}{8}\\&-\tfrac{1}{4(k-a)}\int_a^k[f(t,b,c)+f(t,l,c)+f(t,b,m)+f(t,l,m)]\,dt\\&-\tfrac{1}{4(l-b)}\int_b^l[f(a,s,c)+f(k,s,c)+f(a,s,m)+f(k,s,m)]\,ds\\&-\tfrac{1}{4(m-c)}\int_c^m[f(a,b,u)+f(a,l,u)+f(k,b,u)+f(k,l,u)]\,du\\&+\tfrac{1}{2(k-a)(l-b)}\int_a^k\int_b^l[f(t,s,m)+f(t,s,c)]\,ds\,dt\\&+\tfrac{1}{2(l-b)(m-c)}\int_b^l\int_c^m[f(k,s,u)+f(a,s,u)]\,du\,ds\\&+\tfrac{1}{2(k-a)(m-c)}\int_a^k\int_c^m[f(t,l,u)+f(t,b,u)]\,du\,dt\\&-\tfrac{1}{(k-a)(l-b)(m-c)}\int_a^k\int_b^l\int_c^m f(t,s,u)\,du\,ds\,dt|\\&\leq\tfrac{1}{8}\int_a^k\int_b^l\int_c^m|\tfrac{\partial^3 f(t,s,u)}{\partial t\partial s\partial u}|\,du\,ds\,dt.\end{aligned}\tag{2.20}$$

It is clear that inequality (24) is just the same as inequality (4), and thus we may regard that Theorem 5 is a generalization of Theorem 3.

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